ON COMPLEXITY OF MODEL-CHECKING FOR THE TQL LOGIC

Iovka Boneva, Jean-Marc Talbot

*Laboratoire d'Informatique Fondamentale de Lille, France
INRIA team Mostrare*

Abstract In this paper we study the complexity of the model-checking problem for the tree logic introduced as the basis for the query language TQL [Cardelli and Ghelli, 2001]. We define two distinct fragments of this logic: TL containing only spatial connectives and TL^3 containing spatial connectives and quantification. We show that the combined complexity of TL is PSPACE-hard. We also study data complexity of model-checking and show that it is linear for TL , hard for all levels of the polynomial hierarchy for TL^3 and PSPACE-hard for the full logic. Finally we devise a polynomial space model-checking algorithm showing this way that the model-checking problem for the TQL logic is PSPACE-complete.

1. Introduction

The development of the WEB made semi-structured data prominent. Semi-structured data can be represented by various models such as graphs [Abiteboul et al., 2000], unranked-ordered trees [Neven, 2002] (as for XML documents) or unranked-unordered trees. In [Cardelli, 2000], Cardelli noticed the similarity between this model of unranked-unordered trees and processes from the ambient calculus [Cardelli and Gordon, 2000b]. He also suggested that the ambient logic [Cardelli and Gordon, 2000a] could be a formalism for querying such data representation. Following those ideas, Cardelli and Ghelli proposed in [Cardelli and Ghelli, 2001] TQL (Tree Query Language). The main features of TQL are a semi-structured data representation based on (static) ambient processes and a query formalism based on a tree logic which is an extension of the spatial fragment of the ambient logic. The TQL logic is a very expressive logic which contains Boolean operations, spatial connectives, quantification over labels and trees and a fixed point operator.

The model-checking problem is to decide for a tree and a formula, whether the tree satisfies the formula. In the context of databases, this problem (also called Boolean query) can be used to test whether a database satisfies some integrity constraints. We focus in this paper on the model-checking problem

for the TQL logic and two natural fragments of it: we investigate in particular the combined and data complexity for these fragments; we recall that the combined complexity of model-checking is the complexity of the model-checking problem when both the tree and the formula are inputs of the problem. The data complexity is the complexity considering only the tree as input of the problem.

One of our main results is that combined complexity for model-checking for the tree logic is PSPACE-complete. To prove complexity upper bound, we design a polynomial space model-checking algorithm inspired from the local model-checking of the μ -calculus [Stirling and Walker, 1991]. Combined complexity of model-checking for the tree logic is already known to be PSPACE-hard [Charatonik et al., 2001]. For the proof of this latter, the authors use quantification over labels. We show here that PSPACE-hardness of model-checking is obtained even for the smallest spatial fragment of the tree logic, that is, the fragment containing only spatial connectives and Boolean operators (we call this fragment TL). This result is surprising as it shows that quantification and fixed point operator, although adding expressive power to the logic, do not make model-checking harder.

Our second contribution concerns the data complexity of model-checking: we show that the data complexity is linear for the TL fragment and PSPACE-hard for the full logic. We obtain also an intermediate result: the data complexity of the fragment obtained by removing fixed point operators from the full logic is hard for any level of the polynomial hierarchy.

The paper is organized as follows: in Sect. 2, we introduce information trees (a representation as nested multisets for edge-labeled, unranked and unordered trees). We give also the syntax and the semantics of the tree logic proposed by Cardelli and Ghelli and define two fragments of this logic. Sect. 3 is devoted to the study of complexity lower bounds of the model-checking problem for the tree logic and its fragments. Finally, we present in Sect. 4 a polynomial space model-checking algorithm for the tree logic.

2. Definitions

2.1 The Tree Model

We consider edge-labeled, unranked and unordered trees represented, following [Cardelli and Ghelli, 2001], as nested multisets. This is a quite natural interpretation when dealing with unordered unranked trees.

We denote $\{\}$ the empty multiset and $\uplus$ the multiset union. We consider a countable set of labels Λ and we define the set $\mathcal{D}$ of *information trees*¹ as the

¹In [Cardelli and Ghelli, 2001], the authors considered additionally the notion of *info-terms*, which correspond to ambient processes. They also define a denotation for info-terms in terms of information trees. Here we omit this notion of info-terms to speak directly about information trees.

least one satisfying: (i) the empty multiset $\{\}$ is in $\mathcal{D}$, (ii) if m is a label from Λ and A belongs to $\mathcal{D}$ then $\{m[A]\}$ belongs to $\mathcal{D}$ and (iii) if A, A' belong to $\mathcal{D}$ then $A \uplus A'$ belongs to $\mathcal{D}$.

For any tree A , we denote $fn(A)$ the set of labels appearing in A .

We define the *component relation* as a partial ordering relation over information trees recursively defined as follows: a tree A is a component of a tree A' (denoted $A \sqsubseteq A'$) if either A is a subset of A' or, for some element $m[B]$ in A' , A is a component of B . The component relation on information trees differs from the usual subtree relation: considering the tree structure of some information tree A , a subtree can be identified by a single node in A whereas a component of A is characterized by a *set* of sibling nodes. We use a different term to avoid counter-intuitions. Note also that the number of subtrees for some tree A is linear in the size of A whereas the number of components of A may be exponential in the size of this tree.

2.2 The Tree Logics $TL_\nu^\exists$, $TL^\exists$ and TL

This section presents the tree logic defined in [Cardelli and Ghelli, 2001], that we denote $TL_\nu^\exists$, as well as its two fragments $TL^\exists$ and TL .

Syntax. We assume a countable set $\mathcal{L}$ of *label variables* ranging over by x, y , a countable set $\mathcal{T}$ of *tree variables* ranging over by X, Y and a countable set $\mathcal{R}$ of *recursion variables* ranging over by ξ . Let η be a label constant or a label variable. Formulas of the logic $TL_\nu^\exists$ are defined recursively as :

- $\mathbf{0}$ (empty tree), $\top$ (true), X (tree variable), ξ (recursion variable) and $\eta = \eta'$ (label equality test) are formulas from $TL_\nu^\exists$;
- if φ, ψ are formulas from $TL_\nu^\exists$ then $\eta[\varphi]$ (location), $\varphi | \psi$ (composition), $\neg\varphi$ (negation), $\varphi \vee \psi$ (disjunction), $\exists x.\varphi$ (quantification over label variables) $\exists X.\varphi$ (quantification over tree variables) and $\nu\xi.\varphi$ (greatest fixed point) are formulas from $TL_\nu^\exists$.

To guarantee the existence of the greatest fixed point $\nu\xi.\varphi$, we require that any occurrence of the recursion variable ξ in the formula φ is in the scope of an even number of negations.

The derived operators $\varphi \wedge \psi$, $\forall x.\varphi$, $\forall X.\varphi$ and $\mu\xi.\varphi$ are defined as usual.

Binding operators are $\exists x$ for label variables, $\exists X$ for tree variables and $\nu\xi$ for recursion variables. Let $fv(\varphi)$ be the set of free variables of the formula φ . We say that a formula is *closed* if it does not contains free variables.

We denote $fn(\varphi)$ the set of labels occurring in the formula φ . >From now on, we assume that in formulas free variables are disjoint from bounded variables and moreover, that two distinct occurrences of binders bind different variables.

Semantics and Satisfiability. We consider *valuations* ρ and δ defined over a finite subset of respectively $\mathcal{L} \cup \mathcal{T}$ and $\mathcal{R}$. The valuation ρ maps label variables to labels and tree variables to trees whereas the valuation δ maps recursion variables to subsets of $\mathcal{D}$. For ρ (resp. δ), we denote $\text{dom}(\rho)$ (resp. $\text{dom}(\delta)$) its domain.

The set of free label- and tree-variables of a formula φ under a valuation ρ denoted $\text{fv}(\varphi, \rho)$ is defined as $(\text{fv}(\varphi) \cap (\mathcal{L} \cup \mathcal{T})) \setminus \text{dom}(\rho)$. Respectively, the set of free recursion variables of a formula φ under a valuation δ denoted $\text{fv}(\varphi, \delta)$ is defined as $(\text{fv}(\varphi) \cap \mathcal{R}) \setminus \text{dom}(\delta)$. We say that a formula φ is closed under a valuation ρ , or simply ρ -closed (resp. under a valuation δ , or δ -closed) if $\text{fv}(\varphi, \rho) = \emptyset$ (resp. $\text{fv}(\varphi, \delta) = \emptyset$).

The interpretation of the formula φ is given by a mapping $\llbracket \varphi \rrbracket_{\rho, \delta}$ which associates with φ a subset of $\mathcal{D}$. This mapping is parametrized by two valuations ρ and δ such that φ is both ρ - and δ -closed. We assume that ρ is extended on labels from Λ and maps any label to itself; thus, we can write $\rho(\eta)$ for constant labels η . The valuation $\rho[x \mapsto n]$ is identical to ρ except for x which is mapped to n ; the valuations $\rho[X \mapsto A]$ and $\delta[\xi \mapsto S]$ are defined in a similar way.

Interpretation of formulas

$\llbracket \text{O} \rrbracket_{\rho, \delta} = \{\{\emptyset\}\}$	$\llbracket \eta[\varphi] \rrbracket_{\rho, \delta} = \{\{\rho(\eta)[A]\} \mid A \in \llbracket \varphi \rrbracket_{\rho, \delta}\}$
$\llbracket \top \rrbracket_{\rho, \delta} = \mathcal{D}$	$\llbracket \varphi \mid \psi \rrbracket_{\rho, \delta} = \{A \uplus B \mid A \in \llbracket \varphi \rrbracket_{\rho, \delta}, B \in \llbracket \psi \rrbracket_{\rho, \delta}\}$
$\llbracket \neg \varphi \rrbracket_{\rho, \delta} = \mathcal{D} \setminus \llbracket \varphi \rrbracket_{\rho, \delta}$	$\llbracket \varphi \vee \psi \rrbracket_{\rho, \delta} = \llbracket \varphi \rrbracket_{\rho, \delta} \cup \llbracket \psi \rrbracket_{\rho, \delta}$
$\llbracket \exists x. \varphi \rrbracket_{\rho, \delta} = \bigcup_{n \in \Lambda} \llbracket \varphi \rrbracket_{\rho[x \mapsto n], \delta}$	$\llbracket \eta = \eta' \rrbracket_{\rho, \delta} = \mathcal{D}$ if $\rho(\eta) = \rho(\eta')$, $\emptyset$ otherwise
$\llbracket X \rrbracket_{\rho, \delta} = \{\rho(X)\}$	$\llbracket \exists X. \varphi \rrbracket_{\rho, \delta} = \bigcup_{A \in \mathcal{D}} \llbracket \varphi \rrbracket_{\rho[X \mapsto A], \delta}$
$\llbracket \xi \rrbracket_{\rho, \delta} = \delta(\xi)$	$\llbracket \nu \xi. \varphi \rrbracket_{\rho, \delta} = \bigcup \{S \subseteq \mathcal{D} \mid S \subseteq \llbracket \varphi \rrbracket_{\rho, \delta[\xi \mapsto S]}\}$

DEFINITION 1 (SATISFIABILITY) For any tree A , any formula φ , and any valuations ρ, δ such that φ is ρ - and δ -closed, the tree A satisfies the formula φ under the valuations ρ and δ , written $A \models_{\rho, \delta} \varphi$, if A is in the set $\llbracket \varphi \rrbracket_{\rho, \delta}$.

In the notations $A \models_{\rho, \delta} \varphi$ and $\llbracket \varphi \rrbracket_{\rho, \delta}$ we sometimes omit ρ (resp. δ) whenever the formula φ contains no free label- and tree-variables (resp. no free recursion variables).

Fragments of the Logic. In the following of this paper, we will consider two fragments of the tree logic $TL_{\nu}^{\exists}$; the first fragment, denoted TL , is the smallest spatial logic included in $TL_{\nu}^{\exists}$ as it is defined only with Boolean operators, the empty tree, composition and location. The second fragment, denoted $TL^{\exists}$, is obtained by removing fixed point operator from the logic $TL_{\nu}^{\exists}$. It is easy to see that $TL \subseteq TL^{\exists} \subseteq TL_{\nu}^{\exists}$.

2.3 Comparison with Other Logics

We show in this section that the monadic second-order logic over trees (MSO) is more expressive than the TL fragment but that the logic $TL^{\exists}$ can express properties that can not be expressed in MSO.

We consider $\Lambda' \subseteq \Lambda$ a finite set of labels. Let τ be the signature $\{\text{label}_n \mid n \in \Lambda'\} \cup \{<\}$ where the label_n 's are unary predicates and $<$ is a binary predicate. An information tree A (with $\text{fn}(A) \subseteq \Lambda'$) can be represented by a finite τ -structure $T^A = \langle E^A, \{\text{label}_n^A \mid n \in \Lambda'\}, <^A \rangle$ where E^A is a finite set of edges, label_n^A associates the label n with edges, and $<^A$ is a binary relation over edges such that $u < u'$ holds iff the destination of the edge u and the source of the edge u' coincide.

We consider MSO over the signature τ . Edge variables range over u, v and set variables range over U, V . For any TL formula φ , we define an MSO formula $R_\varphi(U)$ parametrized by the set of edges U , such that for any information tree A , $A \models \varphi$ iff $T^A \models \forall U. ((\forall u. u \in U) \rightarrow R_\varphi(U))$.

Encoding of TL in MSO

$R_\top(U)$	$:= \text{true}$
$R_{\neg\varphi}(U)$	$:= \neg R_\varphi(U)$
$R_{\varphi \vee \psi}(U)$	$:= R_\varphi(U) \vee R_\psi(U)$
$R_\emptyset(U)$	$:= (U = \emptyset)$
$R_{n[\varphi]}(U)$	$:= \forall V, U' (\text{src}(U, V) \wedge U' = U \setminus V \rightarrow (\text{singleton}(V) \wedge R_\varphi(U')))$
$R_{\varphi\psi}(U)$	$:= \exists U', U'' \text{dunion}(U, U', U'') \wedge R_\varphi(U') \wedge R_\psi(U'') \wedge$ $\quad \forall V, V', V'' (\text{src}(U, V) \wedge \text{src}(U', V') \wedge \text{src}(U'', V'')) \rightarrow \text{dunion}(V, V', V'')$
$\text{src}(U, V)$	$:= \forall v, v \in V \leftrightarrow (v \in U \wedge \neg(\exists v' \in U (v' < v)))$
$\text{dunion}(U, U', U'')$	$:= U' \cap U'' = \emptyset \wedge U = U' \cup U''$

Now, consider the $TL^\exists$ formula $\exists X(a[X] \mid b[X])$: it expresses that the root of the tree has two direct subtrees (reachable by a and b) and that these subtrees are isomorphic. It is well-known that this property can not be expressed in MSO.

Some other logics express properties of unranked and unordered trees: for instance, the monadic second order logic with counting (CMSO) [Courcelle, 1990b], the Presburger monadic second order logic (PMSO) [Seidl et al., 2003], the sheaves logic [Dal-Zilio and Lugiez, 2003]. We strongly believe that the tree logic we consider in this paper is strictly more powerful than those logics.

3. Complexity of Model-Checking

The model-checking problem is given an information tree A , a valuation ρ^2 and a ρ -closed formula φ without free recursion variable, decide whether $A \models_\rho \varphi$. We study in this section lower bounds of the combined complexity and the data complexity of model-checking for the tree logic $TL^\exists_\rho$ and its fragments. We establish that combined complexity is PSPACE-hard for the less expressive fragment TL and that data-complexity is linear for TL , hard for any level of the polynomial hierarchy for $TL^\exists$ and PSPACE-hard for the full logic $TL^\exists_\rho$.

For PSPACE-hardness (resp. hardness for levels in the polynomial hierarchy) proof, we use reductions of the validity problem of closed quantified

²The valuation ρ is considered to be empty for the logic TL .

Boolean formulas (QBF) (resp. of closed QBF with fixed quantifier alternation depth) to the model-checking problem.

3.1 Quantified Boolean Formulas

A quantified Boolean formula (QBF) is a formula $Q_1 v_1 \dots Q_n v_n. G$ where each Q_i is a quantifier among $\exists$ and $\forall$ and G is a prepositional logic formula built on the set of variables $\{v_1, \dots, v_n\}$. We can assume without loss of generality that G is in disjunctive normal form and follows the syntax:

$$G ::= \theta_1 \vee \dots \vee \theta_k \quad \theta ::= l_1 \wedge \dots \wedge l_r \quad l ::= v_i \mid \bar{v}_i, \quad i \in \{1, \dots, n\}$$

It is well known that validity of closed QBF is a PSPACE-complete problem [Stockmeyer, 1976].

The quantifier alternation depth $\text{depth}(\mathcal{G})$ of a QBF $\mathcal{G}$ is defined as: (i) for a quantifier-free formula $\mathcal{G}$, $\text{depth}(\mathcal{G}) = 0$ and $\text{depth}(Qx.\mathcal{G}) = 1$ (for $Q \in \{\exists, \forall\}$), (ii) $\text{depth}(Qx.Q'x'.\mathcal{G}) = \text{depth}(Q'x'.\mathcal{G})$ if $Q = Q'$ and (iii) $\text{depth}(Qx.Q'x'.\mathcal{G}) = 1 + \text{depth}(Q'x'.\mathcal{G})$ if $Q \neq Q'$.

The validity problem of a closed QBF of alternation depth n whose outermost quantifier is $\forall$ (resp. $\exists$) is complete for the universal (resp. existential) n^{th} level of the polynomial hierarchy PH , that is Π_n^P -complete (resp. Σ_n^P -complete) [Stockmeyer, 1976].

3.2 Combined Complexity

The combined complexity of the model-checking problem $A \models_\rho \varphi$ is the complexity of the problem considering A , φ and ρ as inputs of the problem.

It has already been shown in [Charatonik et al., 2001] that the combined complexity of $TL^\exists$ is PSPACE-hard by encoding the validity problem of QBF into the model-checking of $TL^\exists$. This encoding uses only Boolean connectives, label comparison and existential quantification over labels. We show here that the combined complexity is PSPACE-hard even for the less expressive logic TL .

With any QBF $\mathcal{F}$, we associate the TL formula $\langle \mathcal{F} \rangle$ defined as:

Encoding of QBF into TL

$\langle v_i \rangle$	$= v_i[t[0]]$	$\langle \bar{v}_i \rangle$	$= v_i[f[0]]$
$\langle \exists v_i. \mathcal{F} \rangle$	$= v_i[t] \mid \langle \mathcal{F} \rangle$	$\langle \forall v_i. \mathcal{F} \rangle$	$= \neg(v_i[t] \mid \neg \langle \mathcal{F} \rangle)$
$\langle l_1 \wedge \dots \wedge l_k \rangle$	$= \langle l_1 \rangle \mid \top \wedge \dots \wedge \langle l_k \rangle \mid \top$	$\langle \theta_1 \vee \dots \vee \theta_k \rangle$	$= \langle \theta_1 \rangle \vee \dots \vee \langle \theta_k \rangle$

where t, f and the v_i are labels from Λ .

Let T and F be respectively the information trees $\{\{t[\{\{\}\}]\}\}$ and $\{\{f[\{\{\}\}]\}\}$. We consider in the sequel the n -variables QBF $\mathcal{G} = Q_1 v_1 \dots Q_n v_n. G$. Let $P_{\mathcal{G}}$ be the constant information tree (note that the construction of $P_{\mathcal{G}}$ is poly-

nomial time)

$$P_G = \{\{v_1[T], v_1[F], v_2[T], v_2[F], \dots, v_n[T], v_n[F]\}\}$$

LEMMA 2 *For any closed quantified Boolean formula G , $P_G \models (G)$ iff G is valid.*

We don't give here the full proof of the lemma, but just an intuitive idea about the encoding. Remark first that the tree P_G is the multiset of the elements $v_i[T]$ and $v_i[F]$ for all the variables v_i occurring in G . On the other hand, a valuation for the variables v_i in G can be viewed as a component P' of P_G in which each of the v_i occurs only once either as $v_i[T]$ or as $v_i[F]$, and T and F determine the valuation of the variable v_i . The tree P' can be constructed from P_G by n successive eliminations of one of the elements $v_i[T]$ and $v_i[F]$ for each i in $\{1, \dots, n\}$. Hence, the model-checking problem $P_G \models (G)$ can be viewed as the construction of a valuation for the set of variables $\{v_1, \dots, v_n\}$ by elimination of elements in P_G , followed by the verification that this valuation renders the formula G true. The elimination of the element $v_i[Z]$ for $Z \in \{T, F\}$ is done considering P_G as the union $\{\{v_i[Z]\} \uplus P'\}$. Thus, model-checking $(\exists v_i. \mathcal{F}) = v_i[T] \mid (\mathcal{F})$ leads us to eliminate non-deterministically one of the elements ($v_i[T]$ or $v_i[F]$) while model-checking $(\forall v_i. \mathcal{F}) = \neg(v_i[T] \mid \neg(\mathcal{F}))$ leads us to consider both cases in which $v_i[T]$ and $v_i[F]$ are eliminated.

THEOREM 3 *The combined complexity of model-checking the tree logic TL is PSPACE-hard.*

3.3 Data Complexity

The data complexity of the model-checking problem $A \models_\rho \varphi$ is the complexity of the problem for some fixed closed formula φ and valuation ρ . That is, only the tree A is considered as input for the problem. We show in the following that the data complexity of model-checking is PSPACE-hard for TL_ν^3 , hard for any level of the polynomial hierarchy for TL^3 and linear for TL .

Let us first introduce some objects which are in common to the next two encodings. Let $G = \theta_1 \vee \dots \vee \theta_k$ be a Boolean formula in disjunctive normal form constructed over the set of variables $\bigcup_{i \in I} \{v_i\}$ for some finite set I . Moreover, we require that the θ_j are not trivially false, that is, for any j in $1..k$, there is no i in I such that both v_i and $\bar{v}_i$ occur in θ_j .

For any i in I , we define the sets $C_i^{\text{true}} \subseteq \{1, \dots, k\}$ and $C_i^{\text{false}} \subseteq \{1, \dots, k\}$ as: j is in C_i^{true} if the variable v_i does not occur negatively in θ_j and j is in C_i^{false} if the variable v_i does not occur positively in θ_j . Assuming that c_j , for j in $1..k$, is a label from Λ , we define the information trees $C_i^t = \biguplus_{j \in C_i^{\text{true}}} \{\{c_j[\{\}]\}\}$ and $C_i^f = \biguplus_{j \in C_i^{\text{false}}} \{\{c_j[\{\}]\}\}$.

A valuation γ for G is a mapping from I to $\{\text{true}, \text{false}\}$. It is easy to see that the formula G is valid for some valuation γ iff there exists some $j \in 1..k$ such that j is in $C_i^{\gamma(i)}$ for any i in I .

$TL_\nu^{\exists}$. We show here that the data complexity of the model-checking problem for the tree logic $TL_\nu^{\exists}$ is PSPACE-hard, that is, there exists a fixed $TL_\nu^{\exists}$ formula Φ such that the model-checking problem “given the information tree A , does $A \models \Phi$ hold” is PSPACE-hard.

The PSPACE-hardness proof is done using a reduction of the validity problem of a closed QBF into the model-checking problem for $TL_\nu^{\exists}$. With any QBF $\mathcal{G}$, we associate the information tree $A_{\mathcal{G}}$ such that $A_{\mathcal{G}} \models \Phi$ iff $\mathcal{G}$ is valid.

Consider the n -variables QBF $\mathcal{G} = Q_1 v_1 \dots Q_n v_n. G$. Let v_i for $i \in 1..n$, c_j for $j \in 1..k$, $v_0, q_v, q_{\exists}, \text{tt}, \text{ff}, \text{pred}, \text{quant}$ and sat be labels. Let

$$A_{\mathcal{G}} = \{\{v_0[\{\}\}], V_1^{\text{true}}, V_1^{\text{false}}, \dots, V_n^{\text{true}}, V_n^{\text{false}}\}$$

where, for any $i \in 1..n$, the trees V_i^{true} and V_i^{false} are

$$\begin{aligned} V_i^{\text{true}} &= v_i[\{\{\text{tt}[\{\}]\}, \text{quant}[\{\{q_v, [\{\}]\}\}], \text{pred}[\{\{v_{i-1}[\{\}]\}\}], \text{sat}[C_i^{\text{t}}]\}] \\ V_i^{\text{false}} &= v_i[\{\{\text{ff}[\{\}]\}, \text{quant}[\{\{q_v, [\{\}]\}\}], \text{pred}[\{\{v_{i-1}[\{\}]\}\}], \text{sat}[C_i^{\text{f}}]\}] \end{aligned}$$

Intuitively, for any variable v_i occurring in the QBF $\mathcal{G}$, the information tree $A_{\mathcal{G}}$ encodes the kind of quantification of v_i (in the component $\text{quant}[\{\{q_v, [\{\}]\}\}]$) and the rank of the variable (giving the previous variable in the component pred). The information trees C_i^{t} and C_i^{f} encode the formula G .

The $TL_\nu^{\exists}$ formula Φ is defined as

$$\Phi = \mu\xi. (\neg \text{IsValuation} \wedge \text{ConstructVal}) \vee (\text{IsValuation} \wedge \text{Valid})$$

The mechanism used here is similar to the one used to prove combined complexity lower bound of TL (Sect. 3.2). Intuitively, the formula Φ constructs all possible valuations for the variables v_i regarding their quantification in $\mathcal{G}$ and then verifies, for each valuation, whether it values the formula G to true.

To any valuation γ corresponds exactly a component of $A_{\mathcal{G}}$ denoted $A_{\mathcal{G}}\{\gamma\}$ and defined as $\{v_0[\{\}]\}, V_1^{\gamma(1)}, V_2^{\gamma(2)}, \dots, V_n^{\gamma(n)}\}$. Note that in $A_{\mathcal{G}}\{\gamma\}$, any of the v_i occurs only once. Therefore, the formula IsValuation verifying whether a component of $A_{\mathcal{G}}$ corresponds to a valuation is defined as:

$$\text{IsValuation} = \neg(\exists x_v. x_v[\top] \mid x_v[\top] \mid \top)$$

Now, a valuation can be constructed from $A_{\mathcal{G}}$ by n successive eliminations of one of V_i^{true} or V_i^{false} , for any $i \in \{1, \dots, n\}$. All possible valuations of the variables v_i , according to the quantifications of these variables, are enumerated by the $TL_\nu^{\exists}$ subformula ConstructVal which is the recursive part of Φ . In the

following, we let $Forall(x_v) = x_v[\text{quant}[q_v[0]] \mid \top] \mid \top$ and $Exists(x_v) = x_v[\text{quant}[q_\exists[0]] \mid \top] \mid \top$.

$$\begin{aligned} ConstructVal &= \exists x_v. FirstNonVal(x_v) \wedge \left(\begin{array}{c} Forall(x_v) \wedge \neg(x_v[\top] \mid \neg\xi) \\ \vee \\ Exists(x_v) \wedge x_v[\top] \mid \xi \end{array} \right) \\ FirstNonVal(x_v) &= \exists x'_v. \left(\begin{array}{c} \neg(x'_v[\top] \mid x'_v[\top] \mid \top) \\ \wedge \\ x_v[\top] \mid x_v[\top] \mid \top \\ \wedge \\ x_v[pred[x'_v[\top]] \mid \top] \mid \top \end{array} \right) \end{aligned}$$

Finally, according to the definitions of the sets C_i^{true} and C_i^{false} , to verify the validity of G under the valuation γ amounts to test the existence of some $j \in \{1, \dots, k\}$ such that $c_j[\{\{\}\}]$ is a component of $C_i^{\gamma(i)}$ for any i . This is done by the formula *Valid* which is the base case for Φ .

$$\begin{aligned} Valid &= v_0[\top] \mid \exists x_c. Everywhere(x_c) \\ Everywhere(x_c) &= \forall x_v. (x_v[\top] \mid \top \rightarrow x_v[\text{sat}[x_c[0]] \mid \top] \mid \top) \mid \top \end{aligned}$$

Lemma 4 embodies the correctness of the reduction and implies Theorem 5 simply by noticing that the size of the tree $A_{\mathcal{G}}$ is polynomial in the size of the QBF $\mathcal{G}$.

LEMMA 4 *The closed quantified Boolean formula $\mathcal{G}$ is valid iff $A_{\mathcal{G}} \models \Phi$.*

THEOREM 5 *The data complexity of model-checking the tree logic $TL_{\nu}^{\exists}$ is PSPACE-hard.*

$TL^{\exists}$. We show here that the data complexity of the model-checking problem for the tree logic $TL^{\exists}$ is hard for any level of the polynomial hierarchy PH . That is, we show that for any universal (resp. existential) level of PH Π_n^P (resp. Σ_n^P), there exists a fixed formula $\Phi_n^{\forall}$ (resp. $\Phi_n^{\exists}$) such that the model-checking problem “given the information tree A , does $A \models \Phi_n^{\forall}$ (resp. $A \models \Phi_n^{\exists}$) hold” is Π_n^P -hard (resp. Σ_n^P -hard).

To prove Π_n^P -hardness (resp. Σ_n^P -hardness), we present a reduction of the validity problem of closed QBF with alternation depth n and universal (resp. existential) outermost quantifier into the model-checking problem for $TL^{\exists}$ with some fixed formula $\Phi_n^{\forall}$ (resp. $\Phi_n^{\exists}$). More precisely, with any closed QBF $\mathcal{G}$ of alternation depth n we associate the information tree $A_{\mathcal{G}}$ and we show that $A_{\mathcal{G}} \models \Phi_n^{\forall}$ (resp. $A_{\mathcal{G}} \models \Phi_n^{\exists}$) iff the outermost quantifier of $\mathcal{G}$ is $\forall$ (resp. $\exists$) and $\mathcal{G}$ is valid.

Consider the closed QBF with alternation depth n ($Q_i \neq Q_{i+1}$ for i in $1..n-1$).

$$\mathcal{G} = Q_1 v_{(1,1)} \dots v_{(1,n_1)}. Q_2 v_{(2,1)} \dots v_{(2,n_2)} \dots Q_n v_{(n,1)} \dots v_{(n,n_n)}. G$$

Let $v_{(h,i)}$ for any $h \in 1..n$ and any $i \in 1..n_h$, c_j for any $j \in 1..k$, quant , sat and q_i for any $i \in 1..n$ be labels from Λ . The information tree $A_{\mathcal{G}}$ is:

$$A_{\mathcal{G}} = \{\{V_{(1,1)}^{\text{true}}, V_{(1,1)}^{\text{false}}, \dots, V_{(1,n_1)}^{\text{true}}, V_{(1,n_1)}^{\text{false}}, V_{(2,1)}^{\text{true}}, V_{(2,1)}^{\text{false}}, \dots, V_{(n,n_n)}^{\text{true}}, V_{(n,n_n)}^{\text{false}}\}\}$$

where for any $h \in 1..n$ and for any $i \in 1..n_h$, $V_{(h,i)}^{\text{true}}$ and $V_{(h,i)}^{\text{false}}$ are

$$\begin{aligned} V_{(h,i)}^{\text{true}} &= \{\{\text{tt}[\{\}\}], \text{quant}[\{\{q_h[\{\}]\}\}], \text{sat}[C_{(h,i)}^t]\} \\ V_{(h,i)}^{\text{false}} &= \{\{\text{ff}[\{\}\}], \text{quant}[\{\{q_h[\{\}]\}\}], \text{sat}[C_{(h,i)}^f]\} \end{aligned}$$

The formulas $\Phi_n^{\forall}$ and $\Phi_n^{\exists}$ are defined respectively as $\Psi_n^{\forall}(1)$ and $\Psi_n^{\exists}(1)$ where for any $h \in 1..n$, the formulas $\Psi_n^{\forall}(h)$ and $\Psi_n^{\exists}(h)$ are defined recursively:

$$\begin{aligned} \Psi_n^{\forall}(h) &= \neg(\text{Remove}(h) \mid (\text{IsValuation}(h) \wedge \neg\Psi_n^{\exists}(h+1))) \\ \Psi_n^{\exists}(h) &= \text{Remove}(h) \mid (\text{IsValuation}(h) \wedge \Psi_n^{\forall}(h+1)) \end{aligned}$$

and $\Psi_n^{\forall}(n+1) = \Phi_n^{\exists}(n+1) = \text{Valid}'$.

The intuition for this encoding is similar to the one used in previous section. The formulas $\Phi_n^{\forall}$ and $\Phi_n^{\exists}$ construct all possible valuations for the variables of the formula $\mathcal{G}$ according to their quantification and check, for each valuation, whether it evaluates G to true. More precisely, for any $h \in 1..n$, the formula $\Psi_n^{\forall}(h)$ (resp. $\Psi_n^{\exists}(h)$) constructs all possible valuations of the variables $v_{(h,1)}, v_{(h,2)}, \dots, v_{(h,n_h)}$ according to the universal (resp. existential) quantification and reiterates the same mechanism to $h+1$. Therefore, $\text{IsValuation}(h)$ and $\text{Remove}(h)$ are defined as:

$$\begin{aligned} \text{IsValuation}(h) &= \neg\exists x_v. (x_v[\text{quant}[q_h[0]] \mid \top] \mid x_v[\text{quant}[q_h[0]] \mid \top] \mid \top) \\ \text{Remove}(h) &= \forall x_v. (x_v[\top] \mid \top \rightarrow x_v[\text{quant}[q_h[0]] \mid \top] \mid \top) \wedge \text{IsValuation}(h) \end{aligned}$$

The formulas $\Psi_n^{\forall}(n+1)$ and $\Psi_n^{\exists}(n+1)$ check validity of G for some valuation, so the formula Valid' is defined as $\exists x_c. \text{Everywhere}(x_c)$.

The correctness of the encoding we presented is given by Lemma 6 and the complexity result in Theorem 7 is an easy consequence of this lemma using the fact that the size of the tree $A_{\mathcal{G}}$ is polynomial in the size of the QBF $\mathcal{G}$.

LEMMA 6 *Let $\mathcal{G}$ be a closed quantified Boolean formula of alternation depth n whose outermost quantifier is universal (resp. existential). The formula $\mathcal{G}$ is valid iff $A_{\mathcal{G}} \models \Phi_n^{\forall}$ (resp. $A_{\mathcal{G}} \models \Phi_n^{\exists}$).*

THEOREM 7 *The data complexity of model-checking the tree logic $TL^{\exists}$ is hard for any level of the polynomial hierarchy.*

TL. We show here that the data complexity of the model-checking problem for the tree logic TL is linear time using the encoding of TL formulas into MSO presented in Sect. 2.3.

Remark first that by [Courcelle, 1990a], the data complexity of model-checking is linear for MSO interpreted over finite graphs of bounded tree-width which are a generalization of unranked unordered trees. Consider the model-checking problem $A \models \varphi$ where the formula φ is fixed. Let τ be the signature $\{\text{label}_n \mid n \in \text{fn}(A) \cup \text{fn}(\varphi)\} \cup \{<\}$, and let T^A be the τ -structure corresponding to A . It is easy to see that the construction of T^A is linear time in the size of A . Moreover, as φ is fixed, the construction of $R_\varphi(U)$ is constant time.

THEOREM 8 *The data complexity of model-checking the tree logic TL is linear time.*

4. A Model-Checking Algorithm

Model-checking for the tree logic $TL_\nu^{\exists}$ is non trivial for two reasons. On one hand, the existential quantification operator quantifies label and tree variables over infinite sets. We show here how to handle this problem. On the other hand, the (greatest) fixed point operator from $TL_\nu^{\exists}$ differs from the kind of fixed points used to extend the first-order logic (FO) to LFP. In LFP, fixed points are defined over the lattice of relations on nodes of a tree (or graph) whereas in $TL_\nu^{\exists}$, fixed points are defined over the lattice of sets of trees.

We use an adaptation of the local model-checking for the modal μ -calculus given in [Stirling and Walker, 1991] to devise a polynomial space model-checking algorithm. Following the ideas from [Winskel, 1991], our algorithm will actually perform model-checking for a slight extension of $TL_\nu^{\exists}$: the syntax of greatest fixed point operator is now $\nu\xi(M).\varphi$ where M is a finite subset of $\mathcal{D}$ and its semantics is given by $\bigcup\{S \subseteq \mathcal{D} \mid S \subseteq \llbracket \varphi \rrbracket_{\rho, \delta[\xi \mapsto S]} \cup M\}$.

One can notice that the logic $TL_\nu^{\exists}$ corresponds to formulas where M is the empty set for all fixed point operators. We extend the definition of labels occurring in a formula by letting $\text{fn}(\nu\xi(M).\varphi) = \text{fn}(\varphi) \cup \bigcup_{A \in M} \text{fn}(A)$.

4.1 Properties of Satisfaction

The two propositions here after establish that only a finite number of labels and a finite number of information trees have to be considered while model-checking a formula with quantifiers.

Let $\text{fn}(S) = \bigcup_{A \in S} \text{fn}(A)$ for some $S \subseteq \mathcal{D}$, $\text{fn}(\rho) = \bigcup_{x \in \text{dom}(\rho) \cap \mathcal{L}} \{\rho(x)\} \cup \bigcup_{X \in \text{dom}(\rho) \cap \mathcal{T}} \text{fn}(\rho(X))$ for some valuation ρ and $\text{fn}(\delta) = \bigcup_{\xi \in \text{dom}(\delta)} \text{fn}(\delta(\xi))$ for some valuation δ . For a formula φ considered under some valuations ρ and δ , we define $\text{fn}(\varphi, \rho, \delta)$ as $\text{fn}(\varphi) \cup \text{fn}(\rho) \cup \text{fn}(\delta)$.

PROPOSITION 9 *Let A be a tree and $\exists X.\varphi$ be a closed formula under some valuations ρ and δ . Let n be a label which does not belong to $\text{fn}(A)$. Then $A \models_{\rho, \delta} \exists X.\varphi$ iff there exists some tree B in $\{C \mid C \sqsubseteq A\} \cup \{\{n[\{\}\}]\}$ such that $A \models_{\rho[X \mapsto B], \delta} \varphi$.*

SKETCH OF PROOF. The right-to-left implication is an easy consequence of the definition of satisfaction. For the left-to-right implication: if $A \models_{\rho, \delta} \exists X.\varphi$ then, by definition of satisfaction, there exists a tree B such that $A \models_{\rho[X \mapsto B], \delta} \varphi$. If this tree B is a component of A , then the statement is obvious. Otherwise, we show that for any tree C such that $C \not\subseteq A$, it holds that $A \models_{\rho[X \mapsto C], \delta} \text{iff } A \models_{\rho[X \mapsto B], \delta}$. As $\{\{n[\{\}\}]\} \not\subseteq A$, we have $A \models_{\rho[X \mapsto \{\{n[\{\}\}]\}, \delta} \varphi$.

PROPOSITION 10 *Let A be a tree and $\exists x.\varphi$ be a closed formula under some valuations ρ and δ . Let n be a label such that $n \notin \text{fn}(A) \cup \text{fn}(\varphi, \rho, \delta)$. Then $A \models_{\rho, \delta} \exists x.\varphi$ iff there exists some label p in $\text{fn}(A) \cup \text{fn}(\varphi, \rho, \delta) \cup \{n\}$ such that $A \models_{\rho[x \mapsto p], \delta} \varphi$.*

SKETCH OF PROOF. The right-to-left direction is straightforward by the definition of satisfaction. For the left-to-right direction: as $A \models_{\rho, \delta} \exists x.\varphi$, by definition of satisfaction, there exists some label p' in Λ such that $A \models_{\rho[x \mapsto p'], \delta} \varphi$. If $p' \in \text{fn}(A) \cup \text{fn}(\varphi, \rho, \delta)$ then the property obviously holds. Otherwise, we show that for any label m such that $m \notin \text{fn}(A) \cup \text{fn}(\varphi, \rho, \delta)$, it holds that $A \models_{\rho[x \mapsto m], \delta} \varphi$ iff $A \models_{\rho[x \mapsto p'], \delta} \varphi$. Therefore, $A \models_{\rho[x \mapsto m], \delta} \varphi$.

4.2 Polynomial Space Model-checking Algorithm

We consider Δ , a (possibly empty) sequence of the form $[\xi_1 \rightarrow \varphi_1] \dots [\xi_n \rightarrow \varphi_n]$ where the φ_i 's are formulas from the tree logic and $\xi_i \neq \xi_j$ whenever $i \neq j$. For all i , $[\xi_i \rightarrow \varphi_i]$ is the substitution which, when applied to some formula ψ , replaces the free occurrences of the recursion variable ξ_i in ψ by the formula φ_i . The empty sequence is denoted ϵ and considered as the identity. The domain of a sequence of substitutions Δ (written $\text{dom}(\Delta)$) is defined as (i) $\emptyset$ if $\Delta = \epsilon$ and (ii) $\{\xi_1, \dots, \xi_n\}$ if $\Delta = [\xi_1 \rightarrow \varphi_1] \dots [\xi_n \rightarrow \varphi_n]$.

For a formula φ , $\Delta(\varphi)$ denotes the formula $[\xi_1 \rightarrow \varphi_1] \circ \dots \circ [\xi_n \rightarrow \varphi_n](\varphi)$. For a sequence of substitutions Δ , $\Delta(\xi \leftarrow \varphi)$ denotes the sequence of substitution (i) $\Delta[\xi \rightarrow \varphi]$ if $\xi \notin \text{dom}(\Delta)$ and (ii) $[\xi_1 \rightarrow \varphi_1] \dots [\xi_i \rightarrow \varphi] \dots [\xi_n \rightarrow \varphi_n]$ if $\Delta = [\xi_1 \rightarrow \varphi_1] \dots [\xi_n \rightarrow \varphi_n]$ and $\xi = \xi_i \in \text{dom}(\Delta)$.

We extend the mapping fn to sequences of substitutions Δ : $\text{fn}(\epsilon) = \emptyset$ and $\text{fn}(\Delta[\xi \rightarrow \varphi]) = \text{fn}(\varphi) \cup \text{fn}(\Delta)$. For any formula φ , any valuation ρ and any sequence of substitutions Δ , we define $\text{fn}(\varphi, \rho, \Delta) = \text{fn}(\varphi) \cup \text{fn}(\rho) \cup \text{fn}(\Delta)$.

We present an algorithm $\text{check}(A, \varphi, \rho, \Delta)$ with A an information tree, φ a tree logic formula, ρ a valuation for label and tree variables and Δ a sequence of substitutions. This algorithm is correct in the sense of Theorem 11 below.

Model-checking algorithm

$\text{check}(A, \varphi, \rho, \Delta)$ is

case φ of

0 :	return $A = \{\{\}\}$;
$\eta[\varphi']$:	return $A = \{\{\rho(\eta)[A']\}\}$ and $\text{check}(A', \varphi', \rho, \Delta)$;

```

 $\varphi' \mid \varphi''$  :   for all  $A', A''$  such that  $A = A' \uplus A''$ 
                  if  $\text{check}(A', \varphi', \rho, \Delta)$  and  $\text{check}(A'', \varphi'', \rho, \Delta)$ 
                  then return true;
                  return false;
 $\top$  :           return true;
 $\neg \varphi'$  :       return not  $\text{check}(A, \varphi', \rho, \Delta)$ ;
 $\varphi' \vee \varphi''$  :  return  $\text{check}(A, \varphi', \rho, \Delta)$  or  $\text{check}(A, \varphi'', \rho, \Delta)$ ;
 $\exists x.\varphi'$  :       let  $m$  be a label neither in  $\text{fn}(A)$  nor in  $\text{fn}(\varphi', \rho, \Delta)$  in
                  for  $n \in \text{fn}(A) \cup \text{fn}(\varphi', \rho, \Delta) \cup \{m\}$  do
                    if  $\text{check}(A, \varphi', \rho[x \rightarrow n], \Delta)$  then return true;
                  return false;
 $X$  :           return  $A = \rho(X)$ ;
 $\exists X.\varphi'$  :      let  $m$  be a label not in  $\text{fn}(A)$  in
                  for  $B \in \{C \mid C \subseteq A\} \cup \{\{m\}\{\}\}$  do
                    if  $\text{check}(A, \varphi', \rho[X \rightarrow B], \Delta)$  then return true;
                  return false;
 $\eta = \eta'$  :      return  $\rho(\eta) = \rho(\eta')$ ;
 $\nu\xi(M).\varphi'$  :  if  $A \in M$  then return true;
                  else return  $\text{check}(A, \varphi', \rho, \Delta(\xi \leftarrow \nu\xi(M \cup \{A\}).\varphi'))$ ;
 $\xi$  :           if  $\xi \notin \text{dom}(\Delta)$  return error;
                  let  $[\xi \rightarrow \nu\xi(M).\varphi']$  occurring in  $\Delta$  in
                  if  $A \in M$  then return true;
                  else return  $\text{check}(A, \varphi', \rho, \Delta(\xi \leftarrow \nu\xi(M \cup \{A\}).\varphi'))$ ;

```

THEOREM 11 (CORRECTNESS) Let φ be a TL_ν^3 formula without free recursion variable and ρ be a valuation such that φ is ρ -closed, and let A be an information tree. Then the evaluation of $\text{check}(A, \varphi, \rho, \epsilon)$ terminates and computes true if $A \models_\rho \varphi$ holds and false otherwise.

PROPOSITION 12 (COMPLEXITY) For any valuation ρ , any ρ -closed TL_ν^3 formula φ without free recursion variables and any information tree A , the evaluation of $\text{check}(A, \varphi, \rho, \epsilon)$ requires polynomial space in the size of the model-checking problem $A \models_\rho \varphi$.

SKETCH OF PROOF. Any recursive call $\text{check}(A', \varphi', \rho', \Delta')$ generated by the evaluation of $\text{check}(A, \varphi, \rho, \epsilon)$ has a polynomial-size representation. Moreover, the length of any chain of recursive calls of the algorithm check starting from $\text{check}(A, \varphi, \rho, \epsilon)$ is polynomially bounded. It is then sufficient to show that the evaluation of $\text{check}(A', \varphi', \rho', \Delta')$ can be done in polynomial space. For most of the cases of the “case” statement, it is obvious. The two non trivial cases are for φ' being $\psi' \mid \psi''$ and $\exists X.\psi$. For the former, one has to consider all couples of trees A', A'' such that $A' \uplus A'' = A$. These couples correspond to all bipartitions of the multiset A and so can be enumerated using linear space. For the latter, the set of all components of A have to be considered. Remark that a component of A corresponds to a subset of the set of nodes of A . This correspondence is not bijective, but one can determine in polynomial space whether

a subset of the set of nodes of A corresponds to a component of A . Therefore, the set of all components of A can be enumerated in polynomial space. Note finally that all iterations of the evaluation loop reuse the same space. $\square$

As a consequence of Theorem 3, Proposition 12 and the inclusion $TL \subseteq TL^{\exists} \subseteq TL_{\nu}^{\exists}$ we have

THEOREM 13 *The model-checking problem for the tree logics TL , $TL^{\exists}$ and $TL_{\nu}^{\exists}$ is PSPACE-complete.*

References

- [Abiteboul et al., 2000] Abiteboul, S., Buneman, P., and Suciu, D. (2000). *Data on the Web*. Morgan Kaufmann Publishers.
- [Cardelli, 2000] Cardelli, L. (2000). Semistructured Computation. In *7th International Workshop on Database Programming Languages, DBPL'99*, LNCS, pages 1–16. Springer.
- [Cardelli and Ghelli, 2001] Cardelli, L. and Ghelli, G. (2001). A Query Language Based on the Ambient Logic. In *European Symposium on Programming (ESOP'01)*, volume 2028 of LNCS, pages 1–22. Springer.
- [Cardelli and Gordon, 2000a] Cardelli, L. and Gordon, A.D. (2000a). Anytime, Anywhere: Modal Logics for Mobile Ambients. In *27th ACM Symposium on Principles of Programming Languages (POPL'00)*, pages 365–377.
- [Cardelli and Gordon, 2000b] Cardelli, L. and Gordon, A.D. (2000b). Mobile Ambients. *Theoretical Computer Science*, 240:177–213.
- [Charatonik et al., 2001] Charatonik, W., Dal Zilio, S., Gordon, A. D., Mukhopadhyay, S., and Talbot, J.-M. (2001). The Complexity of Model Checking Mobile Ambients. In *Foundations of Software Science and Computation Structures (FoSSaCS'01)*, volume 2030 of LNCS, pages 152–167. Springer.
- [Courcelle, 1990a] Courcelle, B. (1990a). Graph Rewriting: An Algebraic and Logic Approach. In *Handbook of Theoretical Computer Science*, volume B. Elsevier.
- [Courcelle, 1990b] Courcelle, B. (1990b). The Monadic Second-Order Logic of Graphs. I. Recognizable Sets of Finite Graphs. *Information and Computation*, 85(1): 12–75.
- [Dal-Zilio and Lugiez, 2003] Dal-Zilio, S. and Lugiez, D. (2003). XML Schema, Tree Logic and Sheaves Automata. In *Rewriting Techniques and Applications, 14th International Conference, RTA 2003*, LNCS, pages 246–263. Springer.
- [Neven, 2002] Neven, F. (2002). Automata, Logic and XML. In *Annual Conference of the European Association for Computer Science Logic (CSL '02)*, volume 2471 of LNCS, pages 2–26. Springer.
- [Seidl et al., 2003] Seidl, H., Schwentick, T., and Muscholl, A. (2003). Numerical Document Queries. In *Twenty-Second ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems*, pages 155–166. ACM.
- [Stirling and Walker, 1991] Stirling, C. and Walker, D. (1991). Local model checking in the modal μ -calculus. *Theoretical Computer Science*, 89:161–177.
- [Stockmeyer, 1976] Stockmeyer, L. J. (1976). The Polynomial-time Hierarchy. *Theoretical Computer Science*, 3(1): 1–22.
- [Winskel, 1991] Winskel, G. (1991). A note on model checking the modal ν -calculus. *Theoretical Computer Science*, 83:157–167.