CHEAP CONTROL PROBLEM OF LINEAR SYSTEMS WITH DELAYS: A SINGULAR PERTURBATION APPROACH

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Abstract A quadratic cheap control of linear systems with multiple state delays is considered. This optimal control problem is transformed to an optimal control problem of singularly perturbed systems. A composite suboptimal control of the latter is designed based on its asymptotic decomposition into two much simpler parameter-free subproblems, the slow and fast ones. Using this composite control, a suboptimal control of the original cheap control problem is constructed and justified for two classes of the initial function for the state variable. An illustrative example is presented.

keywords: time-delayed system, cheap control, singular perturbation, composite control.

1. Introduction

The cheap control problem, i.e. an optimal control problem with a small control cost (with respect to a state cost) in the cost functional, is of considerable importance in such topics of control theory as: singular (degenerate) optimal control and its regularization, limitations of linear and nonlinear regulators, high gain control, inverse control problems, linear optimal filtering with a small noise in the observation, robust controllability of systems with disturbances, and some others.

The smallness of the control cost yields the singular perturbation in the Hamilton-Jacobi-Bellman equation, as well as in the Hamilton boundary-value problem, associated with the original problem by control optimality conditions.

The cheap control problem for differential equations without delays has been extensively investigated in the open literature (see e.g. [1-3] and references therein), while there are only few works in the open literature devoted to analysis of the cheap control problem with a delayed dynamics (see [4-6]).

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In [4], a finite horizon linear-quadratic optimal control problem was considered with a small cost of the control in the cost functional, and with single point-wise and distributed state delays in the dynamics. It was assumed that the delay is small of order of the control cost, and that the initial function for the state variable is continuous. The zero-order asymptotic solution to the singularly perturbed set of ordinary and partial functional-differential equations of Riccati type, associated with the original problem by the control optimality conditions, has been derived. Based on this solution, the zero-order asymptotic expansion of the optimal trajectory, as well as the suboptimal feedback control, were obtained. In [5,6], a similar cheap control problem was studied for the case of a nonsmall delay. The zero-order asymptotic solution to the singularly perturbed set of Riccati-type functional-differential equations, associated with the original problem, has been obtained in [5]. Using this asymptotic solution, a suboptimal control was constructed. In [6], based on a proper transformation of the original problem and, then, on an asymptotic decomposition of the resulting one into two much simpler parameter-free subproblems, a suboptimal state-feedback control of the original problem was designed.

In this paper, a finite-horizon linear-quadratic cheap control problem with multiple point-wise and distributed state delays in the dynamics is analyzed. Two classes of the initial function for the state variable are considered: (1) measurable square-integrable functions; (2) measurable essentially bounded functions. A direct method of suboptimal solution of the original cheap control problem is proposed. This method is based on: (i) an equivalent transformation of the original problem to a control problem with singularly perturbed dynamics; (ii) an asymptotic decomposition of the resulting problem into two much simpler parameter-free subproblems, the slow and fast ones. It should be noted that the fast state variable of the control problem, obtained after the transformation, becomes a control in the slow subproblem. Thus, the slow subproblem contains not only the state delays in the dynamics, but also the control ones. In spite of the latter, the set of Riccati-type functional-differential equations, associated with the slow subproblem, is much simpler than the one associated with the original cheap control problem. The fast subproblem does not contain delays, and it is solved analytically. Using the optimal feedback controls of the slow and fast subproblems, a composite control for the transformed problem and, then, a suboptimal state-feedback control for the original problem are designed.

The following main notations are applied in the paper: (1) E^n is the n -dimensional real Euclidean space; (2) $\|\cdot\|$ denotes the Euclidean norm either of a vector or of a matrix; (3) the prime denotes the transposition of a matrix A , (A') or of a vector x , (x') ; (4) $L^2[b, c; E^n]$ is the space of n -dimensional vector-valued functions $x(t)$ defined, measurable and square-integrable on the interval $[b, c)$, $\|x(\cdot)\|_{L^2} = \left(\int_b^c x'(t)x(t)dt \right)^{1/2}$ denotes the norm in this space;

(5) $L^\infty[b, c; E^n]$ is the space of n -dimensional vector-valued functions $x(t)$ defined, measurable and essentially bounded on the interval $[b, c]$, $\|x(\cdot)\|_{L^\infty} = \text{ess sup}_{t \in [b, c]} \|x(t)\|$ denotes the norm in this space; (6) I_n is the n -dimensional identity matrix.

2. Problem statement

Consider the controlled system

$$dz(t)/dt = \sum_{i=0}^N A_i z(t-h_i) + \int_{-h}^0 G(\tau) z(t+\tau) d\tau + Bu(t), \quad t \in [0, T], \quad (2.1)$$

where $z(t) \in E^n$, $u(t) \in E^r$, ($n \geq r$), (u is a control); $N > 0$ is an integer; $0 = h_0 < h_1 < \dots < h_N = h$ are given constant time-delays; A_i , ($i = 0, 1, \dots, N$), $G(\tau)$ and B are given time-invariant matrices of corresponding dimensions; B has full rank r ; the matrix-valued function $G(\tau)$ is piece-wise continuous for $\tau \in [-h, 0]$; $T > 0$ is a given finite duration of the control process.

Using that $\text{rank } B = r$ and [7], one can transform (2.1) to an equivalent linear controlled system with state delays, in which the matrix of coefficients for the control has the form $\begin{pmatrix} 0 \\ I_r \end{pmatrix}$. Therefore, in the sequel we assume (without a loss of generality) that B has such a form.

Initial conditions for (2.1) have the form

$$z(\tau) = \varphi(\tau), \quad \tau \in [-h, 0]; \quad z(0) = \varphi_0, \quad (2.2)$$

where $\varphi(\tau)$, $\tau \in [-h, 0]$ and φ_0 are given vector-valued function and vector, respectively. In the sequel, the following two cases are considered: (1) $\varphi(\cdot) \in L^2[-h, 0; E^n]$; (2) $\varphi(\cdot) \in L^\infty[-h, 0; E^n]$.

Let partition $z(t)$, A_i , ($i = 0, 1, \dots, N$) and $G(\tau)$ in the accordance with the block-form of B

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A_i = \begin{pmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{pmatrix}, \quad G(\tau) = \begin{pmatrix} G_1(\tau) & G_2(\tau) \\ G_3(\tau) & G_4(\tau) \end{pmatrix}, \quad (2.3)$$

where $x(t) \in E^{n-r}$, $y(t) \in E^r$; A_{i1} and $G_1(\tau)$ are of the dimension $(n-r) \times (n-r)$, while A_{i4} and $G_4(\tau)$ are of the dimension $r \times r$.

Using (2.3) and the block form of B , one can rewrite (2.1) as follows

$$\begin{aligned} dx(t)/dt &= \sum_{i=0}^N [A_{i1}x(t-h_i) + A_{i2}y(t-h_i)] \\ &+ \int_{-h}^0 [G_1(\tau)x(t+\tau) + G_2(\tau)y(t+\tau)] d\tau, \end{aligned} \quad (2.4)$$

$$dy(t)/dt = \sum_{i=0}^N [A_{i3}x(t-h_i) + A_{i4}y(t-h_i)] + \int_{-h}^0 [G_3(\tau)x(t+\tau) + G_4(\tau)y(t+\tau)] d\tau + u(t), \quad t \in [0, T]. \quad (2.5)$$

For system (2.4)-(2.5) with initial conditions (2.2) the following performance index is considered

$$J_\varepsilon(u) \triangleq \int_0^T [x'(t)D_x x(t) + y'(t)D_y y(t) + \varepsilon^2 u'(t)Mu(t)] dt \rightarrow \min_u, \quad (2.6)$$

where D_x is a symmetric positive semi-definite matrix, while D_y and M are symmetric positive definite ones; ε is a small positive parameter.

In the sequel, (2.4)-(2.6), (2.2) is called the original problem (OP).

The objective of the paper is to construct a suboptimal state-feedback control for (2.4)-(2.6), (2.2) uniformly valid for all sufficiently small $\varepsilon > 0$.

3. Control optimality conditions for the OP

Using results of [8] yields that, for a given $\varepsilon > 0$, the optimal state-feedback control and the optimal value of the cost functional of the OP have the form

$$u_\varepsilon^*[z(t), z_h(t), t] = -\varepsilon^{-2} M^{-1} B' \left[P(t)z(t) + \int_{-h}^0 Q(t, \tau)z(t+\tau) d\tau \right], \quad (3.1)$$

$$J_\varepsilon^* = \varphi_0' P(0) \varphi_0 + 2\varphi_0' \int_{-h}^0 Q(0, \tau) \varphi(\tau) d\tau + \int_{-h}^0 \int_{-h}^0 \varphi'(\tau) R(0, \tau, \theta) \varphi(\theta) d\tau d\theta, \quad (3.2)$$

where $z_h(t) = \{z(t+\tau) \mid \forall \tau \in [-h, 0]\}$, and $\{P(t), Q(t, \tau), R(t, \tau, \theta)\}$ is the unique solution of the following set of Riccati-type functional-differential equations in the domain $\Omega = \{(t, \tau, \theta) : t \in [0, T], \tau \in [-h, 0], \theta \in [-h, 0]\}$

$$dP(t)/dt = -P(t)A_0 - A_0'P(t) - Q(t, 0) - Q'(t, 0) + P(t)S_\varepsilon P(t) - D, \quad (3.3)$$

$$(\partial/\partial t - \partial/\partial \tau)Q(t, \tau) = -A_0'Q(t, \tau) - P(t) \left[\sum_{i=1}^{N-1} A_i \delta(\tau + h_i) + G(\tau) \right] - R(t, 0, \tau) + P(t)S_\varepsilon Q(t, \tau), \quad (3.4)$$

$$(\partial/\partial t - \partial/\partial \tau - \partial/\partial \theta)R(t, \tau, \theta) = - \left[\sum_{i=1}^{N-1} A_i' \delta(\tau + h_i) + G'(\tau) \right] Q(t, \theta) - Q'(t, \tau) \left[\sum_{i=1}^{N-1} A_i \delta(\theta + h_i) + G(\theta) \right] + Q'(t, \tau)S_\varepsilon Q(t, \theta), \quad (3.5)$$

$$P(T) = 0, \quad Q(T, \tau) = 0, \quad R(T, \tau, \theta) = 0, \quad (3.6)$$

$$Q(t, -h) = P(t)A_N, \quad R(t, -h, \theta) = A_N'Q(t, \theta), \quad R(t, \tau, -h) = Q'(t, \tau)A_N. \quad (3.7)$$

Here, $S_\varepsilon = \varepsilon^{-2}BM^{-1}B'$, $D = \text{diag}(D_x, D_y)$, $\delta(\cdot)$ is the delta-function.

For a given $\varepsilon > 0$, problem (3.3)-(3.7) is very complicated. Moreover, if $\varepsilon \rightarrow 0$, it becomes an ill-posed one and, therefore, much more difficult.

In this paper, an approach to constructing a suboptimal state-feedback control for this problem is proposed. This approach directly uses the singular perturbation nature of the OP, and it does not require to solve problem (3.3)-(3.7).

The suboptimal control is constructed in four stages. At the first stage, the OP is transformed equivalently to a control problem with singularly perturbed dynamics and ε -free cost functional. At the second stage, the resulting (transformed) problem is decomposed asymptotically into two much simpler ε -free subproblems, the slow and fast ones. At the third stage, based on the optimal feedback controls of the slow and fast subproblems, a composite state-feedback control for the transformed problem is designed. Finally, at the fourth stage, a suboptimal control for the original problem is obtained based on the composite control of the transformed system.

4. Transformation of the OP

By the control transformation

$$u(t) = (1/\varepsilon)v(t), \quad (4.1)$$

where v is a new control, (2.4)-(2.6) becomes

$$\begin{aligned} dx(t)/dt = & \sum_{i=0}^N [A_{i1}x(t-h_i) + A_{i2}y(t-h_i)] \\ & + \int_{-h}^0 [G_1(\tau)x(t+\tau) + G_2(\tau)y(t+\tau)] d\tau, \end{aligned} \quad (4.2)$$

$$\begin{aligned} \varepsilon dy(t)/dt = & \varepsilon \left\{ \sum_{i=0}^N [A_{i3}x(t-h_i) + A_{i4}y(t-h_i)] \right. \\ & \left. + \int_{-h}^0 [G_3(\tau)x(t+\tau) + G_4(\tau)y(t+\tau)] d\tau \right\} + v(t), \quad t \in [0, T], \end{aligned} \quad (4.3)$$

$$J(v) \triangleq \int_0^T [x'(t)D_x x(t) + y'(t)D_y y(t) + v'(t)Mv(t)] dt \rightarrow \min_v. \quad (4.4)$$

In the sequel, (4.2)-(4.4), (2.2) is called the transformed problem (TP).

5. Asymptotic decomposition of the TP

5.1 Slow subproblem

The slow subproblem is obtained from the TP by setting there formally $\varepsilon = 0$ and redenoting x, y, v and J by x_s, y_s, v_s and J_s , respectively. Thus, one obtains

$$dx_s(t)/dt = \sum_{i=0}^N [A_{i1}x_s(t-h_i) + A_{i2}y_s(t-h_i)] \\ + \int_{-h}^0 [G_1(\tau)x_s(t+\tau) + G_2(\tau)y_s(t+\tau)]d\tau, \quad t \in [0, T], \quad (5.1)$$

$$v_s(t) = 0, \quad t \in [0, T], \quad (5.2)$$

$$J_s(y_s) = \int_0^T [x'_s(t)D_x x_s(t) + y'_s(t)D_y y_s(t)]dt \rightarrow \min_{y_s}, \quad (5.3)$$

$$x_s(\tau) = \varphi_x(\tau), \quad y_s(\tau) = \varphi_y(\tau), \quad \tau \in [-h, 0]; \quad x_s(0) = \varphi_{0x}, \quad (5.4)$$

where $\varphi_x(\tau)$ and $\varphi_y(\tau)$ are the upper and lower blocks of $\varphi(\tau)$ of the dimensions $n-r$ and r , respectively; φ_{0x} is the upper block of φ_0 of the dimension $n-r$.

The slow subproblem consists of equation (5.2) and the problem of minimizing the cost functional $J_s(y_s)$ along trajectories of (5.1), (5.4). This problem is called in the sequel the reduced-order problem (ROP). In the ROP, y_s is a control, i.e. this problem has time-delays in state and control.

Due to [8], the optimal feedback control of the ROP exists, is unique and has the form

$$y_s^*[x_s(t), x_{sh}(t), y_{sh}(t), t] = -D_y^{-1} \{ [A'_{02}P_s(t) + Q'_{s2}(t, 0)]x_s(t) \\ + \int_{-h}^0 [A'_{02}Q_{s1}(t, \tau) + R'_{s1}(t, \tau, 0)]x_s(t+\tau)d\tau \\ + \int_{-h}^0 [A'_{02}Q_{s2}(t, \tau) + R_{s2}(t, 0, \tau)]y_s(t+\tau)d\tau \}, \quad (5.5)$$

where $x_{sh}(t) = \{x_s(t+\tau) \forall \tau \in [-h, 0)\}$, $y_{sh}(t) = \{y_s(t+\tau) \forall \tau \in [-h, 0)\}$, and $P_s(t), Q_{sk}(t, \tau), R_{sk}(t, \tau, \theta)$, ($k = 1, 2$) and a matrix $R_{s0}(t, \tau, \theta)$ form the unique solution to the following set of Riccati-type functional-differential equations in Ω

$$dP_s(t)/dt = -P_s(t)A_{01} - A'_{01}P_s(t) - Q_{s1}(t, 0) - Q'_{s1}(t, 0) - D_x \\ + [P_s(t)A_{02} + Q_{s2}(t, 0)]D_y^{-1}[P_s(t)A_{02} + Q_{s2}(t, 0)]', \quad (5.6)$$

$$(\partial/\partial t - \partial/\partial \tau)Q_{s1}(t, \tau) = -A'_{01}Q_{s1}(t, \tau) - P_s \left[\sum_{i=1}^{N-1} A_{i1}\delta(\tau+h_i) + G_1(\tau) \right]$$

$$-R_{s0}(t, 0, \tau) + [P_s(t)A_{02} + Q_{s2}(t, 0)]D_y^{-1}[A'_{02}Q_{s1}(t, \tau) + R'_{s1}(t, \tau, 0)], \quad (5.7)$$

$$(\partial/\partial t - \partial/\partial \tau)Q_{s2}(t, \tau) = -A'_{01}Q_{s2}(t, \tau) - P_s \left[\sum_{i=1}^{N-1} A_{i2}\delta(\tau + h_i) + G_2(\tau) \right]$$

$$-R_{s1}(t, 0, \tau) + [P_s(t)A_{02} + Q_{s2}(t, 0)]D_y^{-1}[A'_{02}Q_{s2}(t, \tau) + R_{s2}(t, 0, \tau)], \quad (5.8)$$

$$(\partial/\partial t - \partial/\partial \tau - \partial/\partial \theta)R_{s0}(t, \tau, \theta) = - \left[\sum_{i=1}^{N-1} A'_{i1}\delta(\tau + h_i) + G'_1(\tau) \right] Q_{s1}(t, \theta)$$

$$-Q'_{s1}(t, \tau) \left[\sum_{i=1}^{N-1} A_{i1}\delta(\theta + h_i) + G_1(\theta) \right]$$

$$+[Q'_{s1}(t, \tau)A_{02} + R_{s1}(t, \tau, 0)]D_y^{-1}[A'_{02}Q_{s1}(t, \theta) + R'_{s1}(t, \theta, 0)], \quad (5.9)$$

$$(\partial/\partial t - \partial/\partial \tau - \partial/\partial \theta)R_{s1}(t, \tau, \theta) = - \left[\sum_{i=1}^{N-1} A'_{i1}\delta(\tau + h_i) + G'_1(\tau) \right] Q_{s2}(t, \theta)$$

$$-Q'_{s1}(t, \tau) \left[\sum_{i=1}^{N-1} A_{i2}\delta(\theta + h_i) + G_2(\theta) \right]$$

$$+[Q'_{s1}(t, \tau)A_{02} + R_{s1}(t, \tau, 0)]D_y^{-1}[A'_{02}Q_{s2}(t, \theta) + R_{s2}(t, 0, \theta)], \quad (5.10)$$

$$(\partial/\partial t - \partial/\partial \tau - \partial/\partial \theta)R_{s2}(t, \tau, \theta) = - \left[\sum_{i=1}^{N-1} A'_{i2}\delta(\tau + h_i) + G'_2(\tau) \right] Q_{s2}(t, \theta)$$

$$-Q'_{s2}(t, \tau) \left[\sum_{i=1}^{N-1} A_{i2}\delta(\theta + h_i) + G_2(\theta) \right]$$

$$+[Q'_{s2}(t, \tau)A_{02} + R_{s2}(t, \tau, 0)]D_y^{-1}[A'_{02}Q_{s2}(t, \theta) + R_{s2}(t, 0, \theta)], \quad (5.11)$$

$$P_s(T) = 0, \quad Q_{sk}(T, \tau) = 0, \quad R_{sj}(T, \tau, \theta), \quad (k = 1, 2; j = 0, 1, 2), \quad (5.12)$$

$$Q_{sk}(t, -h) = P_s(t)A_{Nk}, \quad (k = 1, 2), \quad (5.13)$$

$$R_{s0}(t, -h, \tau) = A'_{N1}Q_{s1}(t, \tau), \quad R_{s0}(t, \tau, -h) = Q'_{s1}(t, \tau)A_{N1}, \quad (5.14)$$

$$R_{s1}(t, -h, \tau) = A'_{N1}Q_{s2}(t, \tau), \quad R_{s1}(t, \tau, -h) = Q'_{s1}(t, \tau)A_{N2}, \quad (5.15)$$

$$R_{s2}(t, -h, \tau) = A'_{N2}Q_{s2}(t, \tau), \quad R_{s2}(t, \tau, -h) = Q'_{s2}(t, \tau)A_{N2}. \quad (5.16)$$

REMARK 1 Although set of equations (5.6)-(5.16), associated with the ROP, does not look like a simple one, it is essentially simpler than set (3.3)-(3.7), associated with the OP. The number of unknown scalar functions in the first set $[2n^2 + n - 2nr + r(r-1)/2]$ is much less than the one $(2n^2 + n)$ in the second set. Moreover, the first set is ε -free and, therefore, it is well-posed.

5.2 Fast subproblem

The fast subproblem is obtained as follows: (1) the slow variable $x(\cdot)$ is removed from equation (4.3) and performance index (4.4) of the TP; (2) the transformation of variables $t = \varepsilon\xi$, $y(\varepsilon\xi) = y_f(\xi)$, $v(\varepsilon\xi) = v_f(\xi)$, $J(v(\varepsilon\xi)) = \varepsilon J_f(v_f(\xi))$ is made in the resulting problem, where ξ , y_f , v_f and J_f are new independent variable, state, control and cost functional, respectively. As a result, one obtains the problem

$$dy_f(\xi)/d\xi = \varepsilon \left[\sum_{i=0}^N A_{i4} y_f(\xi - h_i/\varepsilon) + \int_{-h}^0 G_4(\tau) y_f(\xi + \tau/\varepsilon) d\tau \right] + v_f(\xi), \quad (5.17)$$

$$J_f(v_f) = \int_0^{T/\varepsilon} [y_f'(\xi) D_y y_f(\xi) + v_f'(\xi) M v_f(\xi)] d\xi \rightarrow \min_{v_f}. \quad (5.18)$$

Now, neglecting formally the term with the multiplier ε in (5.17) and replacing T/ε by $+\infty$ in (5.18) yield the fast subproblem

$$dy_f(\xi)/d\xi = v_f(\xi), \quad (5.19)$$

$$J_f(v_f) = \int_0^{+\infty} [y_f'(\xi) D_y y_f(\xi) + v_f'(\xi) M v_f(\xi)] d\xi \rightarrow \min_{v_f}. \quad (5.20)$$

Due to [9], the fast subproblem with a given initial value $y_f(0)$ of the state variable has the unique optimal state-feedback control

$$v_f^*[y_f(\xi)] = -M^{-1} P_f y_f(\xi), \quad (5.21)$$

where P_f is the unique symmetric positive definite solution of the algebraic Riccati equation

$$P_f M^{-1} P_f - D_y = 0. \quad (5.22)$$

Moreover, the optimal trajectory $y_f(\xi)$ satisfies the inequality

$$\|y_f(\xi)\| \leq a \exp(-\beta\xi) \|y_f(0)\|, \quad \xi \geq 0, \quad (5.23)$$

where $a > 0$ and $\beta > 0$ are some constants.

6. Composite control for the TP

The design of composite control for problem (4.2)-(4.4), (2.2) consists of two stages. At the first stage, the auxiliary control is constructed

$$v_a[x(t), y(t), x_h(t), y_h(t), t] = v_s(t) + v_f^*[\tilde{y}(t/\varepsilon)], \quad (6.1)$$

where $\tilde{y}(t/\varepsilon)$ is defined as follows

$$\tilde{y}(t/\varepsilon) \triangleq y(t) - y_s^*[x(t), x_h(t), \bar{y}_h(t), t], \quad \bar{y}_h(t) \triangleq y_h(t) - y_{fh}(t/\varepsilon), \quad (6.2)$$

$$y_{fh}(t/\varepsilon) \triangleq \{y_f((t+\tau)/\varepsilon) \quad \forall \tau \in [-h, 0)\}, \quad y_f(\xi) = 0 \quad \forall \xi < 0. \quad (6.3)$$

By substituting (6.2)-(6.3) into (6.1), one has after some rearrangement

$$\begin{aligned} v_a[x(t), y(t), x_h(t), y_h(t), t] = & -M^{-1}P_f\{y(t) \\ & + D_y^{-1}[(A'_{02}P_s(t) + Q'_{s2}(t, 0))x(t) \\ & + \int_{-h}^0 (A'_{02}Q_{s1}(t, \tau) + R'_{s1}(t, \tau, 0))x(t+\tau)d\tau \\ & + \int_{-h}^0 (A'_{02}Q_{s2}(t, \tau) + R_{s2}(t, 0, \tau))(y(t+\tau) - y_f((t+\tau)/\varepsilon))d\tau\}. \end{aligned} \quad (6.4)$$

Expression (6.4) for $v_a[\cdot]$ contains $y_f(\cdot)$. At the second stage, using (5.23) and the limit process for $\varepsilon \rightarrow 0$, we eliminate the term depending on $y_f(\cdot)$ from (6.4). Thus, we obtain the composite state-feedback control for system (4.2)-(4.3), (2.2)

$$\begin{aligned} v_c[x(t), y(t), x_h(t), y_h(t), t] = & -M^{-1}P_f\{y(t) \\ & + D_y^{-1}[(A'_{02}P_s(t) + Q'_{s2}(t, 0))x(t) \\ & + \int_{-h}^0 (A'_{02}Q_{s1}(t, \tau) + R'_{s1}(t, \tau, 0))x(t+\tau)d\tau \\ & + \int_{-h}^0 (A'_{02}Q_{s2}(t, \tau) + R_{s2}(t, 0, \tau))y(t+\tau)d\tau\}. \end{aligned} \quad (6.5)$$

7. Suboptimal control for the OP

By using (4.1) and (6.5), one obtains the state-feedback control for (2.4)-(2.6), (2.2)

$$u_{so}[x(t), y(t), x_h(t), y_h(t), t] = \varepsilon^{-1}v_c[x(t), y(t), x_h(t), y_h(t), t]. \quad (7.1)$$

Let $J_{\varepsilon 1}^{so}$ be the value of the cost functional in (2.6) obtained by employing the control $u_{so}[\cdot]$ in system (2.4)-(2.5), (2.2) in the case $\varphi(\cdot) \in L^2[-h, 0; E^n]$, while $J_{\varepsilon 2}^{so}$ be such a value in the case $\varphi(\cdot) \in L^\infty[-h, 0; E^n]$. Let $J_{\varepsilon 1}^*$ and $J_{\varepsilon 2}^*$ be the optimal values of the cost functional in the OP corresponding to the first and second cases of $\varphi(\cdot)$, respectively.

THEOREM 2 *There exists a positive number ε_0 , such that, for all $\varepsilon \in (0, \varepsilon_0]$, the following inequalities are satisfied:*

$$\begin{aligned} 0 < J_{\varepsilon 1}^{so} - J_{\varepsilon 1}^* & \leq a\varepsilon^{3/2}[\|\varphi_0\| + \|\varphi(\cdot)\|_{L^2}]^2, \\ 0 < J_{\varepsilon 2}^{so} - J_{\varepsilon 2}^* & \leq a\varepsilon^2[\|\varphi_0\| + \|\varphi(\cdot)\|_{L^\infty}]^2, \end{aligned}$$

where $a > 0$ is some constant independent of ε .

8. Example

Consider the scalar cheap control problem

$$dy(t)/dt = y(t) + 2y(t - 0.5) + u(t), \quad (8.1)$$

$$J_\varepsilon(u) \triangleq \int_0^2 [y^2(t) + \varepsilon^2 u^2(t)] dt \rightarrow \min_u \quad (8.2)$$

subject to each of the following versions of the initial conditions

$$y(\tau) = \varphi_1(\tau) = |\tau + 0.251|^{-1/4}, \quad \tau \in [-0.5, 0]; \quad y(0) = \varphi_0 = 1.5, \quad (8.3)$$

$$y(\tau) = \varphi_2(\tau) = \tau + 1.41, \quad \tau \in [-0.5, 0]; \quad y(0) = \varphi_0 = 1.5. \quad (8.4)$$

It is seen that $\varphi_1(\cdot) \in L^2[-0.5, 0; E^1]$, while $\varphi_2(\cdot) \in L^\infty[-0.5, 0; E^1]$.

Due to Section 3, the optimal control for (8.1)-(8.2) with (8.3) (or (8.4)) is

$$u_\varepsilon^*[y(t), y_{0.5}(t), t] = -\varepsilon^{-2} [P(t)y(t) + \int_{-0.5}^0 Q(t, \tau)y(t + \tau)dt], \quad (8.5)$$

where $y_{0.5}(t) = \{y(t + \tau) \mid \forall \tau \in [-0.5, 0]\}$, and $P(t), Q(t, \tau)$ along with a function $R(t, \tau, \theta)$ form the unique solution of the following set of Riccati-type functional-differential equations in the domain $\Omega = \{(t, \tau, \theta) : t \in [0, 2], \tau \in [-0.5, 0], \theta \in [-0.5, 0]\}$

$$dP(t)/dt = -2P(t) - 2Q(t, 0) + \varepsilon^{-2}P^2(t) - 1, \quad P(2) = 0, \quad (8.6)$$

$$(\partial/\partial t - \partial/\partial \tau)Q(t, \tau) = [\varepsilon^{-2}P(t) - 1]Q(t, \tau) - R(t, 0, \tau), \quad Q(2, \tau) = 0, \quad (8.7)$$

$$(\partial/\partial t - \partial/\partial \tau - \partial/\partial \theta)R(t, \tau, \theta) = \varepsilon^{-2}Q(t, \tau)Q(t, \theta), \quad R(2, \tau, \theta) = 0, \quad (8.8)$$

$$Q(t, -0.5) = 2P(t), \quad R(t, -0.5, \tau) = R(t, \tau, -0.5) = 2Q(t, \tau). \quad (8.9)$$

Although all equations of this set are scalar, its solution is a complicated task, especially, for $\varepsilon \rightarrow 0$.

Let construct the suboptimal control for (8.1)-(8.2). By control transformation (4.1) in (8.1)-(8.2) and asymptotic decomposition of the resulting (transformed) problem, one obtains the slow subproblem, which consists of the equation $v_s(t) = 0, t \in [0, 2]$ and the ROP $\int_0^2 y_s^2(t)dt \rightarrow \min_{y_s}$ yielding the solution $y_s^*(t) = 0, t \in [0, 2]$. The fast subproblem is the scalar version of (5.19)-(5.20) with $D_y = M = 1$ yielding the optimal control $v_f^*[y_f(\xi)] = -y_f(\xi)$. Using these solutions of the slow and fast subproblems, and (6.5) and (7.1), we obtain the simple analytical memoryless expression for the suboptimal control $u_{so}[y(t), y_{0.5}(t), t] = -\varepsilon^{-1}y(t)$.

Let $J_{\varepsilon, k}^*$, ($k = 1, 2$) be the optimal values of the cost functional in (8.1)-(8.2) with initial conditions (8.3) and (8.4), respectively. Let $J_{\varepsilon, k}^{so}$, ($k = 1, 2$) be the

values of the cost functional in (8.2) obtained by employing $u_{so}[\cdot]$ in (8.1) with initial conditions (8.3) and (8.4), respectively. In the following table, these values are presented for some values of ε .

Table 1. Values $J_{\varepsilon 1}^*$, $J_{\varepsilon 1}^{so}$ and $J_{\varepsilon 2}^*$, $J_{\varepsilon 2}^{so}$

ε	0.1	0.08	0.06	0.04	0.02
$J_{\varepsilon 1}^*$	0.434	0.311	0.202	0.122	0.054
$J_{\varepsilon 1}^{so}$	0.562	0.383	0.243	0.136	0.057
$J_{\varepsilon 2}^*$	0.346	0.256	0.177	0.109	0.050
$J_{\varepsilon 2}^{so}$	0.404	0.287	0.192	0.114	0.051

By using Table 1, the following inequalities are obtained:

$$J_{\varepsilon 1}^{so} - J_{\varepsilon 1}^* \leq 0.48\varepsilon^{3/2}[\|\varphi_0\| + \|\varphi_1\|_{L^2}]^2, \quad J_{\varepsilon 2}^{so} - J_{\varepsilon 2}^* \leq 0.68\varepsilon^2[\|\varphi_0\| + \|\varphi_2\|_{L^\infty}]^2.$$

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