SHAPE OPTIMIZATION OF CONTACT PROBLEMS WITH SLIP RATE DEPENDENT FRICTION

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Abstract This paper deals with the formulation of a necessary optimality condition for a shape optimization problem of a viscoelastic body in unilateral dynamic contact with a rigid foundation. The contact with Coulomb friction is assumed to occur at a portion of the boundary of the body. The contact condition is described in velocities. The friction coefficient is assumed to be bounded and Lipschitz continuous with respect to a slip velocity. The evolution of the displacement of the viscoelastic body in unilateral contact is governed by a hemivariational inequality of the second order. The shape optimization problem for a viscoelastic body in contact consists in finding, in a contact region, such shape of the boundary of the domain occupied by the body that the normal contact stress is minimized. It is assumed, that the volume of the body is constant. Using material derivative method, we calculate the directional derivative of the cost functional and we formulate a necessary optimality condition for this problem.

keywords: dynamic unilateral problem, shape optimization, necessary optimality condition

1. Introduction

This paper deals with the formulation of a necessary optimality condition for a shape optimization problem of a viscoelastic body in unilateral dynamic contact with a rigid foundation. The contact with a given Coulomb friction [2, 3] is assumed to occur at a portion of the boundary of the body. The contact condition is described in velocities. Usually [2, 3] this contact problem is considered either with a constant or suitable small functional friction coefficient depending on spatial variables. Numerous experiments indicate [5, 7], that in the study of many frictional processes (stick - sleep motions, earthquake modelling, etc.) the friction coefficient has to be considered variable during the slip. The evolution of the body in the unilateral contact is described by a variational inequality of the second order [2, 4]. Under the assumption that the friction coefficient is bounded and Lipschitz continuous with respect to the sliding velocity, the

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existence of solutions to the viscoelastic dynamic contact problems is shown in [3, 5]. For regularity result with respect to time variable see [7].

The shape optimization problem for a viscoelastic body in unilateral contact consists in finding, in a contact region, such shape of the boundary of the domain occupied by the body that the normal contact stress is minimized. It is assumed that the volume of the body is constant. Shape optimization of static contact problems was considered, among others, in [4, 11]. The necessary optimality conditions for shape optimization of dynamic contact problems were formulated in [6, 9].

In this paper we study this shape optimization problem for a viscoelastic body in unilateral dynamical contact with friction coefficient bounded and Lipschitz dependent on a slip rate. Assuming small friction coefficient and suitable regularity of data it can be shown [3] that the solution to dynamic contact problem is enough regular to differentiate it with respect to a parameter. Using material derivative method [11], we calculate the directional derivative of the cost functional and we formulate a necessary optimality condition for this problem. The present paper extends authors results contained in [9].

The following notation will be employed: $\Omega \subset R^2$ will denote the bounded domain with Lipschitz continuous boundary Γ . The time variable will be denoted by t and the time interval $I = (0, T)$, $T > 0$. By $H^k(\Omega)$, $k \in (0, \infty)$, we will denote the Sobolev space of functions having derivatives in all directions of the order k belonging to $L^2(\Omega)$ [1]. For an interval I and a Banach space B , $L^p(I; B)$, $p \in [1, \infty)$, denotes the usual Bochner space [3]. $u_t = du/dt$ and $u_{tt} = d^2u/dt^2$ will denote first and second order derivatives, respectively, with respect to t of function u . u_{tN} and u_{tT} will denote normal and tangential components, respectively, of function u_t . $Q = I \times \Omega$, $\gamma_i = I \times \Gamma_i$, $i = 1, 2, 3$, where Γ_i are pieces of the boundary Γ .

2. Contact problem formulation

Consider deformations of a viscoelastic body occupying domain $\Omega \subset R^2$. The boundary Γ of domain Ω is Lipschitz continuous. The body is subject to body forces $f = (f_1, f_2)$. Moreover surface tractions $p = (p_1, p_2)$ are applied to a portion Γ_1 of the boundary Γ . We assume that the body is clamped along the portion Γ_0 of the boundary Γ and that the contact conditions are prescribed on the portion Γ_2 of the boundary Γ . Moreover $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i, j = 0, 1, 2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

Let us denote by $u = (u_1, u_2)$, $u = u(t, x)$, $x \in \Omega$, $t \in [0, T]$, $T > 0$ the displacement of the body and by $\sigma = \{\sigma_{ij}(u(t, x))\}$, $i, j = 1, 2$, the stress field in the body. We shall consider viscoelastic bodies obeying Kevin - Voigt law [2–4]:

$$\sigma_{ij}(u) = c_{ijkl}^0(x)e_{kl}(u) + c_{ijkl}^1(x)e_{kl}(u_t) \quad x \in \Omega, \quad (1)$$

where $e_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k})$, $i, j, k, l = 1, 2$, $u_{k,l} = \partial u_k / \partial x_l$. The summation convention over repeated indices [2] is used throughout the paper. $c_{ijkl}^0(x)$ and $c_{ijkl}^1(x)$, $i, j, k, l = 1, 2$, are components of elasticity and viscoelasticity tensors. It is assumed that elements c_{ijkl}^0 and c_{ijkl}^1 satisfy usual symmetry, boundedness and ellipticity conditions [2–4]. The displacement field u is a solution of the linear viscoelastic system [2]:

$$u_{tti} - \sigma_{ij}(u(x))_{,j} = f_i(x), \quad (t, x) \in (0, T) \times \Omega, \quad i, j = 1, 2, \quad (2)$$

where $\sigma_{ij}(x)_{,j} = \partial \sigma_{ij}(x) / \partial x_j$, $i, j = 1, 2$. There are given the following boundary conditions for $i, j = 1, 2$:

$$u_i(x) = 0 \quad \text{on } (0, T) \times \Gamma_0, \quad \text{and } \sigma_{ij}(x)n_j = p_i \quad \text{on } (0, T) \times \Gamma_1, \quad (3)$$

as well as the contact and friction conditions on the boundary $(0, T) \times \Gamma_2$,

$$u_{tN} \leq 0, \quad \sigma_N \leq 0, \quad u_{tN}\sigma_N = 0, \quad (4)$$

$$u_{tT} = 0 \quad \Rightarrow \quad |\sigma_T| \leq \mathcal{F} |\sigma_N|, \quad (5)$$

$$u_{tT} \neq 0 \quad \Rightarrow \quad \sigma_T = -\mathcal{F} |\sigma_N| \frac{u_{tT}}{|u_{tT}|}. \quad (6)$$

Here we denote [2]: $u_N = u_i n_i$, $\sigma_N = \sigma_{ij} n_i n_j$, $(u_T)_i = u_i - u_N n_i$, $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, $n = (n_1, n_2)$ is the unit outward vector to the boundary Γ . $\mathcal{F} = \mathcal{F}(x, u_t) = \mathcal{F}(u_t)$ denotes a friction coefficient. The initial conditions are:

$$u_i(0, x) = u_0 \quad u_{ti}(0, x) = u_1, \quad i = 1, 2, \quad x \in \Omega, \quad (7)$$

where u_0 and u_1 are given functions. We shall consider problem (2) – (7) in the variational form. Assume,

$$\begin{aligned} f &\in H^{1/4}(I; (H^1(\Omega; R^2))^*) \cap L^2(Q; R^2), \\ p &\in L^2(I; (H^{1/2}(\Gamma_1; R^2))^*), \\ u_0 &\in H^{3/2}(\Omega; R^2) \quad u_1 \in H^{3/2}(\Omega; R^2), \quad u_1|_{\Gamma_2} = 0, \end{aligned} \quad (8)$$

are given. The space $L^2(Q; R^2)$ and the Sobolev spaces $H^{1/2}(\Gamma_1; R^2)$ as well as $H^{1/4}(I; (H^1(\Omega; R^2))^*)$ are defined in [1]. Note, that from (8) as well as from Sobolev Imbedding Theorem [1] it follows that u_0 and u_1 in (7) are continuous on the boundary of cylinder Q . The friction coefficient $\mathcal{F} : \Gamma_2 \times R_+ \rightarrow R_+$, $\mathcal{F} \in L^\infty(\Gamma_2; R^2)$ has the following properties:

$$|\mathcal{F}(x, u_1) - \mathcal{F}(x, u_2)| \leq L |u_1 - u_2| \quad \forall u_1, u_2 \in [0, +\infty) \quad \text{a.e. } x \in \Gamma_2, \quad (9)$$

$$0 \leq \mathcal{F}(x, u) \leq \mathcal{F}_0 \quad \text{a.e. } x \in \Gamma_2, \quad \forall u \in R_+, \quad (10)$$

and the function $x \rightarrow \mathcal{F}(x, u)$ is measurable for $u \in R_+$. We shall write $\mathcal{F} = \mathcal{F}(u_t) = \mathcal{F}(x, u_t)$. Let us introduce:

$$F = \{z \in L^2(I; H^1(\Omega; R^2)) : z_i = 0 \text{ on } (0, T) \times \Gamma_0, i = 1, 2\}, \quad (11)$$

and the kinematically admissible set

$$K = \{z \in F : z_t = 0 \text{ on } (0, T) \times \Gamma_0, z_{tN} \leq 0 \text{ on } (0, T) \times \Gamma_2\}. \quad (12)$$

The problem (1) - (7) is equivalent to the following variational problem [3]: find a function $u \in L^\infty(I; H^1(\Omega; R^2)) \cap H^{1/2}(I; L^2(\Omega; R^2)) \cap K$ such that $u_t \in L^\infty(I; L^2(\Omega; R^2)) \cap H^{1/2}(I; L^2(\Omega; R^2)) \cap K$ and $u_{tt} \in L^\infty(I; H^{-1}(\Omega; R^2)) \cap (H^{1/2}(I; L^2(\Omega; R^2)))^*$ satisfying the following inequality,

$$\begin{aligned} & \int_Q u_{tti}(v_i - u_{ti}) dx d\tau + \int_Q \sigma_{ij}(u) e_{ij}(v_i - u_{ti}) dx d\tau + \\ & \int_{\gamma_2} \mathcal{F} | \sigma_N(u) | (| v_T | - | u_{tT} |) dx d\tau \geq \int_Q f_i(v_i - u_{ti}) dx d\tau + \quad (13) \\ & \int_{\gamma_1} p_i(v_i - u_{ti}) dx d\tau \quad \forall v \in H^{1/2}(I; H^1(\Omega; R^2)) \cap K. \end{aligned}$$

The existence of solutions to system (1) - (7) was shown in [3, Theorem 5.1.8, p. 286]:

THEOREM 1 Assume : (i) condition (8) holds, (ii) Γ_2 is of class C^β , $\beta > 2$, (iii) tensors $c_{ijkl}^\eta(x)$, $\eta = 0, 1$, are Hölder continuous with respect to x with $\beta' > 1/2$, (iv) the friction coefficient $\mathcal{F}$ is bounded. Then there exists a weak solution to the problem (1) - (7).

Proof. The proof is based on penalization of the inequality (11), friction regularization and employment of localization and shifting technique due to Lions and Magenes. For details of the proof see [3]. $\diamond$

REMARK 2 Assuming that $\mathcal{F}$ satisfies the conditions (9), (10) and using the same arguments as in proof of Theorem 2.1 in [5] one can prove that the solution to (11) is unique.

For the sake of brevity we shall consider the contact problem with the prescribed friction, i.e., we shall assume $g_N \in L^2(I; (H^{1/2}(\Gamma_2))^*)$ is a given normal traction and

$$| \sigma_N | = g_N, \quad \mathcal{F}(u_t) | g_N | = \sigma_T \leq 1. \quad (14)$$

The condition (6) is replaced by the following one,

$$u_{tT} \sigma_T + \mathcal{F} g_N | u_{tT} | = 0, \quad | \sigma_T | \leq 1 \quad \text{on } I \times \Gamma_2. \quad (15)$$

Let us introduce the set,

$$\Lambda = \{\lambda \in L^\infty(I; L^2(\Gamma_2)) : |\lambda| \leq 1 \text{ on } I \times \Gamma_2\}. \quad (16)$$

Taking into account (15) the system (11) takes the form: find $u \in K$ and $\lambda \in \Lambda$ such that,

$$\begin{aligned} & \int_Q u_{tti}(v_i - u_{ti}) dx d\tau + \int_Q \sigma_{ij}(u) e_{ij}(v_i - u_{ti}) dx d\tau - \\ & \int_{\gamma_2} \mathcal{F} g_N \lambda_T (v_T - u_{tT}) dx d\tau \geq \int_Q f_i (v_i - u_{ti}) dx d\tau + \\ & \int_{\gamma_1} p_i (v_i - u_{ti}) dx d\tau \quad \forall v \in H^{1/2}(I; H^1(\Omega; R^2)) \cap K, \end{aligned} \quad (17)$$

$$\int_{\gamma_2} \sigma_T u_{tT} ds d\tau \leq \int_{\gamma_2} \mathcal{F} g_N \lambda_T u_{tT} ds d\tau \quad \forall \lambda_T \in \Lambda. \quad (18)$$

3. Formulation of the shape optimization problem

Consider a family $\{\Omega_s\}$ of the domains Ω_s depending on a parameter s . The domain Ω_s we shall consider as an image of a reference domain Ω under a smooth mapping T_s . To describe the transformation T_s we shall use the speed method [11]. Let us denote by $V(.,.) : [0, \vartheta) \times R^2 \rightarrow R^2$ enough regular vector field depending on the parameter $s \in [0, \vartheta)$, $\vartheta > 0$:

$$V(s, .) \in C^2(R^2, R^2) \quad \forall s \in [0, \vartheta), \quad V(., x) \in C([0, \vartheta), R^2) \quad \forall x \in R^2. \quad (19)$$

Let $T_s(V) : R^2 \ni X \rightarrow x(t, X) \in R^2$ denotes the family of mappings depending on a velocity field V . The vector function $x(., X) = x(., X)$ satisfies the systems of ordinary differential equations:

$$\frac{d}{d\tau} x(\tau, X) = V(\tau, x(\tau, X)), \tau \in [0, \vartheta), \quad x(0, X) = X \in R. \quad (20)$$

The family of domains $\{\Omega_s\}$ depending on parameter $s \in [0, \vartheta)$, $\vartheta > 0$, is defined as follows: $\Omega_0 = \Omega$, and $\Omega_s = T_s(\Omega)(V) = \{x \in R^2 : \exists X \in R^2 \text{ such that, } x = x(s, X), \text{ where the function } x(., X) \text{ satisfies equation (18) for } 0 \leq \tau \leq s\}$.

Consider problem (17) - (18) in the domain Ω_s . Let F_s , K_s , Λ_s be defined, respectively, by (9), (10), (16) with Ω_s rather than Ω . We shall write $u_s = u(\Omega_s)$, $\sigma_s = \sigma(\Omega_s)$. The problem (17) - (18) in the domain Ω_s takes the form: find $u_s \in L^\infty(I; H^1(\Omega_s; R^2)) \cap H^{1/2}(I; L^2(\Omega_s; R^2)) \cap K$ such that $u_{ts} \in L^\infty(I; L^2(\Omega_s; R^2)) \cap H^{1/2}(I; L^2(\Omega_s; R^2)) \cap K$ and $u_{tts} \in$

$L^\infty(I; H^{-1}(\Omega_s; R^2)) \cap (H^{1/2}(I; L^2(\Omega_s; R^2)))^*$ and $\lambda_s \in \Lambda_s$ such that,

$$\begin{aligned} & \int_{Q_s} u_{ttsi} v_i dx d\tau + \int_{Q_s} \sigma_{ij}(u_s) e_{ij}(v_i - u_{ttsi}) dx d\tau - \\ & \int_{\gamma_{s2}} \mathcal{F}_s g_N \lambda_s (v_T - u_{tsT}) dx d\tau \geq \int_{Q_s} f_i (v_i - u_{ttsi}) dx d\tau + \end{aligned} \quad (21)$$

$$\int_{\gamma_{s1}} p_i (v_i - u_{ttsi}) dx d\tau \quad \forall v \in H^{1/2}(I; H^1(\Omega_s; R^2)) \cap K,$$

$$\int_{\gamma_{s2}} \sigma_{sT} u_{tsT} ds d\tau \leq \int_{\gamma_{s2}} \mathcal{F}_s g_N \lambda_s u_{tsT} ds d\tau \quad \forall \lambda_s \in \Lambda_s. \quad (22)$$

Let us formulate the optimization problem. By $\hat{\Omega} \subset R^2$ we denote a hold - all domain such that $\Omega_s \subset \hat{\Omega}$ for all $s \in [0, \vartheta]$, $\vartheta > 0$ and by $P_{\hat{\Omega}}(\Omega)$ a finite perimeter [11] of a domain Ω in $\hat{\Omega}$. Let M be an auxiliary set M determined as $M = \{\phi \in L^\infty(I; H_0^2(\hat{\Omega}; R^2)) : \phi \leq 0 \text{ on } I \times \hat{\Omega}, \|\phi\|_{L^\infty(I; H_0^2(\hat{\Omega}; R^2))} \leq 1\}$. Introduce, for a given $\phi \in M$, the following cost functional:

$$J_\phi(u_s) = J_\phi^1(u_s) + P_{\hat{\Omega}}(\Omega), \quad \text{where } J_\phi^1(u_s) = \int_{\gamma_{s2}} \sigma_{sN} \phi_{tNs} dz d\tau, \quad (23)$$

and ϕ_{tNs} and σ_{sN} are normal components of ϕ_{ts} and σ_s , respectively, depending on a parameter s . The cost functional (23) depends on the solution $(u_s, \lambda_s) \in K_s \times \Lambda_s$ of the system (21) - (22).

We shall consider such family of domains $\{\Omega_s\}$, that every Ω_s , $s \in [0, \vartheta]$, $\vartheta > 0$, has a constant volume $c > 0$, i.e., every Ω_s belongs to the constraint set U given by:

$$U = \{\Omega_s : \int_{\Omega_s} dx = c\}. \quad (24)$$

The set U is assumed to be nonempty. The shape optimization problem has the form:

$$\begin{aligned} & \text{For a given } \phi \in M, \text{ find a domain } \Omega_s \text{ occupied by the body,} \\ & \text{minimizing the cost functional (23) subject to } \Omega_s \in U. \end{aligned} \quad (25)$$

The goal of the shape optimization problem (22) is to find such boundary Γ_2 of the domain Ω occupied by the body that the normal contact stress is minimized. It is known [4] that the normal contact stress attains peak in the contact area. This stress can be significantly reduced and uniformly distributed when the bodies in contact have an optimal shape. The finite perimeter term in (23) is added to ensure the existence of solutions to the shape optimization problem (22). This term implies the compactness of the set (24) in L^2 topology of its characteristic functions. For detailed discussion concerning the conditions ensuring the existence of optimal solutions to shape optimization problems see [4, 11].

4. Shape derivatives of contact problem solution

In order to calculate Euler derivative (42) of the cost functional (23) we have to determine material $(\dot{u}, \dot{\lambda}) \in F \times \Lambda$ and shape derivatives $(u', \lambda') \in F \times \Lambda$ of a solution $(u_s, \lambda_s) \in K_s \times \Lambda_s$ of the system (21)–(22). Recall the notion of the material derivative [11]:

DEFINITION 3 *The material derivative $\dot{u} \in F$ of the function $u_s \in F_s$ at a point $X \in \Omega$ is determined by :*

$$\lim_{s \rightarrow 0} \| [(u_s \circ T_s) - u_0]/s - \dot{u} \|_F = 0, \quad (26)$$

where $u_0 \in F$, $u_s \circ T_s \in F$ is an image of function $u_s \in F_s$ in the space F under the mapping T_s .

Recall [11], that if the shape derivative $u' \in F$ of the function $u_s \in F_s$ exists, then it holds:

$$u' \stackrel{\text{def}}{=} \dot{u} - \nabla u V(0), \quad (27)$$

where $\dot{u} \in F$ is material derivative of the function $u_s \in F_s$. Taking into account Definition 3 we can calculate the material derivative of a solution to the system (21), (22):

LEMMA 4 *The material derivatives $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$ of a solution $(u_s, \lambda_s) \in K_s \times \Lambda_s$ to the system (21) – (22) are determined as a unique solution to the following system:*

$$\begin{aligned} & \int_Q \{ (\dot{u}_{tt}\eta + u_{tt}\dot{\eta} + u_{tt}\eta \operatorname{div} V(0) + (DV(0)u)_{tt}\eta + u_{tt}(DV(0)\eta) \\ & \quad - \dot{f}\eta - f\dot{\eta} + (\sigma_{ij}(u)e_{kl}(\eta) - f\eta) \operatorname{div} V(0) \} dx d\tau - \\ & \int_{\gamma_1} (\dot{p}\eta + p\dot{\eta} + p\eta D) dx d\tau - \int_{\gamma_2} \{ (\frac{\partial \mathcal{F}}{\partial u_t} \dot{u}_t g_N \lambda \eta_{tT} + \mathcal{F} g_N \dot{\lambda} \eta_{tT}) + \\ & \quad (\mathcal{F} g_N \lambda \dot{\eta}_{tT} + \mathcal{F} g_N \lambda \eta_{tT} D) \} dx d\tau \geq 0 \quad \forall \eta \in K_1, \end{aligned} \quad (28)$$

$$\begin{aligned} & \int_{\gamma_2} \{ (\frac{\partial \mathcal{F}}{\partial u_t} \dot{u}_t g_N \lambda + \mathcal{F} g_N \dot{\lambda} - \mu) u_{tT} + (\mathcal{F} g_N \lambda - \dot{\mu}) u_{tT} + \\ & \quad (\mathcal{F} g_N \lambda - \mu) \dot{u}_{tT} + \mathcal{F} \lambda u_{tT} D \} g_N dx d\tau \quad \forall \mu \in L_1, \end{aligned} \quad (29)$$

where $V(0) = V(0, X)$, $DV(0)$ denotes the Jacobian of the matrix $V(0)$ and D is the expression

$$D = \operatorname{div} V(0) - (DV(0)n, n). \quad (30)$$

Moreover the sets K_1 and L_1 are determined by

$$\begin{aligned} K_1 = \{ \xi \in F : \xi = u - DVu \text{ on } \gamma_0, \xi n \geq n DV(0)u \text{ on } A_1, \\ \xi n = n DV(0)u \text{ on } A_2 \}, \end{aligned} \quad (31)$$

$$L_1 = \{\xi \in \Lambda : \xi \geq 0 \text{ on } B_2, \xi \leq 0 \text{ on } B_1, \xi = 0 \text{ on } B_0\}, \quad (32)$$

where the sets $A_i, B_i, i = 0, 1, 2$ are given by: $A_0 = \{x \in \gamma_2 : u_{tN} = 0\}$, $A_1 = \{x \in A_0 : \sigma_N = 0\}$, $A_2 = \{x \in A_0 : \sigma_N < 0\}$, $B_0 = \{x \in \gamma_2 : \lambda_T = 1, u_{tT} \neq 0\}$, $B_1 = \{x \in \gamma_2 : \lambda_T = -1, u_{tT} = 0\}$, $B_2 = \{x \in \gamma_2 : \lambda_T = 1, u_{tT} = 0\}$.

Proof: is based on approach proposed in [11]. First we transport the system (21) – (22) to the fixed domain Ω . Let $u^s = u_s \circ T_s \in F$, $u = u_0 \in F$, $\lambda^s = \lambda_s \circ T_s \in \Lambda$, $\lambda = \lambda_0 \in \Lambda$. Since in general $u^s \notin K(\Omega)$ we introduce a new variable $z^s = DT_s^{-1}u^s \in K$. Moreover $\dot{z} = \dot{u} - DV(0)u$ [11]. Using this new variable z^s as well as the formulae for transformation of the function and its gradient into reference domain Ω [11] we write the system (21) – (22) in the reference domain Ω . Using the estimates on time derivative of function u [11] the Lipschitz continuity of u and λ satisfying (21) – (22) with respect to s can be proved. Applying to this system the result concerning the differentiability of solutions to variational inequality [11] we obtain that the material derivative $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$ satisfies the system (28) – (29). Moreover from the ellipticity condition of the elasticity coefficients by a standard argument [11] it follows that $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$ is a unique solution to the system (28) – (29). $\diamond$

Assume

$$\nabla u V(0) \in F, \quad \nabla \lambda_T V(0) \in \Lambda, \quad (33)$$

where the spaces F and Λ are determined by (9) and (16) respectively. Integrating by parts system (28), (29) and taking into account (27), (33) we obtain the similar system to (28) – (29) determining the shape derivative $(u', \lambda'_T) \in F \times L$ of the solution $(u_s, \lambda_{sT}) \in K_s \times L_s$ of the system (21) – (22):

$$\begin{aligned} & \int_Q [u'_{tt}\eta + u_{tt}\eta' + (DV(0) + {}^*DV(0))u_{tt}\eta] dx d\tau + \\ & \int_\gamma u_{tt}\eta V(0) n dx d\tau + \int_Q \sigma_{ij}(u') e_{kl}\eta dx d\tau - \\ & \int_{\gamma_2} \left\{ \frac{\partial \mathcal{F}}{\partial u_t} u'_t g_n \lambda \eta_{tT} + \mathcal{F} g_N \lambda' \eta_{tT} + \mathcal{F} g_N \lambda \eta'_{tT} \right\} dx d\tau + \\ & I_1(u_t, \eta) + I_2(\lambda, u, \eta) \geq 0 \quad \forall \eta \in N_1, \end{aligned} \quad (34)$$

$$\int_{\gamma_2} \left[\frac{\partial \mathcal{F}}{\partial u_t} u'_t g_n \lambda u_{tT} u'_{tT} (\mu - \mathcal{F} g_N \lambda) - u_{tT} \mathcal{F} g_N \lambda' \right] dx d\tau + \quad (35)$$

$$I_3(u, \mu - \lambda) \geq 0 \quad \forall \mu \in L_1, \quad (36)$$

$$N_1 = \{\eta \in F : \eta = \lambda - DuV(0), \lambda \in K_1\}, \quad (37)$$

$$I_1(\varphi, \phi) = \int_\gamma \{\sigma_{ij}(\varphi) e_{kl} \phi - f \phi -$$

$$((\nabla p n)\phi + (p\nabla\phi)n + p\phi\kappa)V(0)n\}dx d\tau, \quad (38)$$

$$I_2(\mu, \varphi, \phi) = \int_{\gamma_2} \left\{ \frac{\partial \mathcal{F}}{\partial \varphi_t} \nabla \varphi_t g_n \mu \varphi_{tT} + (\nabla \mu \mathcal{F} g_N) n \nabla \phi + \mathcal{F} g_N \mu (\nabla(\nabla \varphi n)) \varphi + \mathcal{F} g_N \mu \nabla \varphi_{tT} H + \mathcal{F} g_N \mu \nabla \varphi n \right\} V(0) n dx d\tau, \quad (39)$$

$$I_3(\varphi, \mu - \lambda) = \int_{\gamma_2} \left[\frac{\partial \mathcal{F}}{\partial \varphi_t} \nabla \varphi_t g_n \mu + (\varphi n)(\mu - \mathcal{F} g_N \lambda) + \varphi(\nabla \mu n) - \varphi(\nabla \mathcal{F} g_N \lambda n) + \varphi(\mu - \mathcal{F} g_N \lambda) \kappa \right] V(0) n dx d\tau. \quad (40)$$

where κ denotes a mean curvature of the boundary Γ [11].

5. Necessary optimality condition

Our goal is to calculate the directional derivative of the cost functional (23) with respect to the parameter s . We will use this derivative to formulate a necessary optimality condition for the optimization problem (22). First, recall from [11] the notion of Euler derivative of the cost functional depending on a domain Ω :

DEFINITION 5 *Euler derivative $dJ(\Omega; V)$ of the cost functional J at a point Ω in the direction of the vector field V is given by:*

$$dJ(\Omega; V) = \limsup_{s \rightarrow 0} [J(\Omega_s) - J(\Omega)]/s. \quad (41)$$

By direct application of this Definition we get the form of the directional derivative $dJ_\phi(u; V)$ of the cost functional (23):

LEMMA 6 *The directional derivative $dJ_\phi(u; V)$ of the cost functional (23), for $\phi \in M$ given, at a point $u \in K$ in the direction of vector field V is determined by:*

$$\begin{aligned} dJ_\phi(u; V) = & \int_Q [u'_{tt}\eta + u_{tt}\eta' + (DV(0) + {}^*DV(0))u_{tt}\eta] dx d\tau + \\ & \int_\gamma u_{tt}\eta V(0) n dx d\tau + \int_Q (\sigma'_{ij} e_{kl}(\phi)) dx d\tau + \\ & \int_\Gamma (\sigma_{ij} e_{kl}(\phi) - f\phi) V(0) n ds - \int_{\Gamma_1} [(\nabla p \phi V(0) + \\ & p \nabla \phi V(0) + p\phi D)] ds - \int_{\Gamma_2} \lambda' \phi_T ds + I_1(u, \phi) - I_2(\lambda, u, \phi), \end{aligned} \quad (42)$$

where σ' is a shape derivative of the function σ_s with respect to s . ∇p is a gradient of function p with respect to x . Moreover $V(0) = V(0, X)$, ϕ_T and

σ_T are tangent components of ϕ and σ , respectively, as well as D is given by (30). $DV(0)$ denotes the Jacobian of the matrix $V(0)$.

In order to eliminate the shape derivative (u', λ') from (42) we introduce an adjoint state $(r, q) \in K_2 \times L_2$ defined as follows:

$$\begin{aligned} & \int_Q r_{tt} \zeta dx d\tau + \int_Q \sigma_{ij}(\zeta) e_{kl}(\phi + r) dx d\tau + \\ & \int_{\gamma_2} \frac{\partial \mathcal{F}}{\partial u_t} g_N r_{tT} (q - \lambda) \zeta dx d\tau = 0, \quad \forall \zeta \in K_2, \end{aligned} \quad (43)$$

with

$$r(\mathcal{T}, x) = 0, \quad r_t(\mathcal{T}, x) = 0,$$

$$\int_{\gamma_2} \mathcal{F} g_N (r_{tT} + \phi_{tT} - u_{tT}) \delta dx d\tau = 0, \quad \forall \delta \in L_2, \quad (44)$$

where $K_2 = \{\zeta \in K_1 : \zeta n = 0 \text{ on } A_0\}$ and $L_2 = \{\delta \in \Lambda : \delta = 0 \text{ on } A_0 \cap B_0\}$. Since $\phi \in M$ is a given element, then by the same arguments as used to show the existence of the solution $(u, \lambda) \in K \times L$ to the system (21)-(22) we can show the existence of the solution $(r, q) \in K_2 \times L_2$ to the system (43) - (44). From (42), (34), (35), (43), (44) we obtain:

$$dJ_\phi(u; V) = I_1(u, \phi + r) + I_2(\lambda, u, \phi + r) + I_3(u, q - \lambda). \quad (45)$$

The necessary optimality condition has a standard form [2, 9, 11]:

THEOREM 7 *Let $\Omega^* \subset U$ be an optimal solution to the problem (22). Then there exists Lagrange multiplier $\mu \in R$ such that for all vector fields V determined by (17), (18) the following condition holds:*

$$dJ_\phi(u^*; V) + \mu \int_{\Gamma^*} V(0) n ds + dP_{\hat{\Omega}}(\Omega^*; V) \geq 0, \quad (46)$$

where $dJ_\phi(u^*; V)$ is given by (45), Γ^* denotes the boundary of the optimal domain Ω^* , u^* is a solution to the state system (17) - (18) in Ω^* , and $dP_{\hat{\Omega}}(\Omega^*; V)$ denotes Euler derivative of $P_{\hat{\Omega}}(\Omega)$ at Ω^* in a direction V .

6. Conclusions

The directional derivative of the cost functional with respect to the perturbation of the domain occupied by a viscoelastic body in unilateral contact with a rigid foundation was calculated and the necessary optimality condition for the shape optimization problem for the dynamical contact problem was formulated. This shape optimization problem has been numerically solved. Preliminary numerical results can be found in [10].

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