A SEMIGROUP APPROACH TO STOCHASTIC DYNAMICAL BOUNDARY VALUE PROBLEMS

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Abstract In many physical applications, the evolution of the system is endowed with *dynamical boundary conditions*, i.e., with boundary operators containing time derivatives. In this paper we discuss a generalization of such systems, where stochastic perturbations affect the way the system evolves in the interior of the domain as well as on the boundary.

keywords: Stochastic differential equations, boundary noise, semigroup theory, dynamical boundary conditions.

1. Introduction

In this paper we apply the technique of product spaces and operator matrices to solve stochastic evolution equations with randomly perturbed dynamic boundary conditions. Similar results for deterministic problems were recently reached for instance by [1, 10]; in their approach, the starting point is to convert an equation with inhomogeneous boundary conditions into an abstract Cauchy problem, and therefore use semigroup theory. Here, we combine these techniques with stochastic analysis to solve some "model" problems.

In the following lines, we present an appropriate abstract setting for our program. Let X and ∂X be two Hilbert spaces, called the *state space* and *boundary space*, respectively, and $\mathcal{X} = X \times \partial X$ their product space.

We consider the following linear operators:

- $A_m : D(A_m) \subset X \to X$, called *maximal operator*, and $B : D(B) \subset \partial X \to \partial X$;
- $L : D(A_m) \to \partial X$, called *boundary operator*;
- $\Phi : D(\Phi) \subset X \to \partial X$, called *feedback operator*; we assume that $D(A_m) \subset D(\Phi)$.

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With these operators, we consider the *stochastic dynamic boundary value problem*

$$\begin{cases} \mathrm{d}u(t) = [A_m u(t) + F(u(t))]\,\mathrm{d}t + G(u(t))\,\mathrm{d}W(t), & t > 0 \\ x(t) := Lu(t), & \\ \mathrm{d}x(t) = [Bx(t) + \Phi u(t)]\,\mathrm{d}t + \Gamma(x(t))\,\mathrm{d}V(t), & t > 0 \\ u(0) = u_0, \quad x(0) = x_0. & \end{cases} \tag{1}$$

In the next section, we introduce the abstract setting, and discuss sufficient conditions in order to solve (1); then in section 3 we discuss some examples that fit into our framework.

In general, we apply the abstract theory of section 2 by checking, first, that the leading operators of the internal and boundary dynamics both generate a strongly continuous (or even analytic) semigroup, while the stochastic perturbation is defined by a Lipschitz mapping. Then, we need to control the feedback operator Φ; in several cases, for instance if it is a bounded operator or if the boundary space is finite dimensional (compare example 3.1), the generation results (Lemma 7 and 8) hold. Then the existence and uniqueness result for the solution of equation (1) are given by Lemma 6.

2. Abstract setting for dynamic boundary value problems

Given Hilbert spaces U, V, we shall denote $\mathcal{L}(U;V)$ (resp. $\mathcal{L}(U)$) the space of linear bounded operators from U into V (resp. into U itself) and $\mathcal{L}_2(U;V)$ the space of Hilbert-Schmidt operators from U into V.

In order to consider the evolution of the system with dynamic boundary conditions, we start by introducing the operator A_0, defined by

$$\begin{cases} D(A_0) = \{f \in D(A_m) \mid Lf = 0\} \\ A_0 f = A_m f \text{ for all } f \in D(A_0). \end{cases}$$

We are in the position to formulate the main set of assumptions on the deterministic dynamic of the system.

ASSUMPTION 1

1. *A_0 is the generator of a strongly continuous semigroup $(T_0(t))$, $t \geq 0$, on the space X;*

2. *B is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on the space ∂X;*

3. *$L : D(A_m) \subset X \to \partial X$ is a surjective mapping;*

4. *the operator $\binom{A_m}{L} : D(A_m) \subset X \to \mathcal{X} = X \times \partial X$ is closed.*

2.1 No boundary feedback

In order to separate difficulties, in this section we consider the case $\Phi \equiv 0$. In order to treat (1) using semigroup theory, we consider the operator matrix $\mathcal{A}$ on $\mathcal{X}$ given by

$$\mathcal{A} = \begin{pmatrix} A_m & 0 \\ 0 & B \end{pmatrix}, \qquad D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A_m) \times D(B) \mid Lu = x \right\}$$

Our first step is to introduce the *Dirichlet operator* D_μ. This construction is justified by [9, Lemma 1.2]. For given $\mu \in \rho(A_0)$, assume that the stationary boundary value problem

$$\mu w - A_m w = 0, \qquad Lw = x$$

has a unique solution $D_\mu x := w \in D(A_m)$ for arbitrary $x \in \partial X$. Then D_μ is the Green (or Dirichlet) mapping associated with A_m and L. For $\mu \in \rho(A_0)$ we define the operator matrix

$$\mathcal{D}_\mu = \begin{pmatrix} I_X & -D_\mu \\ 0 & I_{\partial X} \end{pmatrix};$$

from [12, Lemma 2] we obtain the representation

$$(\mu - \mathcal{A}) = (\mu - \mathcal{A}_0)\mathcal{D}_\mu$$

where $\mathcal{A}_0$ is the diagonal operator matrix

$$\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & B \end{pmatrix}$$

on $D(\mathcal{A}_0) = D(A_0) \times D(B)$.

Using [1, Theorem 2.7 and Corollary 2.8], we are in the position to characterize the generation property of $(\mathcal{A}, D(\mathcal{A}))$.

LEMMA 2 *Assume that A_0 is invertible. Then $\mathcal{A}$ generates a C_0-semigroup $\mathcal{T}(t)$ on $\mathcal{X}$ if and only if the operator $Q_0(t) : D(B) \subset \partial X \to X$,*

$$Q_0(t)y := -A_0 \int_0^t T_0(t-s) D_0 S(s) y \, \mathrm{d}s, \tag{2}$$

has an extension to a bounded operator on ∂X, satisfying

$$\limsup_{t \searrow 0} \|Q_0(t)\| < +\infty. \tag{3}$$

Moreover, in this case we can also give a representation of $\mathcal{T}(t)$:

$$\mathcal{T}(t) = \begin{pmatrix} T_0(t) & Q_0(t) \\ 0 & S(t) \end{pmatrix}. \tag{4}$$

COROLLARY 3 *Assume that A_0 and B generate analytic semigroups on X and ∂X, respectively; then $\mathcal{A}$ generates an analytic semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$.*

COROLLARY 4 *If $B \in \mathcal{L}(\partial X)$ is bounded, then $\mathcal{A}$ generates a C_0 semigroup on $\mathcal{X}$; in particular, if A_0 is invertible and $B = 0$, then $Q_0(t) = (I - T_0(t))D_0$.*

We are now in position to write the original problem (1) in an equivalent problem as a *stochastic abstract Cauchy problem*

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = [\mathcal{A}\mathbf{x}(t) + \mathcal{F}(\mathbf{x}(t))]\,\mathrm{d}t + \mathcal{G}(\mathbf{x}(t))\,\mathrm{d}\mathcal{W}(t), \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{5}$$

In order to solve this stochastic problem, we introduce some assumptions on the nonlinear and stochastic terms which appear in (1); these assumptions, in turn, will be reflected to the operators $\mathcal{F}$ and $\mathcal{G}$ in (5).

ASSUMPTION 5

We are given a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$;

1 *W and V are Wiener noises on X and ∂X, respectively, and $\mathcal{W} = (W, V)$ is a Q-Wiener process on $\mathcal{X}$, with trace class covariance operator Q;*

2 *the mappings $F : X \to X$ and $G : X \to \mathcal{L}_2(X, X)$ are Lipschitz continuous*

$$|F(x) - F(y)| + \|G(x) - G(y)\| \leq C|x - y|,$$

with linear growth bound

$$|F(x)|^2 + \|G(x)\|^2 \leq C(1 + |x|^2);$$

3 *the mapping $\Gamma : \partial X \to \mathcal{L}_2(\partial X, \partial X)$ is Lipschitz continuous with linear growth bound:*

$$\|\Gamma(x) - \Gamma(y)\| \leq C|x - y|, \qquad \|\Gamma(x)\|^2 \leq C(1 + |x|^2).$$

In order to make this paper self-contained, let us recall the relevant result from [3].

LEMMA 6 *Assume that $\mathcal{A}$ is the generator of a C_0 semigroup on $\mathcal{X}$ and Assumption 5 holds. Then for every $\mathbf{x}_0 \in \mathcal{X}$, there exists a unique mild solution to (5); moreover, it has a continuous modification.*

2.2 Boundary feedback

We are now in the position to include the feedback operator Φ into our discussion. In order to simplify the exposition, and in view of the examples below, we choose to concentrate on two cases, which are far from being general.

We shall prove some generation results for the operator matrix

$$\mathcal{A} = \begin{pmatrix} A_m & 0 \\ \Phi & B \end{pmatrix}, \qquad D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in D(A_m) \times D(B) \mid Lu = x \right\},$$

where we assume that A_0 is the generator of a strongly continuous semigroup with $0 \in \rho(A_0)$.

As in Section 2.1 we write

$$\mathcal{A} = \mathcal{A}_0 \mathcal{D}_0^\Phi,$$

where the operator matrix $\mathcal{D}_0^\Phi$ is given by

$$\mathcal{D}_0^\Phi = \begin{pmatrix} I_X & -D_0 \\ B^{-1}\Phi & I_{\partial X} \end{pmatrix} = I_{\mathcal{X}} + \begin{pmatrix} 0 & -D_0 \\ B^{-1}\Phi & 0 \end{pmatrix}.$$

The first result can be proved as in [1, Section 4].

LEMMA 7 *Assume that the feedback operator $\Phi : X \to \partial X$ is bounded. Then the matrix operator $\mathcal{A}$ is the generator of a C_0 semigroup.*

Next, we consider a generation result in case Φ is unbounded, which is the case in several applications (see for instance [2]). This case may be treated using the techniques of *one-sided coupled operators*, compare [8, Theorem 3.13 and Corollary 3.17].

LEMMA 8 *Assume that A_0 is the generator of an analytic semigroup, that $B \in \mathcal{L}(\partial X)$ and $D_0\Phi$ is a compact operator; then the matrix operator $\mathcal{A}$ is the generator of an analytic semigroup.*

Assume that the boundary space ∂X is finite dimensional. Then $B \in \mathcal{L}(\partial X)$ is bounded and $D_0\Phi$ is a finite rank operator, hence it is compact, so that $\mathcal{A}$ is the generator of an analytic semigroup thanks to previous lemma.

3. Motivating examples

We are concerned with the following examples. The first two are also considered in the paper [2]; notice that the first one has some applications in mathematical biology (for instance, to study impulse propagation along a neuron). The third example was considered in the paper [4] and (in the special case discussed here) in the paper [6].

3.1 Impulse propagation with boundary feedback

A widely accepted model for a dendritic spine with passive electric activity can be described by means of the following equation for the potential

$$\frac{\partial u}{\partial t}(t,\xi) = \frac{\partial^2 u}{\partial \xi^2}(t,\xi) + f(\xi, u(t,\xi)), \qquad t > 0, \quad \xi > 0;$$

the extremal point $\xi = 0$ denotes the cellular *soma*, where the potential evolves with a different dynamic; setting $x(t) = u(t,0)$, the following equation is a possible model for this dynamic

$$\mathrm{d}x(t) = [-bx(t) + cu'(t,0)]\,\mathrm{d}t + \sigma(x(t))\,\mathrm{d}W(t),$$

where $W(t)$ is a real standard brownian motion.

In order to set the problem in an abstract setting, we consider the spaces $X = L^2(\mathbf{R}_+)$ and $\partial X = \mathbf{R}$; the matrix operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ c\frac{\partial}{\partial x}\big|_{x=0} & -b \end{pmatrix}.$$

Since the boundary space ∂X is finite dimensional and the leading operator $\frac{\partial^2}{\partial x^2}$ on $\mathbf{R}_+$ with Dirichlet boundary condition generates an analytic semigroup, then so does $\mathcal{A}$ on $\mathcal{X} = X \times \partial X$. Therefore, we write our problem in the equivalent form

$$\mathrm{d}\mathbf{x}(t) = [\mathcal{A}\mathbf{x}(t) + \mathcal{F}(\mathbf{x}(t))]\,\mathrm{d}t + \mathcal{G}(\mathbf{x}(t))\,\mathrm{d}\mathcal{W}(t)$$

and we obtain existence and uniqueness of the solution thanks to Lemma 6.

3.2 Dynamic on a domain with mixed boundary conditions

In previous example, the boundary space was finite dimensional. Here, we shall be concerned with a dynamical system which evolves in a bounded region $\mathcal{O} \subset \mathbf{R}^d$, with smooth boundary $\Gamma = \partial\mathcal{O}$. We assume that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, where Γ_i are open subsets of Γ with $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Let $\mathcal{O}$ represent a solid body; suppose that a classical heat diffusion process occurs inside $\mathcal{O}$, so if $u = u(t,x)$ represents the temperature at point x and time t, the process can be modelled by the classical heat equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x)$$

where the thermal conductivity $\rho > 0$ is taken to be 1 for simplicity. This equation needs to be completed by boundary conditions and initial data. Inspired by physical considerations, different sorts of boundary conditions exist in the literature. In this paper, we consider dynamical ones, that is

$$\mathrm{d}x_1(t) = Bx_1(t)\,\mathrm{d}t + \Gamma(x_1(t))\,\mathrm{d}V(t),$$

where x_1 is the valuation of u on the (portion of) boundary Γ_1 and a stochastic perturbation acts on the boundary behavior. This kind of boundary conditions appears when the boundary material has a large thermal conductivity and sufficiently small thickness. Hence, the boundary material is regarded as the boundary of the domain. For instance, one considers an iron ball in which water and ice coexists. Notice that we can cover the case where the boundary conditions are dynamical only on a part of the boundary.

We are concerned with the Sobolev spaces $H^s(\mathcal{O})$, $s > 0$ (see for instance [11] for the definition). The construction of the Sobolev spaces $H^s(\Gamma)$ for functions defined on the boundary $\Gamma = \partial\mathcal{O}$ is given in terms of the Laplace-Beltrami operator $B := \Delta_\Gamma$ on Γ; indeed we have

$$H^s(\Gamma) = \text{ domain of } (-\Delta_\Gamma)^s.$$

Denote $B^s(\Gamma) = H^{s-\frac{1}{2}}(\Gamma)$, for $s > \frac{1}{2}$, and similarly for $B^s(\Gamma_i)$. Then the trace mapping γ (and similarly for γ_i) is continuous from $H^s(\mathcal{O})$ into $B^s(\Gamma)$, for $s > \frac{1}{2}$.

In order to state the equation in an abstract setting, we introduce, on the Hilbert space $X = L^2(\mathcal{O})$, the operator

$$\begin{aligned}&A_m u := \Delta u(x), \qquad \text{with domain}\\ &D(A_m) = \{\varphi \in H^{1/2}(\mathcal{O}) \cap H^2_{loc}(\mathcal{O}) \mid A_m\varphi \in X\}.\end{aligned}$$

We also consider the normal boundary derivative

$$\mathcal{B}_2(x,\partial) = \sum_{k,j=1}^{d} a_{kj}(x)\nu_k\gamma_2\frac{\partial}{\partial x_j}, \qquad x \in \Gamma_2,$$

where $\nu = (\nu_1, \dots, \nu_d)$ is the outward normal vector field to Γ. Then we consider the following linear equation

$$\begin{cases} \mathrm{d}u(t) = A_m u(t)\,\mathrm{d}t, \\ x_1 = \gamma_1 u, \qquad \mathrm{d}x_1(t) = Bx_1(t)\,\mathrm{d}t + \Gamma(x_1(t))\,\mathrm{d}V(t), \\ x_2 = \mathcal{B}_2 u, \qquad \mathrm{d}x_2(t) = 0, \end{cases} \tag{6}$$

with the initial conditions

$$u(0) = u_0, \qquad x_1(0) = \gamma_1 u_0, \qquad x_2(0) = 0.$$

We shall transform our problem in an abstract Cauchy problem in a larger space. We define the Hilbert space $\mathcal{X} = L^2(\mathcal{O}) \times L^2(\Gamma_1) \times L^2(\Gamma_2)$, and we denote $\mathbf{x} \in \mathcal{X}$ the column vector with components (u, x_1, x_2). On the product space $\mathcal{X}$ we introduce the matrix operator $\mathcal{A}$, defined as

$$\mathcal{A} := \begin{pmatrix} A_m & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

on

$$D(\mathcal{A}) = \Big\{ \mathbf{x} = (u, x_1, x_2) \mid u \in H^2(\mathcal{O}),\ x_1 = \gamma_1 u,\ x_1 \in D(B), \\ x_2 = \mathcal{B}_2 u,\ x_2 = 0 \Big\}.$$

Then $\mathcal{A}$ satisfies the assumptions of Corollary 3, hence it is the generator of an analytic semigroup; we solve problem (6) in the equivalent form

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = \mathcal{A}\mathbf{x}(t)\,\mathrm{d}t + \mathcal{G}(\mathbf{x}(t))\,\mathrm{d}\mathcal{W}(t) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

using Assumption 5 and Lemma 6.

3.3 Inhomogeneous boundary conditions

Let us consider, with the notation of previous example, the case when the boundary conditions on Γ_2 are given by $x_2(t) = f(t)$ for a function $f : \mathbf{R}_+ \to \Gamma_2$ that is continuously differentiable in time. The boundary space ∂X is given by the product $\partial X_1 \times \partial X_2$; denote Π_1, resp. Π_2, the immersions on ∂X_1 (resp. ∂X_2) into ∂X. We define the boundary operator $L : D(A_m) \to \partial X$ as $Lu = \binom{\gamma_1 u}{\mathcal{B}_2 u}$; the operator $\underline{B}$ is the operator matrix $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ on ∂X. Let R be a bounded operator from U into ∂X_1; in order to separate the difficulties we consider the following form of (1)

$$\begin{cases} \mathrm{d}u(t) = A_m u(t)\,\mathrm{d}t, & t > 0, \\ x(t) = Lu(t), & t \geq 0, \\ \mathrm{d}x(t) = [\underline{B}x(t) + \Pi_2 f'(t)]\,\mathrm{d}t + \Pi_1 R\,\mathrm{d}V(t), & t > 0, \\ u(0) = u_0, \quad x(0) = x_0, \quad f(0) = f_0. \end{cases} \tag{7}$$

We define the abstract problem

$$\begin{cases} \mathrm{d}\mathbf{x}(t) = [\mathcal{A}\mathbf{x}(t) + \underline{f}'(t)]\,\mathrm{d}t + \mathcal{R}\,\mathrm{d}V(t), \\ \mathbf{x}(0) = \zeta, \end{cases} \tag{8}$$

where $\mathcal{R} \in L(U, \mathcal{X})$ is defined by $\mathcal{R} \cdot h := \binom{0}{\Pi_1 R \cdot h}$ for all h in the Hilbert space U, $\underline{f}(t) = \binom{0}{\Pi_2 f(t)}$ and $\zeta = (u_0, x_0, f_0)^*$.

The solution in *mild form* is given by the formula

$$\begin{aligned}\mathbf{x}(t) = \mathbf{x}(t,\zeta) = \mathcal{T}(t)\zeta + \int_0^t \mathcal{T}(t-s)\begin{pmatrix}0\\ \Pi_2 f'(s)\end{pmatrix}\mathrm{d}s \\ + \int_0^t \mathcal{T}(t-s)\mathcal{R}\,\mathrm{d}V(s).\end{aligned} \tag{9}$$

We consider first the middle integral in (9); using the representation of the semigroup $\mathcal{T}(t)$ given by formula (4) we obtain

$$\begin{aligned}\int_0^t \mathcal{T}(t-s)\begin{pmatrix}0\\ \Pi_2 f'(s)\end{pmatrix}\mathrm{d}s = -\mathcal{T}(t)\underline{f}(0) + \underline{f}(t) \\ - \begin{pmatrix} A_0 \int_0^t T_0(t-\sigma)D_0\Pi_2 f(\sigma)\,\mathrm{d}\sigma \\ 0\end{pmatrix}.\end{aligned}$$

We then write (9) in the form

$$\begin{aligned}\mathbf{x}(t) = \mathcal{T}(t)\begin{pmatrix}u_0\\ \Pi_1 x_0\end{pmatrix} + \begin{pmatrix}0\\ \Pi_2 f(t)\end{pmatrix} - \begin{pmatrix} A_0 \int_0^t T_0(t-\sigma)D_0\Pi_2 f(\sigma)\,\mathrm{d}\sigma \\ 0\end{pmatrix} \\ + \int_0^t \mathcal{T}(t-s)\mathcal{R}\,\mathrm{d}V(s);\end{aligned}$$

notice that we do not need anymore the differentiability condition on f.

In the next statement, we are concerned with the properties of the stochastic convolution process

$$W_{\mathcal{A}}^{\mathcal{R}}(t) = \int_0^t \mathcal{T}(t-s)\mathcal{R}\,\mathrm{d}V(s). \tag{10}$$

COROLLARY 9 *Under the assumptions of Proposition 3, assume*

$$\int_0^t \|\mathcal{T}(s)\mathcal{R}\|_{HS}^2\,\mathrm{d}s < +\infty \qquad \forall\, t \in [0,T]. \tag{11}$$

Then $W_{\mathcal{A}}^{\mathcal{R}}$ is a gaussian process, centered, with covariance operator defined by

$$\operatorname{Cov} W_{\mathcal{A}}^{\mathcal{R}}(t) = Q_t := \int_0^t [\mathcal{T}(s)\mathcal{R}\mathcal{R}^*\mathcal{T}^*(s)]\mathrm{d}s \tag{12}$$

and $Q_t \in \mathcal{L}_2(H)$ for every $t \in [0,T]$.

REMARK 10 *Condition (11) is verified whenever $R \in \mathcal{L}_2(U, \partial X)$ and, in particular, in case $R \in \mathcal{L}(U, \partial X)$ and U is finite dimensional.*

3.4 Stochastic boundary conditions

In several papers, the case of a white-noise perturbation on the boundary $f(t) = \dot{V}(t)$ is considered; oure interest is motivated by the results in [4–6].

Following [4], we shall define as mild solution of (9) in the interior of the domain the process $u(t)$ given by

$$u(t) = T_0(t)u_0 + Q_0(t)x_0 - A_0 \int_0^t T_0(t-\sigma)D_0(\Pi_2 1)\,\mathrm{d}V(\sigma) \qquad (13)$$

We study this example with state space $\mathcal{O} = [0,1]$ and mixed boundary condition on $\partial\mathcal{O}$, that is, $X = L^2(\mathcal{O})$, $D(A_0) = \{u \in X : u(0) = 0,\ u'(1) = 0\}$, $A_0u(\xi) = u''(\xi)$, $U = \mathbf{R}$, $Q = I$, $R = I$ and we choose

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad S(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix},$$

which determine the boundary condition at 0, while there exists a continuous function $f(t)$ which determines the boundary condition $\frac{\mathrm{d}}{\mathrm{d}x}u(t,1) = f(t)$.

In this case we can explicitly work out the solution. At first, notice that $(D_0\Pi_2\alpha)(x) = \alpha x$; next, we construct the orthogonal basis $\{g_k : k \in \mathbf{N}\}$, setting $g_k(x) = \sin((\pi/2 + k\pi)x)$, to which it correspond the eigenvalues $\lambda_k = -(\pi/2 + k\pi)^2$. Since

$$T_0(t)(x) = \sum_{k=1}^{\infty} \langle x, g_k(x)\rangle e^{\lambda_k t} g_k(x)$$

and

$$\int_0^1 x \sin((\pi/2 + k\pi)x)\,\mathrm{d}x = \frac{(-1)^k}{|\lambda_k|},$$

we obtain

$$-A_0 \int_0^t T_0(t-\sigma)D_0(\Pi_2 1)\,\mathrm{d}V(\sigma) = -\sum_{k=1}^{\infty} (-1)^{k+1} g_k(x) \int_0^t e^{\lambda_k(t-\sigma)}\,\mathrm{d}V(\sigma).$$

In this setting estimate (11) is verified due to the choice of $\partial X = \mathbf{R}^2$, see Remark 10. Also, the new stochastic term is well defined for every $t \geq 0$, since

$$\mathbf{E}\left| A_0 \int_0^t T_0(t-\sigma)D_0(\Pi_2 1)\,\mathrm{d}V(\sigma)\right|^2 \leq C \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < +\infty, \qquad (14)$$

with a constant C independent from t.

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