ON A VARIATIONAL PROBLEM OF ULAM

S. Villa¹

¹*DIMA, Università di Genova, Genova, Italy, villa@dima.unige.it*

Abstract Using the concept of well-posedness under perturbations we give an answer to the following question posed by Stanislaw Ulam : “when it is true that solutions of two problems in the calculus of variations which corresponds to “close” physical data must be close to each other?” (“A collection of mathematical problems”, 1960).

keywords: well-posedness, integral functionals, bounded Hausdorff convergence

1. Introduction

The origin of the problem studied in this paper can be found in the following question posed by Stanislaw Ulam in 1960 in the book “A collection of mathematical problems” (see [14]).

“If we consider a typical elementary problem, that of finding an extremum y_0 of

$$I(y) = \int_a^b F(x, y(x), y'(x)) dx \quad (1)$$

[...] a question arises, namely, the conditions which guarantee that for every $\epsilon > 0$ there exists a $d > 0$ such that for all sufficiently “regular” $G(x, y, z)$ with $|G - F| < d$, there exists a minimum y_1 for

$$J(y) = \int_a^b G(x, y(x), y'(x)) dx, \quad (2)$$

where $|y_1 - y_0| < \epsilon$.

We assume merely the proximity of F and G and nothing is assumed about the proximity of their partial derivatives, occurring in the Lagrangian equations.

Speaking descriptively, the question is: when it is true that solutions of two problems in the calculus of variations which corresponds to “close” physical data must be close to each other?”

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The aim of the paper is to give an answer to this question in the more general case of multiple integrals defined by

$$J(u) = \int_{\Omega} L(x, u(x), Du(x)) dx \quad (3)$$

on a suitable Sobolev space and to locate it in an appropriate context.

The first remark is that in this form the question has always a negative answer; in fact Bobylev showed in [4] that whatever the functional I defined by (1) under study would be, it is possible to destroy all its minimum points by disturbing to be as small as wished.

EXAMPLE 1 *[[4]] Let $L : [0, 1] \times \mathbb{R} \rightarrow [0, +\infty)$ be a three times differentiable function and let us consider the associated integral functional*

$$J(u) = \int_0^1 L(x, u(x), u'(x)) dx,$$

where $u \in C^1(0, 1)$ and $u(0) = u(1) = 0$. Let us suppose that J is Tikhonov well-posed with unique minimizer $u_0 = 0$.

Consider the one-parameter family of functionals

$$J_{\epsilon}(u) = \int_0^1 \left[L(x, u(x), u'(x)) + \epsilon \cos\left(\frac{u'(x)}{\epsilon}\right) \right] dx.$$

For a sufficiently small ϵ the functional J_{ϵ} has no minimum points in the unit ball

$$B = \{u(x) \in C^1(0, 1) \mid \|u\| \leq 1\}.$$

Anyway Bobylev showed that it is possible to restrict the class of admissible perturbations in order to ensure that every perturbed problem has a solution. In fact he showed that considering only strictly convex and sufficiently regular integrands, the problem of Ulam is solved positively.

The same approach is developed in [13], where it is proved the existence of minimizers for problems in a neighborhood of a given one, if a uniform strictly subdifferentiability property is satisfied.

Since we do not want to impose convexity, we choose another way to treat this problem, taking into account the stability of approximate minimizers instead of that of the minimizers. In fact, since the given functional I has a minimizer, it follows from the proximity of F and G that the infimum of J (defined by (2)) is finite. Therefore for every $\epsilon > 0$, surely there exist functions y_{ϵ} such that $J(y_{\epsilon}) < \inf J + \epsilon$. So the question posed by Ulam can be modified into

“when it is true that approximate solutions of two problems in the calculus of variations which corresponds to “close” physical data must be close to each other?”

Clearly in the case that the minimizers of close problems exist, an affirmative answer to this question implies an affirmative answer to the original one.

This question is intimately related to the well-posedness properties of the minimization problem of the functional I as Ulam himself wrote in his book: “Affirmative theorems of this sort would ensure the stability of the solutions of physical problems even with respect to “hidden” parameters.[...] It seems desirable in many mathematical formulations of physical problems to add still another requirement to the well-known desiderata of Hadamard of existence, uniqueness and continuity of the dependence of solutions on the initial parameters.

Specifically one should have a stability in the strongest sense illustrated by us above: the solutions should vary continuously even when the operator itself is subject to “small” variations”.

This is exactly the point of view that we adopt throughout this work. Let us better explain what we mean. Given an optimization problem, arising in the calculus of variations or not, it is possible to see it as an element of a “variational system” according to the definition given by Rockafellar and Wets in [11].

More precisely we consider a fixed space $\mathcal{A}$ of parameters, endowed with a convergence structure, and we associate to each parameter an optimization problem. Moreover we of course assume that the problem we are starting from corresponds to a fixed $a_0 \in \mathcal{A}$. In this way we generate a family of problems having the same structure of the starting one.

Remarkable examples are the cases in which the space of parameters is the space of admissible initial data or the space of integrands satisfying some given properties.

Once that the given minimization problem is embedded in a suitable family, we can speak of well-posedness under perturbations of it. This concept, introduced by Zolezzi in [20], rigorously expresses the generalization of the Hadamard’s idea of well-posedness mentioned by Ulam, in the more general form, which takes into account not only the behavior of the minimizers, but also that of the approximate minimizers (see Definition 3), requiring the continuous dependence of the approximate solutions on the parameters.

This notion is the right instrument to reformulate the problem of Ulam in a better way. The final version of the problem is therefore:

It is possible to find sufficient conditions on the integrand which guarantees well-posedness under perturbations of the functional I ?

In this new form the example proposed by Bobylev is no more a counterexample (see Example 10), since we are considering approximate minimizers. Moreover it is somehow misleading in the sense that we explain below.

Looking at Example 1 it turns out in fact that the existence of a minimizer is a very unstable property if we consider perturbations of the integrand with re-

spect to the uniform metric (which is a very strong one) but Ioffe and Zaslavski proved that for several variational systems (among them integral functionals parameterized by the integrands) well-posedness under perturbations is a generic property (with respect to a suitable topology).

This result is not completely satisfactory since it only tells us that there are a lot of well-posed problems without giving a method to find them.

In the literature there are not many constructive results on this subject, and the existing ones often require strong regularity properties on the integrand and convexity (see [22, 20, 4, 12]). Therefore our aim is to find some weaker sufficient conditions still guaranteeing well-posedness of a given problem of the calculus of variations.

The last thing that we want to point out is the choice of the distance between two problems. The uniform convergence of the integrands proposed by Ulam is too strong to treat this kind of problems.

The right choice seems to be one of the so called “variational convergences”, as Γ , Mosco and bounded Hausdorff convergence. We choose the last one because it gives us the useful link between Tikhonov well-posedness and well-posedness under perturbations that we mentioned above (see Theorem 4), and moreover gives the possibility of finding quantitative bounds on the distance between minima and minimizers of two given problems.

The paper is organized as follows: the answer to the question analyzed here will be given in Section 3, whereas Section 2 contains some preliminaries, as the introduction of the well-posedness concepts and the relations between them.

The last section is devoted to some dominated convergence theorems for the bounded Hausdorff convergence of integral functionals.

The proofs of the results presented here can be found in the papers [16, 17, 15, 18].

2. Well-posedness concepts and preliminaries

We start introducing the definitions that we need. We only review the well-posedness concepts whereas for the definition of bounded Hausdorff (bH) convergence we refer to the book [3] and to the papers [1, 2].

DEFINITION 2 *We say that a function $J : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is Tikhonov well-posed if it satisfies the following conditions:*

- a) there exists a unique global minimizer u_0 of J ;*
- b) if u_h is any minimizing sequence, i.e. a sequence such that $J(u_h) \rightarrow J(u_0)$, then $u_h \rightarrow u_0$.*

For a more general definition, and for several characterizations of Tikhonov well-posedness, we refer to [6].

Now we consider also a convergence space $\mathcal{A}$ and a fixed point $f \in \mathcal{A}$. We are given the proper extended real-valued functions

$$J : X \rightarrow (-\infty, +\infty], \quad I : \mathcal{A} \times X \rightarrow (-\infty, +\infty]$$

such that

$$J(u) = I(f, u), \quad u \in X.$$

The corresponding value function is given by

$$V(g) = \inf \{I(g, u) \mid u \in X\}, \quad g \in \mathcal{A}.$$

DEFINITION 3 ([21]) *The problem of minimizing J on X is called well-posed under perturbations (with respect to the embedding defined by I) iff*

- 1 $V(g) > -\infty$ for all $g \in \mathcal{A}$,
- 2 there exists a unique global minimizer u_0 for J ,
- 3 for every sequences $f_h \rightarrow f$ in $\mathcal{A}$ and $u_h \in X$ such that

$$I(f_h, u_h) - V(f_h) \rightarrow 0 \text{ as } h \rightarrow +\infty \quad (4)$$

we have $u_h \rightarrow u_0$ in X .

Sequences satisfying condition (4) are called asymptotically minimizing sequences.

Let $L(X) = \{G : X \rightarrow \mathbb{R} \cup \{+\infty\}, \text{ bounded below, proper and l.s.c.}\}$ and $\psi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\lim_{\|u\| \rightarrow +\infty} \psi(u) = +\infty$. Set

$$\mathcal{A} = \{G \in L(X) \mid G(u) \geq \psi(u) \text{ for each } u\}, \quad (5)$$

endowed with the bH-convergence, and define

$$\begin{aligned} I : \mathcal{A} \times X &\rightarrow (-\infty, +\infty] \\ I(G, u) &= G(u). \end{aligned} \quad (6)$$

The following result is a rewriting in different terms of Theorem 3.5 of [16].

THEOREM 4 *Let X be a Banach space, $\mathcal{A}$ and I defined by (5) and (6). Assume that $J \in \mathcal{A}$ is Tikhonov wellposed. Then J is wellposed under perturbations with respect to the embedding defined by I .*

Theorem 4 cannot be extended without further restrictions on the space $\mathcal{A}$ of admissible perturbations putting on $\mathcal{A}$ a weaker convergence as Example 4.9 of [10] shows and this is why we choose bH-convergence to study this problem.

We conclude this section recalling the definition of strict convexity at a point in order to illustrate Theorem 6. In the sequel, given a function $L : \mathbb{R}^n \rightarrow [0, +\infty]$ we will denote by L^{**} its convex regularization.

In the following we denote by L_r the function defined on $\mathbb{R}^n$ by setting $L_r = L$ on the closed ball $\overline{B}(u_0, r)$ with center u_0 and radius r and $L_r = +\infty$ otherwise.

DEFINITION 5 *Let U be a closed convex subset of $\mathbb{R}^n$ and $u_0 \in U$. We say that a function $L : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex at u_0 with respect to the set U if*

$$\sum_{i=1}^m c_i L(v_i) \geq L(u_0) \quad (7)$$

for every $m > 0$, for every $v_i \neq u_0$ and $c_i > 0$ such that $v_i \in U$, $\sum_{i=1}^m c_i = 1$ and $\sum_{i=1}^m c_i v_i = u_0$.

If inequality (7) is always strict, we say that the function L is strictly convex at u_0 with respect to U .

Clearly, there are many examples of convex functions at a point which are not globally convex; for instance every function is convex at its minimum point, if it has one.

3. Well-posedness results

Following the idea described in in the Introduction, we give an answer to Ulam's question in the following way: given a normal integrand (see [7]) we find some conditions on it guaranteeing Tykhonov well-posedness of the associated integral functional.

More precisely, consider $L : \Omega \times \mathbb{R}^{nm} \rightarrow [0, +\infty]$ a normal integral and define the associated integral functional

$$J(u) = \int_{\Omega} L(x, Du(x)) dx \quad (8)$$

on the space $W_0^{1,1}(\Omega; \mathbb{R}^m)$. The key result of this section is the following theorem which enable us to prove Corollary 9.

THEOREM 6 *Let L and J be defined as in Corollary 6. Suppose that J is coercive having a unique minimum point $u_0 \in W_0^{1,1}(\Omega; \mathbb{R}^m)$.*

Moreover assume that $v \mapsto L(x, v)$ is strictly convex at the point $Du_0(x)$ for almost every $x \in \Omega$. Then J is Tikhonov wellposed in $W_0^{1,1}(\Omega; \mathbb{R}^m)$.

Equivalently, if $u_h \in W_0^{1,1}(\Omega; \mathbb{R}^m)$ is any minimizing sequence, then:

$$|u_h - u_0| \rightarrow 0 \text{ in } W_0^{1,1}(\Omega; \mathbb{R}^m).$$

We skip the proof of Theorem 6, which can be found in [17], and we just state the basic tools in order to prove it. The first is Lemma 7, which gives a characterization of the points of strict convexity in terms of their geometric properties.

LEMMA 7 Let $L : \mathbb{R}^n \rightarrow [0, +\infty]$ be lower semicontinuous and proper. Then:

- 1 If $(u_0, L^{**}(u_0))$ is an extreme point of $\text{epi} L^{**}$ then L is strictly convex at u_0 .
- 2 If L is strictly convex at u_0 , then $(u_0, L(u_0))$ is an extreme point of $\text{epi}(L_r)^{**}$ for all $r > 0$.

The second lemma is a generalization to the nonconvex setting of Lemma 3 of [19] which can be applied thanks to a semicontinuity result (see Lemma 4.3 of [17]).

LEMMA 8 Let $L : \Omega \times \mathbb{R}^{mn} \rightarrow [0, +\infty]$ be a normal integrand and $u_h, u_0 \in W^{1,1}(\Omega; \mathbb{R}^m)$. Suppose that the function $v \mapsto L(x, v)$ is convex at the point $Du_0(x)$ for almost every $x \in \Omega$.

If $\|u_h - u_0\|_{L^1} \rightarrow 0$, $Du_h \rightarrow Du_0$ and $J(u_h) \rightarrow J(u_0)$ then:

$$(x \mapsto L(x, Du_h(x))) \rightarrow (x \mapsto L(x, Du_0(x)))$$

in $L^1(\Omega; \mathbb{R})$.

As a consequence of Theorems 4 and 6, we get the following Corollary, which establishes a criterium of well-posedness under perturbations for nonconvex integral functionals.

COROLLARY 9 Let $L, L_h : \Omega \times \mathbb{R}^{nm} \rightarrow [0, +\infty]$ be normal integrands and let J be defined as (8). Suppose that J is coercive having a unique minimum point $u_0 \in W_0^{1,1}(\Omega; \mathbb{R}^m)$.

Moreover assume that $v \mapsto L(x, v)$ is strictly convex at the point $Du_0(x)$ for almost every $x \in \Omega$. Consider the sequence J_h defined by $J_h(u) := \int_{\Omega} L_h(x, Du(x)) dx$ on $W_0^{1,1}(\Omega, \mathbb{R}^m)$. Assume that

- (i) $J_h \rightarrow J$ with respect to the bH-convergence on $W_0^{1,1}(\Omega; \mathbb{R}^m)$;
- (ii) J_h is equicoercive.

Then:

$$\|u_h - u_0\| \rightarrow 0 \text{ in } W_0^{1,1}(\Omega; \mathbb{R}^m)$$

for every asymptotically minimizing sequence u_h . In other words the problem of minimizing J is well-posed under equicoercive perturbations with respect to the bH-convergence.

Referring again to the comments in the Introduction, in the next example we show in which sense Corollary 9 gives an answer to the Ulam's question.

EXAMPLE 10 Let us consider the functionals

$$J(u) = \int_0^1 (u(x)^2 + u'(x)^4) dx$$

and

$$J_h(u) = \int_0^1 (u(x)^2 + (u'(x)^2 - \frac{1}{h})^2) dx,$$

with $u \in W_0^{1,4}(0, 1)$.

It is easy to show that the perturbed functionals J_h do not have a minimizer for any h ; in fact in this case we do not have uniform strict convexity of the perturbed functionals. According to Example 1, this means that even if J is strictly convex and Tykhonov well-posed it is possible to perturb it with perturbations as small as wished and destroy the existence of a minimizer.

Anyway, since J is Tykhonov well-posed and the sequence J_h is convergent with respect to the bounded Hausdorff topology (see Remark 11), every asymptotically minimizing sequence strongly converges.

This shows how the concept of well-posedness under perturbations allow us to avoid the requirement of the existence of the minimizer of the perturbed problems.

4. Dominated convergence theorems

In order to obtain well-posedness results it is useful to obtain criteria which guarantee that a sequence of lower semicontinuous functions converges in the bH-sense to a certain function (see for instance Example 10).

The only known result on this subject is about quadratic functionals of elliptic type and can be found in [5].

In this section we study some classes of integral functionals, and we prove that, under suitable hypotheses, convergence in different senses of the sequence of the integrands is a sufficient condition to assure convergence of the sequence of the associated integrals (with respect to the bounded Hausdorff convergence). The first result is the following remark. The proof is elementary, but we note that sequences of normal integrands satisfying condition (9) are said to be convergent modulo given growth or with respect to the τ_p topology and this family of topologies was introduced in [9] to prove the so called antirelaxation theorem, which establishes the genericity of well-posed problems arising in the calculus of variations with respect to τ_p perturbations of the integrand.

REMARK 11 Fix $p \geq 1$ and let $L_h, L : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be normal integrands.

Assume that for every $\epsilon > 0$ there exist $N_\epsilon \in \mathbb{N}$, $\phi_\epsilon \in L^1(\Omega; \mathbb{R})$ such that $\|\phi_\epsilon\| \leq 1$ and

$$|L_h(x, u, z) - L(x, u, z)| \leq \epsilon(\phi_\epsilon(x) + |u|^p + |z|^p) \quad (9)$$

for every $h > N_\epsilon$, for every $(u, z) \in \mathbb{R}^n \times \mathbb{R}^{nm}$ and for almost every $x \in \Omega$.

Let J, J_h the associated integral functionals on $W^{1,p}(\Omega; \mathbb{R}^m)$. Then $J_h \rightarrow J$ with respect to bH-convergence in $W^{1,p}(\Omega; \mathbb{R}^m)$.

It follows from this Remark that τ_p is the natural convergence to require on the integrands in order to obtain bH-convergence of the associated integrals. Nevertheless, for every $p \geq 1$, uniform convergence modulo given growth is stronger than uniform convergence on bounded subsets, so it would be useful to find a weaker convergence on the integrands still guaranteeing bH-convergence of the integral functionals.

This is the aim of Theorem 11, where we relax the hypotheses on convergence, requiring only pointwise convergence of the integrands, but we pay this weaker requirement imposing a restrictive growth condition (which cannot be weakened), and an equilipschitz property on bounded subsets for the integrands. Moreover we deal with functionals not depending explicitly on u . So in the next theorem, the functionals considered are of the form

$$J(u) := \int_{\Omega} L(x, Du(x)) \, dx.$$

Let us state the assumption that we will make in the following theorem.

- A) Let $L_h : \Omega \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be a sequence of functions. Suppose that for every $R > 0$ there exists $L(\cdot, R) \in L^1(\Omega, \mathbb{R})$ such that

$$|L_h(x, z) - L_h(x, y)| \leq K(x, R)|z - y|$$

for almost every $x \in \mathbb{R}^n$, for every $z, y \in B_R(0)$ and for every $h \in \mathbb{N}$.

We note that this equilipschitz property is always satisfied by a sequence of equibounded finite convex functions.

THEOREM 12 *Let $L, L_h : \Omega \times \mathbb{R}^{nm} \rightarrow [0, +\infty)$ be normal integrands and assume that L, L_h satisfy assumption A). Let $p > 1$ and suppose that there exist $1 < q < p$, $r > \frac{p}{p-q}$ and $\Lambda \in L^r(\Omega, \mathbb{R})$ such that for every $z \in \mathbb{R}^{nm}$ and for almost every $x \in \Omega$.*

$$0 \leq L_h(x, z) \leq \Lambda(x)(|z|^q + 1). \quad (10)$$

If for all $z \in \mathbb{R}^{nm}$ $L_h(\cdot, z) \rightarrow L(\cdot, z)$ almost everywhere, then $J_h \rightarrow J$ with respect to the bounded Hausdorff convergence on $W^{1,p}(\Omega; \mathbb{R}^m)$.

We note that condition (10) cannot be weakened as Example 3.26 in [18] shows.

Other results on this subject and some consequences of Theorem 12 can be found in [15].

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