Properties of Cryptosystem PGM

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Abstract

A cryptographic system, called PGM, was invented in the late 1970's by S. Magliveras. PGM is based on the prolific existence of certain kinds of factorization sets, called *logarithmic signatures*, for finite permutation groups. Logarithmic signatures were initially motivated by C. Sims' bases and strong generators. Statistical properties of random number generators based on PGM have been investigated in [7], [8] and show PGM to be statistically robust. In this paper we present recent results on the algebraic properties of PGM. PGM is an endomorphic cryptosystem in which the message space is $\mathcal{Z}_{|G|}$, for a given finite permutation group G . We show that the set of PGM transformations $\mathcal{T}_G$ is not closed under functional composition and hence not a group. This set is 2-transitive on $\mathcal{Z}_{|G|}$ if the underlying group G is not hamiltonian. Moreover, if $|G| \neq 2^a$, then the set of transformations contains an odd permutation. An important consequence of the above results is that the group generated by the set of transformations is nearly always the full symmetric group.

1 Introduction

A *Cryptosystem* Π is an ordered four-tuple $(\mathcal{M}, \mathcal{K}, \mathcal{C}, T)$, where $\mathcal{M}, \mathcal{K}$, and $\mathcal{C}$ are finite sets called the *message space*, the *key space*, and the *cipher space* respectively, and T is a family of transformations $\{E_k : \mathcal{M} \rightarrow \mathcal{C}, k \in \mathcal{K}\}$ such that for each $k \in \mathcal{K}$, E_k is invertible. We denote the inverse of E_k by D_k . Implicit in a cryptosystem $(\mathcal{M}, \mathcal{K}, \mathcal{C}, T)$ is the mapping $E : k \rightarrow E_k$ which associates to each key $k \in \mathcal{K}$ the

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transformation E_k induced by k . The cryptosystem is said to be *faithful* if E is an injection.

If the message space and cipher space are the same, then the cryptosystem is called *endomorphie* and in this case for every key k , $E_k : \mathcal{M} \rightarrow \mathcal{C}$ is a permutation on $\mathcal{M}$. An endomorphic cryptosystem Π is said to be *closed* if and only if $\mathcal{T}$ is closed under functional composition. In other words, Π is *closed* if and only if for every two keys $i, j \in \mathcal{K}$ there exists a key $k \in \mathcal{K}$ such that $E_i E_j = E_k$. We wish to remark that the term *closed* had been used by Shannon [9] to mean something totally different. Our usage follows current terminology [4]. Since $\mathcal{T}$ is finite it follows that Π is closed if and only if $\mathcal{T}$ forms a group under composition. Let $\mathcal{G}_\Pi = \langle \mathcal{T} \rangle$ be the group generated by $\mathcal{T}$ and $S_{\mathcal{M}}$ the *symmetric group* on $\mathcal{M}$. Then, of course, $\mathcal{G}_\Pi$ is a subgroup of $S_{\mathcal{M}}$.

Although the terminology that follows is not standard, it is natural and extends the terminology used in [4]. A Cryptosystem Π is said to be *t-transitive* if given any ordered t -tuple of distinct messages $(m_1, \dots, m_t) \in \mathcal{M}^t$ and any ordered t -tuple of distinct ciphertexts $(c_1, \dots, c_t) \in \mathcal{C}^t$, there is some $k \in \mathcal{K}$ such that $E_k(m_i) = c_i$ ($1 \leq i \leq t$). Here we also write $E_k(m_1, \dots, m_t) = (c_1, \dots, c_t)$. Note that we speak of a t -transitive system whether or not $\mathcal{T}$ is a group. It is clear that a t -transitive system is $(t-1)$ -transitive.

2 Logarithmic Signatures and PGM

If G is a finite permutation group of degree n we call an ordered collection $\alpha = \{B_i : i = 1, \dots, s\}$ of ordered sets $B_i = \{u(i, j) : j = 1, \dots, r(i)\}$, where $|B_i|$ and s are bounded by a polynomial in n , a *logarithmic signature* of G if each element g of G can be expressed uniquely as a product of the form

$$g = q_s \cdot q_{s-1} \dots q_2 \cdot q_1 \quad (1)$$

with $q_i \in B_i$. The elements q_i are not necessarily elements of G , but could belong to a much larger group in which G is embedded. The B_i are called the *blocks* of α and the vector of block lengths $\mathbf{r} = (r(1), \dots, r(s))$ is called the *type* of α . We also write $\mathbf{r} = (\alpha)$ for the type of α . The logarithmic signature is called *non-trivial* if $s \geq 2$ and $r(i) \geq 2$ for at least two indices i , where $(1 \leq i \leq s)$. Otherwise it is called *trivial*. A logarithmic signature is called *tame* if the factorization (1) can be achieved in time polynomial in the degree n of G , it is called *supertame* if (1) can be achieved in time $O(n^2)$. A logarithmic signature is called *wild* if it is not tame. We denote by Λ the collection of all logarithmic signatures of G .

Let $\gamma : G = G_0 > G_1 > \dots > G_s = 1$ be a chain of subgroups of G , and $\{B_i : i = 1, \dots, s\}$, an ordered collection of subsets of G , where each $B_i = \{u(i, j) :$

$j = 1, \dots, r(i)\}$ is a complete set of right coset representatives of G_i in G_{i-1} . It is easily seen that $\{B_i\}$ forms a logarithmic signature for G . Such a logarithmic signature is called *transversal* with respect to γ . Here the type $\mathbf{r} = (r(1), \dots, r(s))$ has $r(i) = [G_i : G_{i-1}]$. In view of the fact that membership in a permutation group can be tested in time polynomial in the degree [3], it is easily seen that a transversal logarithmic signature is tame. We denote the set of all transversal logarithmic signatures of G with respect to a chain γ by $\Lambda(\gamma)$. In section 4 we indicate how $\Lambda(\gamma)$ is a single regular orbit under the action of a certain monomial group. Existence of supertame logarithmic signatures is established in the following lemma :

Lemma 2.1 *If G is a permutation group then there exists a supertame logarithmic signature for G .*

Proof: Suppose G acts on the letters of $\Omega = \{1, 2, \dots, n\}$. Let $G = G_0 > G_1 > \dots > G_s = 1$ be a chain of nested stabilizers in G . Thus, $G_0 = G$ and for $i \geq 1$, G_i fixes pointwise the letters $1, 2, \dots, i$ of Ω . Suppose now that the orbit of $i \in \Omega$ under G_{i-1} is $B = \{\delta_1 = i, \delta_2, \dots, \delta_{r(i)}\}$ and that $u(i, j) \in G_{i-1}$ moves δ_1 to δ_j , then $G_{i-1} = G_i u(i, 1) + G_i u(i, 2) + \dots + G_i u(i, r(i))$. Consider the logarithmic signature

$$\alpha = [u(1, 1), \dots, u(1, r(1)); u(2, 1), \dots, u(2, r(2)); \dots; u(s, 1), \dots, u(s, r(s))]$$

Now, note that an element h in G_{i-1} belongs to the coset $G_i u(i, j)$ if and only if h moves $\delta_1 = i$ to δ_j . Thus determining the right G_i coset in G_{i-1} to which h belongs requires $O(1)$ operations. The element $h \cdot u(i, j)^{-1}$ fixes $1, 2, \dots, i-1, i$, and therefore belongs to G_i . Computing $h \cdot u(i, j)^{-1}$ requires $O(n)$ operations. Recursively, given any element $g \in G$, we descend in at most n steps and have $g \cdot u(1, j_1)^{-1} \cdot u(2, j_2)^{-1} \dots u(s, j_s)^{-1} = 1$. Inverting yields the unique factorization $g = u(s, j_s) \dots u(2, j_2) \cdot u(1, j_1)$. $\square$.

For the remaining of this paper it is more convenient to write $\{u(i, j) : 0 \leq j < r(i); 1 \leq i \leq s\}$ rather than $\{u(i, j) : 1 \leq j \leq r(i); 1 \leq i \leq s\}$ for a logarithmic signature of a group. Before we describe PGM we introduce some notation. By $\alpha[i; j]$ we mean the j 'th element of the i 'th block of α . Also, if $\mathbf{r} = (\alpha) = (r(1), \dots, r(s))$ and $(p_1, \dots, p_s) \in \mathcal{Z}_{r(1)} \times \dots \times \mathcal{Z}_{r(s)}$ then $\alpha(p_1, \dots, p_s) := \alpha[s; p_s] \dots \alpha[2; p_2] \cdot \alpha[1; p_1]$.

If $\mathbf{r} = (r(1), \dots, r(s))$ is the type of a logarithmic signature α , define the integers $m_i, i = 1, 2, \dots, s$ by :

$$m_1 = 1, \quad m_i = \prod_{j=1}^{i-1} r(j), \quad i = 2, \dots, s.$$

Let λ be the bijection from $\mathcal{Z}_{r(1)} \times \dots \times \mathcal{Z}_{r(s)}$ onto $\mathcal{Z}_{|G|}$, defined by

$$\lambda(p_1, \dots, p_s) = \sum_{i=1}^s p_i m_i$$

then for $x \in \mathcal{Z}_{|G|}$, $\lambda^{-1}(x)$ is efficiently computable by successive subtractions [representation of x with respect to mixed base $(r(1), \dots, r(s))$]. For a group G and a logarithmic signature $\alpha = \{\alpha[i; j] : j = 0, \dots, r(i) - 1; i = 1, \dots, s\}$ define the bijection $\Theta_\alpha : \mathcal{Z}_{r(1)} \times \dots \times \mathcal{Z}_{r(s)} \rightarrow G$ by

$$\Theta_\alpha(p_1, \dots, p_s) = \alpha(p_1, p_2, \dots, p_s)$$

Next, define a map $\hat{\alpha} : \mathcal{Z}_{|G|} \rightarrow G$ by $\hat{\alpha} = \lambda^{-1}\Theta_\alpha$. The function $\hat{\alpha}$ is always efficiently computable, but $\hat{\alpha}^{-1}$ is not unless α is tame.

Having defined the mappings $\hat{\alpha}$, for $\alpha \in \Lambda$, the basic cryptographic system PGM is defined as follows : For a given pair of logarithmic signatures, α, β with β tame, the *encryption* transformation $E_{\alpha, \beta}$ is the mapping

$$E_{\alpha, \beta} = \hat{\alpha} \cdot \hat{\beta}^{-1} : \mathcal{Z}_{|G|} \rightarrow \mathcal{Z}_{|G|}$$

The corresponding *decryption* transformation is obtained by reversing the order of the pair of logarithmic signatures, that is

$$D_{\alpha, \beta} = E_{\alpha, \beta}^{-1} = E_{\beta, \alpha} = \hat{\beta} \cdot \hat{\alpha}^{-1}$$

3 An Example

To clarify the ideas presented in the previous section, we illustrate PGM by means of an example. The group used here is the alternating group on five points, $\mathcal{A}_5$, of order 60. This implies that the message space $\mathcal{M}$ and the ciphertext space $\mathcal{C}$ are the set $\{0, 1, \dots, 59\}$. A supertame logarithmic signature α , with respect to a chain of stabilizer subgroups is obtained by Knuth's algorithm [5] using the generators (1 2 3 4 5) and (1 2 3)(4)(5). Another supertame logarithmic signature β is obtained by applying the procedure *shuffle* to α . The blocks of α consist of right coset representatives in a chain of subgroups in G . Procedure *shuffle* consists of changing the coset representatives and their relative order within each block. The number of blocks is $s = 3$, and the vector of block lengths is $\mathbf{r} = (5, 4, 3)$. The integers m_i are computed to be 1, 5, and 20 respectively. The two logarithmic signatures along with the appropriate knapsack $\mathbf{v}$ needed to compute λ and λ^{-1} efficiently, are shown in Figure (3).

Let us now demonstrate the operation of enciphering. If, for example, the message is 49, then it can be decomposed uniquely with respect to $\mathbf{v}$ as, $49 = (4 + 5 + 40)$. This process determines the vector of row-indices $\lambda^{-1}(49) = (4, 1, 2)$. We next compute $\pi = \Theta_\alpha(4, 1, 2) = \alpha[12;] \cdot \alpha[7;] \cdot \alpha[5;] = (1 \ 5 \ 4)(2)(3)$. We then compute $\Theta_\beta^{-1}(\pi)$, that is, the representation of π with respect to β . Since β is supertame and $\pi(1) = 5$, we locate the element in block 1 of β that sends 1 to 5. This element is $\beta[5;]$. So,

α	v	β
(1)(2)(3)(4)(5)	0	(1 4 2 3 5)
(1 2 3 4 5)	1	(1)(2)(3 5 4)
(1 3 5 2 4)	2	(1 2 5 4 3)
(1 4 2 5 3)	3	(1 3)(2 4)(5)
(1 5 4 3 2)	4	(1 5 3 4 2)
(1)(2)(3)(4)(5)	0	(1)(2 3)(4 5)
(1)(2 3)(4 5)	5	(1)(2 5 3)(4)
(1)(2 4 3)(5)	10	(1)(2 4 3)(5)
(1)(2 5 3)(4)	15	(1)(2)(3)(4)(5)
(1)(2)(3)(4)(5)	0	(1)(2)(3)(4)(5)
(1)(2)(3 4 5)	20	(1)(2)(3 5 4)
(1)(2)(3 5 4)	40	(1)(2)(3 4 5)

Figure 1: The two logarithmic signatures and the knapsack

$\pi = h_1 \cdot \beta[5;]$ for some $h_1 \in G_1$. Solving for h_1 , yields $h_1 = \pi \cdot \beta[5;]^{-1} = (1)(2 4)(3 5)$. Now h_1 fixes 1 and sends 2 to 4. Now, we locate the element of the second block of β which sends 2 to 4, namely $\beta[8;]$. Hence, $h_1 = h_2 \cdot \beta[8;]$ which yields $h_2 = (1)(2)(3 5 4)$. Continuing in this manner we completely factor π with respect to β and get $\pi = \beta[11;] \cdot \beta[8;] \cdot \beta[5;]$. This determines the vector of row pointers for β to be $(4, 2, 1) \in \mathcal{Z}_5 \times \mathcal{Z}_4 \times \mathcal{Z}_3$ and $\lambda(4, 2, 1)$ is $4 + 10 + 20 = 34$. Thus, we have $E_{\alpha,\beta}(49) = 34$. The reader can easily verify that $D_{\alpha,\beta}(34) = E_{\beta,\alpha}(34) = 49$.

4 Transformations on logarithmic signatures

Suppose that $\beta = \{B_i : i = 1, \dots, s\}$ is a transversal logarithmic signature of a group G with respect to the chain of subgroups $\gamma : G = G_0 > G_1 \cdots > G_s = 1$. Note that while $\beta = \{B_i : i = 1, \dots, s\}$ is a logarithmic signature for $G = G_0$, the set of blocks $\beta(k) = \{B_{k+1}, \dots, B_s\}$ is a logarithmic signature for G_k . If the element $u(i, j) \in B_i$ of $\beta(k)$ is replaced by $h \cdot u(i, j)$, where $h \in G_i$, the resulting collection $\beta(k)^*$ forms a new logarithmic signature for G_k . Moreover, any rearrangement of the elements of a block $B_i \in \beta(k)$ yields a new logarithmic signature for G_k . We call this procedure *shuffle*.

Procedure *shuffle* for generating new logarithmic signatures from a given one can be concisely described by considering a certain group action. If $\beta = \{B_i : i = 1, \dots, s\}$, $B_i = \{u(i, j) : j = 0, \dots, r(i) - 1\}$ is a transversal logarithmic signature of G with respect to the chain $\gamma : G = G_0 > G_1 > \cdots > G_s = 1$ of subgroups, let M be the group of all matrices of the form :

$$M = \begin{bmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & H_s \end{bmatrix}$$

where H_i is an $r(i) \times r(i)$ monomial matrix with entries in G_i . This means that H_i can be thought of as an $r(i) \times r(i)$ permutation matrix whose unity entries have been replaced by arbitrary elements of G_i . The procedure described above for obtaining new logarithmic signatures of G corresponds to acting on

$$(u(1,0), \dots, u(1, r(1) - 1); u(2,0), \dots, u(2, r(2) - 1); \dots; u(s,0), \dots, u(s, r(s) - 1))$$

on the left by some $M \in \mathbf{M}$, that is,

$$(v(1,0), \dots, v(s, r(s)))^T = M \cdot (u(1,0), \dots, u(s, r(s)))^T$$

Thus, the totality $\Lambda(\gamma)$ of logarithmic signatures with respect to γ is an $\mathbf{M}$ -orbit. We observe that since only the identity of $\mathbf{M}$ fixes a logarithmic signature in $\Lambda(\gamma)$, $\mathbf{M}$ acts regularly on $\Lambda(\gamma)$. This implies that

$$|\Lambda(\gamma)| = |\mathbf{M}| = \prod_{i=1}^s |G_i|^{r(i)} r(i)! = \prod_{i=1}^s \left(\prod_{j=i+1}^s r(j) \right)^{r(i)} r(i)!$$

Now we define another group action on Λ . Let

$$\mathbf{T} = G \times \mathcal{S}_n \times \mathcal{S}_n \times \cdots \times \mathcal{S}_n \times G$$

be a direct product, where n is the degree of G and the symmetric group $\mathcal{S}_n$ occurs $s - 1$ times. For $q = (g_0, g_1, \dots, g_s) \in \mathbf{T}$ and

$$\alpha = (\alpha[1;0], \dots, \alpha[1; r(1) - 1]; \dots; \alpha[s;0], \dots, \alpha[s; r(s) - 1]) \in \Lambda$$

let q act on α by $(q, \alpha) \rightarrow \alpha^q$ where

$$\alpha^q = (g_1^{-1}, \dots, g_1^{-1}; \dots; g_s^{-1}, \dots, g_s^{-1}) \cdot \alpha \cdot (g_0, \dots, g_0; \dots; g_{s-1}, \dots, g_{s-1})$$

This means that all the elements of the first block are multiplied by g_1^{-1} on the left and by g_0 on the right, the elements of the second block are multiplied by g_2^{-1} on the left and g_1 on the right and so on. Finally the elements in block s are multiplied by g_s^{-1} on the left and g_{s-1} on the right. It is easily seen that $\alpha^1 = \alpha$ and $(\alpha^q)^t = \alpha^{qt}$, hence $\mathbf{T}$ acts on the collection of logarithmic signatures in Λ having s blocks. If $g_0 = g_s = 1$, we say that α^q is a *sandwich* of α . If $g_1 = g_2 = \cdots = g_s = 1$ then we effectively multiply only the elements of the first block on the right by g_0 . We call this transformation a *right translation* of α . On the other hand if

$g_0 = g_1 = \dots = g_{s-1} = 1$ then we call the transformed logarithmic signature a *left translate* of α . Let S be the subgroup of T which consists of the elements of the form $(1, g_1, \dots, g_{s-1}, 1)$ of T , and H the subgroup of S which consists of all elements $(1, g_1, \dots, g_{s-1}, 1)$ with $g_i \in G_i$.

Suppose that α is a logarithmic signature for a group G of type $(\alpha) = (r(1), \dots, r(i), r(i+1), \dots, r(s))$. We can create a new logarithmic signature β by *fusing* two consecutive blocks of α , say B_i and B_{i+1} of lengths $r(i)$ and $r(i+1)$ to a single block of length $r(i) \cdot r(i+1)$. Thus, if $g = q_s \cdots q_{i+1} \cdot q_i \cdots q_2 \cdot q_1$ is the factorization of g with respect to α , then the factorization of g with respect to β will be $g = q_s \cdots q_{i+2} \cdot t \cdot q_{i-1} \cdots q_2 \cdot q_1$ where $t = q_{i+1} \cdot q_i$. In this case we say that α is a *refinement* of β . The refinement relation defines a partial order on Λ and we write $\alpha < \beta$ to denote that α is a refinement of β .

Finally, observe that if $g = q_s \cdot q_{s-1} \cdots q_2 \cdot q_1$ then we have $g^{-1} = q_1^{-1} \cdot q_2^{-1} \cdots q_s^{-1}$. This implies that if

$$\alpha = \{\alpha[i; j] : j = 0, \dots, r(i) - 1; i = 1, \dots, s\}$$

is a logarithmic signature then

$$\alpha' = \{\alpha[i; j]^{-1} : j = r(i) - 1, \dots, 0; i = s, \dots, 1\}$$

is also a logarithmic signature. We call α' the *inversion* of α . Inversion induces a duality on Λ_G . For example, a system of right coset representatives in a chain γ of subgroups (*right transversal* logarithmic signature), is transformed into a system of left coset representatives (*left transversal* logarithmic signature). In this paper we do not study properties of inversion.

Definition 4.1 Two logarithmic signatures α, β are said to be equivalent if $\hat{\alpha} = \hat{\beta}$.

We denote the equivalence of two logarithmic signatures α and β by $\alpha \sim \beta$. If α, β and γ are logarithmic signatures of a group G then it follows from the definition that $E_{\alpha, \gamma} = E_{\beta, \gamma}$ if and only if $\alpha \sim \beta$. Also, we see that if α is a refinement of β then $\alpha \sim \beta$. Hence if two logarithmic signatures are equivalent then they need not have the same type. The concept of equivalence is very important from a cryptanalytic point of view. For a cryptanalyst to break PGM, he only needs to construct logarithmic signatures which are equivalent to the ones specified by the key. The following result was shown by Magliveras and Kreher but its proof has not appeared in print.

Theorem 4.1 Let α and β be two logarithmic signatures of a group G , which have the same type $r = (r(1), \dots, r(s))$. Then α and β are equivalent if and only if they are in the same sandwich S -orbit.

Proof : It is easy to see that if α and β are in the same sandwich S-orbit then they are equivalent. For the converse, let $\alpha = \{B_1, B_2, \dots, B_s\}$ and $\beta = \{B'_1, B'_2, \dots, B'_s\}$, where $B_i = \{u_{i,1}, u_{i,2}, \dots, u_{i,r(i)}\}$ and $B'_i = \{u'_{i,1}, u'_{i,2}, \dots, u'_{i,r(i)}\}$. Now since $\alpha \sim \beta$, we have

$$\alpha(j_1, \dots, j_s) = \beta(j_1, \dots, j_s), \quad 0 \leq j_i < r(i). \quad (2)$$

Specifically, we have $\alpha(j, 0, \dots, 0) = \beta(j, 0, \dots, 0)$ for $0 \leq j < r(s)$. That is, $u_{s,j} \cdot u_{s-1,0} \cdots u_{1,0} = u'_{s,j} \cdot u'_{s-1,0} \cdots u'_{1,0}$. Let $t_{s-1} = u'_{s-1,0} \cdots u'_{1,0} \cdot u_{1,0}^{-1} \cdots u_{s-1,0}^{-1}$, then, $u_{s,j} = u'_{s,j} \cdot t_{s-1}$, and consequently $B_s = B'_s \cdot t_{s-1}$. In particular note that

$$u'_{s,0}{}^{-1} \cdot u_{s,0} = t_{s-1}^{-1}. \quad (3)$$

Next, from (2) we have $\alpha(0, j, 0, \dots, 0) = \beta(0, j, 0, \dots, 0)$ for $0 \leq j_i < r(i)$. This means that, $u_{s,0} \cdot u_{s-1,j} \cdot u_{s-2,0} \cdots u_{1,0} = u'_{s,0} \cdot u'_{s-1,j} \cdot u'_{s-2,0} \cdots u'_{1,0}$ therefore, $u_{s,0} \cdot u_{s-1,j} = u'_{s,0} \cdot u'_{s-1,j} \cdot (u'_{s-2,0} \cdots u'_{1,0} \cdot u_{1,0}^{-1} \cdots u_{s-2,0}^{-1})$. Letting $t_{s-2} = u'_{s-2,0} \cdots u'_{1,0} \cdot u_{1,0}^{-1} \cdots u_{s-2,0}^{-1}$ and using (3) we get $u_{s-1,j} = t_{s-1}^{-1} \cdot u'_{s-1,j} \cdot t_{s-2,j}$, hence $B_{s-1} = t_{s-1}^{-1} \cdot B'_{s-1} \cdot t_{s-2}$. Continuing in this manner we get $\alpha = (t_{s-1}^{-1}, \dots, t_{s-1}^{-1}, 1) \cdot \beta \cdot (1, t_1, \dots, t_{s-1})$. $\square$

It is easy to see that the monomial shuffle and inversion of a transversal logarithmic signature remain transversal. It is also clear that a subgroup involved in a transversal logarithmic signature can be refined in a 'non-transversal' way. Therefore refinement does not preserve the transversal property. In the cases of sandwiching and left and right translations, although the resulting logarithmic signature is not a transversal, we give a polynomial algorithm for constructing a transversal equivalent to the transformed logarithmic signature in Fig (2). In view of this fact we broaden the use of the term *transversal* to include logarithmic signatures for which there exists a polynomial time algorithm to compute an equivalent transversal. The algorithm starts with the last block B_s and *standardizes* it by multiplying on the right by the inverse of the first element $\alpha[s; 1]^{-1}$. The resulting block B'_s is the subgroup G_s with the identity as its first element. We then proceed upwards, standardizing the j 'th block by multiplying on the left by $\alpha[j-1; 1]$ and on the right by $\alpha[j; 1]^{-1}$. We proceed recursively, finally standardizing the first block by multiplying on the left by $\alpha[2; 1]$. It is easily seen that the resulting logarithmic signature β is transversal. Since β is a sandwich of α , we have $\alpha \sim \beta$. Note that we could have used any element of a block for standardization, rather than the first one. An interesting question which arises is whether there exist new types of transformations which would map a tame logarithmic signature to an equivalent wild one. An answer in the affirmative would lead to a public key cryptosystem based on PGM.

The logarithmic signatures we have mentioned so far have all been transversal and hence tame. A conjecture has been made by the first author that wild logarithmic signatures will occur in profusion for arbitrary groups. The second author

Input : A logarithmic signature $\alpha = \{B_1, \dots, B_s\}$

which is the sandwich of a transversal.

Output : A transversal β which is equivalent to α .

Begin

```

 $x = \alpha[s; 0]^{-1};$ 
 $xinv = \alpha[s; 0];$ 
For  $j = 0$  to  $r(s) - 1$  do
     $\beta[s; j] = \alpha[s; j] \cdot x;$ 
For  $i = s - 1$  to  $2$  do
    begin
         $y = \alpha[i; 0]^{-1};$ 
        For  $j = 0$  to  $r(i) - 1$  do
             $\beta[i; j] = xinv \cdot \alpha[i; j] \cdot y;$ 
         $xinv = y^{-1};$ 
    end;
For  $j = 0$  to  $r(1) - 1$  do
     $\beta[1; j] = xinv \cdot \alpha[1; j];$ 

```

End.

Figure 2: Algorithm to construct an equivalent transversal

G	$ \hat{\Lambda}_G $	$ \mathcal{T}_G $	$\mathcal{G}_G$	$\mu = \mathcal{T}_G / G $	comments
$\mathcal{Z}_4$	16	24	$\mathcal{S}_4$	1.00	all transversal
$\mathcal{V}_4$	24	24	$\mathcal{S}_4$	1.00	all transversal
$\mathcal{Z}_6$	132	500	$\mathcal{S}_6$	0.69	all transversal
$\mathcal{S}_3$	288	702	$\mathcal{S}_6$	0.98	all transversal
$\mathcal{Z}_8$	1152	5568	$\mathcal{S}_8$	0.14	not all transv.
$\mathcal{Z}_4 \times \mathcal{Z}_2$	2304	17088	$\mathcal{S}_8$	0.42	all transversal
$\mathcal{V}_8$	4032	10432	$\mathcal{S}_8$	0.26	all transversal
$\mathcal{Q}_8$	1344	5280	$\mathcal{S}_8$	0.13	all transversal
$\mathcal{D}_8$	3328	32640	$\mathcal{S}_8$	0.81	all transversal
$\mathcal{Z}_9$	648	1224	$\mathcal{S}_9$	0.01	all transversal
$\mathcal{V}_9$	2160	8208	$\mathcal{S}_9$	0.02	all transversal
$\mathcal{A}_4$	304128		$\mathcal{S}_{12}$		all transversal

Figure 3: Logarithmic Signatures of Small Groups

has recently constructed many non-transversal logarithmic signatures for the cyclic group $\mathcal{Z}_8$ and the alternating group $\mathcal{A}_5$. In fact the cyclic group $\mathcal{Z}_8$ turns out to be the smallest group for which there exist non-transversal logarithmic signatures.

Finally, we note that the algorithm in Fig (2) can be modified to yield an efficient algorithm to determine whether a given logarithmic signature is a transversal. At every stage, after standardizing the current block B_i , we check the order of the group generated by $\langle B_i, B_{i+1}, \dots, B_s \rangle$. This can be done in polynomial time from [3] and the fact that the number of generators is polynomial in n . If the order equals $r(i) \cdot r(i+1) \cdots r(s)$, we continue otherwise the logarithmic signature is not a transversal. In Figure (3) we tabulate some facts about logarithmic signatures of small groups.

5 Algebraic properties of PGM

In the case of PGM we see that the message space and the cipher space is $\mathcal{Z}_{|G|}$ where G is the underlying group, hence PGM is endomorphic. The key space is $\Lambda_G \times \Lambda_G$ the collection of all ordered pairs of logarithmic signatures of G . We denote by $\mathcal{T}_G$, the set of transformations defined by the key space and by $\mathcal{G}_G$ the group generated by $\mathcal{T}_G$ under functional composition. Since sandwiching produces equivalent logarithmic signatures we see that PGM is not faithful. We proceed to discuss some important algebraic properties of PGM.

Lemma 5.1 *Given a group G which is not cyclic of prime order, let $x \in \mathcal{Z}_{|G|}$ and $g \in G$. Then there exists a non-trivial logarithmic signature $\alpha \in \Lambda$ such that $\hat{\alpha}(x) = g$.*

Proof: Let $|G| = m \cdot p$ where p is a prime and $m \neq 1$, then there exists a subgroup $P \leq G$ with $|P| = p$. Let γ be a transversal logarithmic signature for G with respect to the chain $G = G_0 > G_1 = P > G_2 = 1$. Clearly, γ has two blocks B_1 and B_2 , and is of type $r = (m, p)$. Now, if $g \in G$, then $g \in Pu$ for some $u \in B_1$, hence $g = qu$ for some $q \in P$. On the other hand, given $x \in \mathcal{Z}_{|G|}$, we have $\lambda^{-1}(x) = (i, j)$ with respect to the type r , where $0 \leq i < m$ and $0 \leq j < p$. Let α be a logarithmic signature which arises from γ by rearranging its elements so that u appears in the i 'th position of the first block and q in the j 'th position of the second block. Then we have $\hat{\alpha}(x) = g$. $\square$

Theorem 5.1 *Given a group G , $x, y \in \mathcal{Z}_{|G|}$, there exist $\alpha, \beta \in \Lambda$ such that $E_{\alpha, \beta}(x) = y$. In other words $\mathcal{T}_G$ is 1-transitive.*

Proof: Let g be an arbitrary element of G . From lemma 5.1 there exist $\alpha, \beta \in \Lambda$ such that $\hat{\alpha}(x) = g$ and $\hat{\beta}(y) = g$. This implies that $E_{\alpha, \beta}(x) = \hat{\alpha}\hat{\beta}^{-1}(x) = y$. $\square$

Theorem 5.2 *If a group G has a proper subgroup H , which is not normal in G , then $\mathcal{T}_G$ is 2-transitive.*

Proof: Although $\mathcal{T}_G$ is not a group, without loss of generality, it will suffice to show that given $x, y \in \mathcal{Z}_{|G|} - \{0\}$, there exist $\alpha, \beta \in \Lambda$ such that $E_{\alpha, \beta}(0, x) = (0, y)$. Let α be a logarithmic signature for G with respect to the chain $G = G_0 > H = G_1 > 1$. Then α has two blocks B_1 and B_2 , and type $r = (k, h)$ where $h = |H|$ and $k = [G : H]$. Given $x, y \in \mathcal{Z}_{|G|} - \{0\}$, let $\lambda^{-1}(x) = (i_x, j_x)$ and $\lambda^{-1}(y) = (i_y, j_y)$ with respect to r , where $0 \leq i_x, i_y < k$ and $0 \leq j_x, j_y < h$. Now we shall consider various cases for the values of i_x, i_y, j_x, j_y and in each case construct a logarithmic signature β with type r such that $E_{\alpha, \beta}(0, x) = (0, y)$.

Case 1: $i_x, i_y, j_x, j_y > 0$

Let β be the same as α except that we exchange the elements in positions i_x and i_y of the first block and the elements in positions j_x and j_y in the second block. So we have $\beta[1; i_y] = \alpha[1; i_x]$ and $\beta[2; j_y] = \alpha[2; j_x]$. It follows that $E_{\alpha, \beta}(0, x) = (0, y)$.

Case 2: $i_x = i_y = 0$ and $j_x, j_y > 0$

This can be dealt with in a similar manner as Case 1. Here we need exchange the elements in the second block only.

Case 3: $i_x, i_y > 0$ and $j_x = j_y = 0$

Again this is similar to Cases 1 and 2. This time we only need to exchange the elements in position i_x and i_y of the first block.

Case 4: $i_x = 0$ and $i_y, j_x, j_y > 0$

Since H is not normal there exist $u \in H$ and $t \in G$ such that $tut^{-1} \notin H$. Rearrange, if necessary, the second block of α so that $\alpha[2; j_x] = u$, and place the identity $1 \in G$ in position $\alpha[1; 0]$ as the coset representative of H and also in position $\alpha[2; 0]$. So we have $\hat{\alpha}(x) = \alpha[2; j_x] \cdot \alpha[1; 0] = u \cdot 1 = u \in H$. Now since $v = u^{-1}tut^{-1} \notin H$, v lies in some coset of H , the coset representative of which is $\alpha[1; i]$ for some $i \neq 0$. Let γ be a logarithmic signature which is obtained from α by (i) replacing $\alpha[1; i]$ by $u^{-1}tut^{-1}$; (ii) exchanging the elements in $\alpha[1; i]$ and $\alpha[1; i_y]$; (iii) exchanging $\alpha[2; j_x]$ and $\alpha[2; j_y]$. We then right-translate γ by t and left-translate it by t^{-1} to obtain a new logarithmic signature β . The resulting β has $\beta[1; 0] = t$, $\beta[1; i_y] = u^{-1}tu$ and $\beta[2; 0] = t^{-1}$, $\beta[2; j_y] = t^{-1}u$. It follows that $\hat{\beta}(0) = 1$ and $\hat{\beta}(y) = u$. Hence $E_{\alpha, \beta}(0, x) = (0, y)$.

Case 5 : $i_y = 0$ and $i_x, j_x, j_y > 0$

By Case 4, there exist two logarithmic signatures α and β such that $E_{\alpha,\beta}(0, y) = (0, x)$. But then we have $E_{\beta,\alpha}(0, x) = (0, y)$.

Case 6 : $i_x, j_y = 0$ and $i_y, j_x > 0$

Again as in case 4 we assume that $\alpha[1;0] = \alpha[2;0] = 1 \in G$ and $\alpha[2;j_x] = u$ where $t^{-1}ut \notin H$. Then we have $\hat{\alpha}(x) = \alpha[2;j_x] \cdot \alpha[1;0] = u \cdot 1 = u$. Since $v = t^{-1}ut \notin H$, v belongs to some coset of H , with coset representative $\alpha[1;i]$ for some $i \neq 0$. Let γ be the same as α except that we replace $\alpha[1;i]$ by $t^{-1}ut$ and then exchange the elements $\alpha[1;i]$ and $\alpha[1;j_y]$. We now right-translate γ by t^{-1} and left-translate by t to get β . The resulting β has $\beta[1;0] = t^{-1}, \beta[1;j_y] = t^{-1}u$ and $\beta[2;0] = t$. It follows that $E_{\alpha,\beta}(0, x) = (0, y)$.

Case 7 : $i_y, j_x = 0$ and $i_x, j_y > 0$

This is symmetric to Case 6 and follows in the same way as case 5 was obtained from case 4.

Case 8 : $j_x = 0$ and $i_x, i_y, j_y > 0$

Again we begin with $\alpha[1;0] = \alpha[2;0] = 1 \in G$. Let $\hat{\alpha}(x) = \alpha[2;0] \cdot \alpha[1;i_x] = v$ where $v \notin H$. We now construct β from α by (i) Exchanging the elements $\alpha[1;i_x]$ and $\alpha[1;j_y]$; (ii) For $\alpha[2;j_y] = u \in H$ replacing $\alpha[1;0]$ by u^{-1} ; this is possible because $u^{-1} \in H$ and $1, u^{-1}$ represent the same right H coset. (iii) And finally exchanging the elements $\alpha[2;0]$ with $\alpha[2;j_y]$. We now have $\beta[1;0] = u^{-1}, \beta[1;j_y] = u, \beta[2;0] = u$ and $\beta[2;j_y] = 1$. It follows that $E_{\alpha,\beta}(0, x) = (0, y)$.

Case 9 : $j_y = 0$ and $i_x, i_y, j_x > 0$

This is symmetric to the previous case. □

Theorem 5.3 *Given a group G , $\mathcal{T}_G$ is not closed in general*

Proof : In Figure (3) we tabulate the size of $\mathcal{T}_G$ and $\mathcal{G}_G$ for several small groups. It is seen that, in general, $\mathcal{T}_G \neq \mathcal{G}_G$. □

Theorem 5.4 *If a group G has a proper subgroup H of odd order then $\mathcal{T}_G$ has a transformation which is an odd permutation in $\mathcal{S}_{|G|}$.*

Proof : Consider a transversal logarithmic signature α with respect to the chain $G = G_0 > G_1 = H > G_2 = 1$. α has two blocks and type $r = (k, h)$, where $h = |H|$

and $k = [G : H]$. Now let β be a second logarithmic signature which is identical to α , except that the first two elements in the first block are interchanged. It is clear that $\hat{\alpha}\hat{\beta}^{-1}$ will be an element of order two and have a factorization as a product of h transpositions. More precisely, we have :

$$\hat{\alpha}\hat{\beta}^{-1} = (0, 1)(k, k+1)(2k, 2k+1) \cdots ((h-1)k, (h-1)k+1)$$

□

In view of the fact that every finite 2-transitive group contains a unique minimal normal subgroup which is elementary abelian or simple [1], [10], the recent classification of finite simple groups yields a classification of finite 2-transitive groups. This in turn leads to the following interesting consequence :

Theorem 5.5 *If G is a finite non-hamiltonian group with $|G|$ different from q , $(1+q^2)$, $(1+q^3)$, $\frac{(q^n-1)}{(q-1)}$, $2^{n-1}(2^n \pm 1)$, 11, 12, 15, 22, 23, 24, 176, 276, where q is the power of a prime and n is a positive integer, then $\mathcal{T}_G$ is 2-transitive and $\mathcal{G}_G \cong \mathcal{S}_{|G|}$.*

Proof : The finite doubly transitive groups of degree m are known and consist of the class of alternating groups $\mathcal{A}_m$, the class of symmetric groups $\mathcal{S}_m$, certain infinite classes of groups of degrees q , $(1+q^2)$, $(1+q^3)$, $\frac{(q^n-1)}{(q-1)}$, $2^{n-1}(2^n \pm 1)$, where q is a power of a prime and n a positive integer, and a finite set of certain sporadic groups of degrees 11, 12, 15, 22, 23, 24, 176, 276. Since the degree of $\mathcal{G}_G$ is $|G|$, it follows from the hypothesis that $\mathcal{G}_G$ must be isomorphic to $\mathcal{A}_{|G|}$ or $\mathcal{S}_{|G|}$. However, since $|G| \neq 2^a$, by Theorem 5.4 there is an odd permutation in $\mathcal{G}_G$. Hence $\mathcal{G}_G \simeq \mathcal{S}_{|G|}$. □

6 Closing Remarks

Many questions still remain unanswered. We have asked earlier whether there exist transformations which convert a tame logarithmic signature to a wild one. Such a transformation would lead to a public key cryptosystem. Besides, the question whether there exist non-transversal logarithmic signatures which are tame needs to be addressed.

Evidence from Figure (3) suggests that the condition that G possess a non-normal subgroup may not be necessary for the 2-transitivity of $\mathcal{T}_G$. It would be interesting to investigate whether this condition can be removed.

For encryption purposes being able to factor all of G is not a requirement. For example, for a given $\epsilon > 0$, it may be possible to find subsets $X_1, X_2, \dots, X_s, Z \subset G$ with $|Z|/|G| < \epsilon$ such that each element $g \in G - Z$ has a unique factorization

$g = x_s \cdots x_2 \cdot x_1$, where $x_i \in X_i$. Such *near-factorizations* have been studied in [2] for $Z = \{1\}$, but not much is known for non-abelian G and $Z \neq \{1\}$.

Moreover, we wish to pose the following questions. Is it true that T_G^k covers S_G , for some fixed integer k independently of G , if so, is $k = 2$. Finally, is it true that T_G can be used as a source of random permutations of very large degree.

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