OPTIMIZATION OF A COUPLED FORCE INTENSITY BY HOMOGENIZATION METHODS

M. Codegone ¹

¹*Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino (Italy), marco.codegone@polito.it*

Abstract

In this paper, in the framework of a problem related to an elastic non homogeneous medium, we deal with a periodic coupled force $(f(x)/\varepsilon^\alpha) \vec{F}(x/\varepsilon)$ with intensity of order $1/\varepsilon^\alpha$. The parameter ε is connected with the period of the non homogeneity of the medium and with the periodicity of the coupled force. The determination of the parameter α is the target of our study to obtain an effect in the microscopic equation. The homogenization technique is used in order to study the equation: $-(\partial/\partial x_j)(a_{ijkh}(x/\varepsilon)e_{kh}(\vec{u}^{\varepsilon,\alpha})) = (f(x)/\varepsilon^\alpha)F_i(x/\varepsilon) + G_i(x, x/\varepsilon)$, where $G_i(x, x/\varepsilon)$ is the volume applied force. The limit, when $\varepsilon \rightarrow 0$, of $\vec{u}^{\varepsilon,\alpha}(x)$, in the sense of two scale convergence, is $(\vec{u}^{0,\alpha}(x), \vec{u}^{1,\alpha}(x, y))$ and the microscopic equation becomes: $-(\partial/\partial y_j)(a_{ijkh}(y)e_{khx}(\vec{u}^{0,\alpha}(x))) - (\partial/\partial y_j)(e_{khy}(\vec{u}^{1,\alpha}(x, y))) = f(x)F_i(y)$ if $\alpha = 1$, $-(\partial/\partial y_j)(a_{ijkh}(y)e_{khx}(\vec{u}^{0,\alpha}(x)) + e_{khy}(\vec{u}^{1,\alpha}(x, y))) = 0$ if $0 < \alpha < 1$. When $\alpha > 1$ the solutions are not uniformly bounded respect to ε .

keywords: Coupled forces, Homogenization, Elasticity.

1. Introduction

In this paper, in the framework of a problem related to an elastic non homogeneous medium (see [2] and [7]), we deal with a periodic coupled force

$$\frac{f(x)}{\varepsilon^\alpha} \vec{F}\left(\frac{x}{\varepsilon}\right)$$

with intensity of order $1/\varepsilon^\alpha$ (see [10], [11] and [12]). The parameter ε is related to the period of the non homogeneity of the medium and to the periodicity of the coupled force. For other questions related to the homogenization theory see [3], [5] and [6]. The determination of the parameter α is the target of our study to obtain an effect in the microscopic equation. Let Ω be a regular domain of $\mathbf{R}^3$ and let $a_{ijkh}(y)$ be the symmetric, coercive and Y -periodic elasticity tensor:

$$a_{ijkh} = a_{jikh} = a_{khij}, \quad a_{ijkh} e_{ij} e_{kh} \geq c e_{ij} e_{ij}. \quad (1)$$

Please use the following format when citing this chapter:

Codegone, M., 2006, in IFIP International Federation for Information Processing, Volume 202, Systems, Control, Modeling and Optimization, eds. Ceragioli, F., Dontchev, A., Furuta, H., Marti, K., Pandolfi, L., (Boston: Springer), pp. 115-125.

We denote by $\vec{F}(y)$ the couple force Y -periodic and such that:

$$\frac{1}{|Y|} \int_Y \vec{F}(y) dy = 0 \quad (2)$$

and by $f(x)$ a continuous scalar function representing the intensity of the coupled force. We indicate by $G(x, y)$ the external force Y -periodic in the y variable. The problem is expressed by the equation:

$$-\frac{\partial}{\partial x_j} \left(a_{ijkh} \left(\frac{x}{\varepsilon} \right) e_{kh}(\vec{u}^{\varepsilon, \alpha}) \right) = \frac{f(x)}{\varepsilon^\alpha} F_i \left(\frac{x}{\varepsilon} \right) + G_i \left(x, \frac{x}{\varepsilon} \right), \quad (3)$$

in Ω , with the boundary condition

$$\vec{u}^{\varepsilon, \alpha} |_{\partial\Omega} = 0. \quad (4)$$

We prove that, if $0 < \alpha \leq 1$ the solutions $u^{\varepsilon, \alpha}(x)$ are uniformly bounded:

$$\|\vec{u}^{\varepsilon, \alpha}(x)\|_{(H_0^1(\Omega))^3} \leq C.$$

Then taking as test function:

$$\vec{v}(x, y) = \vec{v}^0(x) + \varepsilon \vec{v}^1(x, \frac{x}{\varepsilon})$$

with:

$$\vec{v}^0(x) \in (H_0^1(\Omega))^3$$

and with

$$\vec{v}^1(x, y) \in (L^2(\Omega; H_{per}^1(\mathbf{R}))^3)$$

the two-convergence limit gives us the following answers:

if $\alpha = 1$ then

$$\begin{aligned} & \int_{\Omega} \int_Y a_{ijkh}(y) \left[e_{khx}(\vec{u}^{0,1}(x)) + e_{khy}(\vec{u}^{1,1}(x, y)) \right] \\ & \quad \left[e_{khx}(\vec{v}^0(x)) + e_{khy}(\vec{v}^1(x, y)) \right] dx dy + \\ & - \int_{\Omega} \int_Y f(x) \vec{F}(y) \vec{v}^1(x, y) dx dy - \int_{\Omega} \int_Y \vec{G}(x, y) \vec{v}^0(x) dx dy = 0, \end{aligned}$$

moreover if $0 < \alpha < 1$ then

$$\begin{aligned} & \int_{\Omega} \int_Y a_{ijkh}(y) \left[e_{khx}(\vec{u}^{0, \alpha}(x)) + e_{khy}(\vec{u}^{1, \alpha}(x, y)) \right] \\ & \quad \left[e_{khx}(\vec{v}^0(x)) + e_{khy}(\vec{v}^1(x, y)) \right] dx dy + \end{aligned}$$

$$-\int_{\Omega} \int_Y \vec{G}(x, y) \vec{v}^0(x) dx dy = 0,$$

An integration by parts of the preceding formulae gives the following two limit problems: if $\alpha = 1$ then

$$\begin{cases} -\frac{\partial}{\partial y_j} [a_{ijkh}(y) [e_{khx}(\vec{u}^{0,1}(x)) + e_{khy}(\vec{u}^{1,1}(x, y))]] - f(x)F_i(y) = 0 \\ -\frac{\partial}{\partial x_j} [\int_Y a_{ijkh}(y) [e_{khx}(\vec{u}^{0,1}(x)) + e_{khy}(\vec{u}^{1,1}(x, y))] dy] + \\ - \int_Y G_i(x, y) dy = 0, \end{cases}$$

and $0 < \alpha < 1$ then

$$\begin{cases} -\frac{\partial}{\partial y_j} [a_{ijkh}(y) [e_{khx}(\vec{u}^{0,\alpha}(x)) + e_{khy}(\vec{u}^{1,\alpha}(x, y))]] = 0 \\ -\frac{\partial}{\partial x_j} [\int_Y a_{ijkh}(y) [e_{khx}(\vec{u}^{0,\alpha}(x)) + e_{khy}(\vec{u}^{1,\alpha}(x, y))] dy] + \\ - \int_Y G_i(x, y) dy = 0. \end{cases}$$

In order to discuss the influence of the coupled force at the microscopic level we remark and discuss the difference of the two microscopic equations: if $\alpha = 1$

$$-\frac{\partial}{\partial y_j} (a_{ijkh}(y) e_{khx}(\vec{u}^{0,1}(x)) + e_{khy}(\vec{u}^{1,1}(x, y))) = f(x)F_i(y) \quad (5)$$

and if $0 < \alpha < 1$

$$-\frac{\partial}{\partial y_j} (a_{ijkh}(y) e_{khx}(\vec{u}^{0,\alpha}(x)) + e_{khy}(\vec{u}^{1,\alpha}(x, y))) = 0. \quad (6)$$

We remark that the equation (6) is a standard microscopic equation, and there the external force $f(x)F_i(y)$ does not appear. On the contrary the equation (5) has the external force $f(x)F_i(y)$, then we can conclude that the optimum coupled force that works at microscopic level is of type $(f(x)/\varepsilon^\alpha) \vec{F}(x/\varepsilon)$, with $\alpha = 1$. In this case the macroscopic equation becomes:

$$-\frac{\partial}{\partial x_j} (a_{ijkh}^{hom} e_{khx}(\vec{u}^{0,1}(x))) = \Phi_{ij} \frac{\partial}{\partial x_j} f(x) + \tilde{G}_i(x) \quad (7)$$

where Φ_{ij} is obtained starting from the microscopic problem and where a_{ijkh}^{hom} are the classical homogenized coefficients and

$$\tilde{G}_i(x) = \frac{1}{|Y|} \int_Y G(x, y) dx.$$

2. Statement of the problem with $\varepsilon > 0$ and $\alpha > 0$

In order to study the equation (3), we introduce the variational formulation in the Hilbert space $(H_0^1(\Omega))^3$. In this setting the existence is proved of a unique solution for ε and α fixed and a uniform bound is obtained, with respect to ε if $0 < \alpha \leq 1$. In view of these targets we multiply the equation (3) by $\vec{v}^0(x)$ with

$$\vec{v}^0(x) \in (H_0^1(\Omega))^3$$

and we integrate by parts. We get the following weak formulation:

$$\text{find } \vec{u}^{\varepsilon, \alpha}(x) \in (H_0^1(\Omega))^3$$

such that

$$\begin{aligned} \int_{\Omega} a_{ijkh}(x/\varepsilon) e_{khx}(\vec{u}^{\varepsilon, \alpha}) e_{ijx}(\vec{v}^0) dx &= \\ &= \int_{\Omega} \frac{f(x)}{\varepsilon^\alpha} \vec{F}\left(\frac{x}{\varepsilon}\right) \cdot \vec{v}^0 dx + \int_{\Omega} \vec{G}\left(x, \frac{x}{\varepsilon}\right) \cdot \vec{v}^0 dx \end{aligned} \quad (8)$$

We suppose that a_{ijkh} satisfy the hypothesis of ellipticity and symmetry:

$$a_{ijkh} = a_{jikh} = a_{khij} \quad m \xi_{ij} \xi_{ij} \leq a_{ijkh} \xi_{ij} \xi_{kh} \leq M \xi_{ij} \xi_{ij}, \quad (9)$$

with $m > 0$. Moreover $a_{ijkh}(y)$ is Y -periodic of period Y and

$$f(x) \vec{F}(y), \vec{G}(x, y) \in (L^2(\Omega; C_{per}^0(Y)))^3 \quad (10)$$

where $C_{per}^0(Y)$ is the space of continuous periodic functions of period Y . It is known (see [1]) that, if (10) holds, then it follows that:

$$f(x) \vec{F}(x/\varepsilon), \vec{G}(x, x/\varepsilon) \in (L^2(\Omega))^3.$$

By hypothesis (9), we get:

$$m e_{ij}(\vec{u}^{\varepsilon, \alpha}) e_{ij}(\vec{u}^{\varepsilon, \alpha}) \leq \int_{\Omega} a_{ijkh}(x/\varepsilon) e_{kh}(\vec{u}^{\varepsilon, \alpha}) e_{ij}(\vec{u}^{\varepsilon, \alpha}) dx,$$

then, by Korn inequalities (see, for instance, [4] and [8]) we get:

$$\|\vec{u}^{\varepsilon, \alpha}(x)\|_{(H_0^1(\Omega))^3}^2 \leq C \int_{\Omega} a_{ijkh}(x/\varepsilon) e_{kh}(\vec{u}^{\varepsilon, \alpha}) e_{ij}(\vec{u}^{\varepsilon, \alpha}) dx. \quad (11)$$

On the other hand, we need to have an estimate of the right-hand side of the equation (8):

$$\begin{aligned} &\left| \int_{\Omega} \frac{f(x)}{\varepsilon^\alpha} \vec{F}\left(\frac{x}{\varepsilon}\right) \cdot \vec{u}^{\varepsilon, \alpha} dx + \int_{\Omega} \vec{G}\left(x, \frac{x}{\varepsilon}\right) \cdot \vec{u}^{\varepsilon, \alpha} dx \right| \\ &\leq K \|\vec{u}^{\varepsilon, \alpha}\|_{(H_0^1(\Omega))^3}. \end{aligned} \quad (12)$$

Taking $\vec{G}(x, x/\varepsilon) \in \left(L^2(\Omega; \mathcal{C}_{per}^0)\right)^3$ we get

$$\int_{\Omega} \vec{G}\left(x, \frac{x}{\varepsilon}\right) \cdot \vec{u}^{\varepsilon, \alpha} dx \leq K_1 \|\vec{u}^{\varepsilon, \alpha}\|_{(H_0^1(\Omega))^3}.$$

The complete inequality (12) will be proved in the next section.

The estimates (11) and (12), by Lax-Milgram lemma, give the existence of a unique solution, for fixed ε and $0 < \alpha \leq 1$, of the weak formulation (8) and moreover give a uniform, with respect to ε , estimate of following norm of $\vec{u}^{\varepsilon, \alpha}$:

$$\|\vec{u}^{\varepsilon, \alpha}(x)\|_{(H_0^1(\Omega))^3} \leq K. \quad (13)$$

The estimate (13) permits us to apply the result of the two scale convergence method (see, for instance, [1], [4] and [9]) to obtain that there exist two functions

$$\vec{u}^{0, \alpha}(x) \in \left(H_0^1(\Omega)\right)^3 \quad \text{and} \quad \vec{u}^{1, \alpha}(x, y) \in \left(L^2(\Omega, H_{per}^1(\mathbf{R}))\right)^3$$

such that:

$$\vec{u}^{\varepsilon, \alpha}(x) \rightharpoonup \vec{u}^{0, \alpha}(x) \quad \text{in} \quad \left(H_0^1(\Omega)\right)^3 \quad \text{weakly}$$

and

$$e_{ij}(\vec{u}^{\varepsilon, \alpha}) \rightharpoonup e_{ijx}(\vec{u}^{0, \alpha}(x)) + e_{ijy}(\vec{u}^{1, \alpha}(x, y))$$

where we wrote:

$$e_{ijx}(\vec{v}) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \text{and} \quad e_{ijy}(\vec{v}) = \frac{1}{2} \left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i} \right).$$

We introduce the Hilbert space

$$\mathbf{V} = \left(H_0^1(\Omega)\right)^3 \times \left(L^2(\Omega, H_{per}^1(\mathbf{R}))\right)^3, \quad (14)$$

with the scalar product

$$\begin{aligned} & \left((\vec{u}^{0, \alpha}, \vec{u}^{1, \alpha}), (\vec{v}^0, \vec{v}^1) \right) = \\ & = \int_{\Omega} \int_Y \left[e_{ijx}(\vec{u}^{0, \alpha}) + e_{ijy}(\vec{u}^{1, \alpha}) \right] \left[e_{ijx}(\vec{v}^0) + e_{ijy}(\vec{v}^1) \right] dx dy. \end{aligned} \quad (15)$$

Multiplying the equation (3) by

$$\vec{v}(x, x/\varepsilon) = \vec{v}^0(x) + \varepsilon \vec{v}^1(x, x/\varepsilon) \quad \text{with} \quad \vec{v}(x, x/\varepsilon) \in \mathbf{V}$$

the problem (8), in the context of the two scale convergence theory, can be written as follows

$$\begin{aligned}
 & \int_{\Omega} a_{ijkh}(x/\varepsilon) e_{khx}(\vec{u}^{\varepsilon, \alpha}(x)) \left[e_{ijx}(\vec{v}^0(x)) + e_{ijy}(\vec{v}^1(x, x/\varepsilon)) + \right. \\
 & \quad \left. + \varepsilon e_{ijx}(\vec{v}^1(x, x/\varepsilon)) \right] dx = \\
 & = \int_{\Omega} \frac{f(x)}{\varepsilon^{\alpha}} \vec{F}\left(\frac{x}{\varepsilon}\right) \cdot \left[\vec{v}^0 + \varepsilon \vec{v}^1(x, x/\varepsilon) \right] dx + \\
 & \quad + \int_{\Omega} \vec{G}\left(x, \frac{x}{\varepsilon}\right) \cdot \left[\vec{v}^0 + \varepsilon \vec{v}^1(x, x/\varepsilon) \right] dx
 \end{aligned} \tag{16}$$

3. Estimate of the term with the couple force

The coupled force $(f(x)/\varepsilon^{\alpha})\vec{F}(x/\varepsilon)$ at the right-hand side of the equation (3) has an intensity that increases as ε goes to zero. Then is not evident that the term with the coupled force may be bounded:

$$\left| \frac{1}{\varepsilon^{\alpha}} \int_{\Omega} f(x) \vec{F}\left(\frac{x}{\varepsilon}\right) \cdot \vec{u}^{\varepsilon, \alpha} dx \right| \leq C_1 \|\vec{u}^{\varepsilon, \alpha}\|_{(H_0^1(\Omega))^3} \tag{17}$$

with C_1 independent of ε . The hypothesis (2) that $\vec{F}(x/\varepsilon)$ has zero mean value is essential. We prove the estimate (17), if

$$0 < \alpha \leq 1,$$

in the following lemma:

LEMMA 1 *Let $f(x) \in C^1(\bar{\Omega})$, $0 < \alpha \leq 1$, $\vec{F}(y) \in \left(H_{per}^1(Y)\right)^3$ and $\vec{u}^{\varepsilon} \in \left(H_0^1(\Omega)\right)^3$ with Ω bounded region of $\mathbf{R}^n$ and $Y = [0, 1]^n$. Moreover let $\vec{F}(y)$ be periodic with period Y and suppose that the condition (2) of zero mean value holds. Then there exists a constant C_1 such that the estimate (17) is fulfilled.*

Proof. We consider the vector $\vec{y}_k \in \mathbf{Z}^n$ and any translation $\vec{y}_k + Y$ of the fundamental period $Y = [0, 1]^n$. We scale down the translated period with the parameter ε and we say $Y_k^{\varepsilon} = \varepsilon(\vec{y}_k + Y)$. There are a finite number N of the Y_k^{ε} periods such that $\bigcup_{k=1}^N Y_k^{\varepsilon} \supseteq \Omega$. There are many period Y_k^{ε} that are strictly included in Ω , for instance $\bigcup_{k=1}^{N-T} Y_k^{\varepsilon} \subset \Omega$, moreover there are a number T of periods Y_k^{ε} such that $\{\bigcup_{k=N-T+1}^N Y_k^{\varepsilon}\} \cap \partial\Omega \neq \emptyset$. We now extend to zero the function $\vec{u}^{\varepsilon, \alpha}$ in the periods $Y_k^{\varepsilon} \cap \{\mathbf{R}^n \setminus \Omega\}$, with $N - T + 1 \leq k \leq N$. Now

we decompose the integral over Ω in a sum of integral over the cells Y_k^ε :

$$\left| \frac{1}{\varepsilon^\alpha} \int_{\Omega} f(x) \vec{F} \left(\frac{x}{\varepsilon} \right) \cdot \vec{u}^{\varepsilon, \alpha}(x) \, dx \right| \leq \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \left| \int_{Y_k^\varepsilon} f(x) \vec{F} \left(\frac{x}{\varepsilon} \right) \cdot \vec{u}^{\varepsilon, \alpha}(x) \, dx \right| \quad (18)$$

In any period Y_k^ε we choose a point x_k where $f(x)$ attains the maximum and adding and subtracting $f(x_k)$ we get:

$$\begin{aligned} & \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \left| \int_{Y_k^\varepsilon} f(x) \vec{F} \left(\frac{x}{\varepsilon} \right) \cdot \vec{u}^{\varepsilon, \alpha}(x) \, dx \right| \leq \\ & \leq \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \left| \int_{Y_k^\varepsilon} f(x_k) \vec{F} \left(\frac{x}{\varepsilon} \right) \cdot \vec{u}^{\varepsilon, \alpha}(x) \, dx \right| + \\ & + \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \int_{Y_k^\varepsilon} |f(x) - f(x_k)| \left| \vec{F} \left(\frac{x}{\varepsilon} \right) \right| |\vec{u}^{\varepsilon, \alpha}(x)| \, dx \end{aligned} \quad (19)$$

In the k -cell Y_k^ε we take the mean value of $\vec{u}^{\varepsilon, \alpha}(x)$:

$$\mathcal{M}(\vec{u}^{\varepsilon, \alpha}(x)) = \frac{1}{|Y_k^\varepsilon|} \int_{Y_k^\varepsilon} \vec{u}^{\varepsilon, \alpha}(x) \, dx. \quad (20)$$

By the condition (2) of zero mean value and by the preceding formula (20), the first term at the right-hand side of inequality (19) becomes

$$\begin{aligned} & \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \left| \int_{Y_k^\varepsilon} f(x_k) \vec{F} \left(\frac{x}{\varepsilon} \right) \cdot \vec{u}^{\varepsilon, \alpha}(x) \, dx \right| = \\ & = \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \left| \int_{Y_k^\varepsilon} f(x_k) \vec{F} \left(\frac{x}{\varepsilon} \right) \cdot (\vec{u}^{\varepsilon, \alpha}(x) - \mathcal{M}(\vec{u}^{\varepsilon, \alpha}(x))) \, dx \right| \end{aligned} \quad (21)$$

Now we apply the Cauchy-Schwartz inequality in the right-hand side of (21) to obtain

$$\begin{aligned} & \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \left| \int_{Y_k^\varepsilon} f(x_k) \vec{F} \left(\frac{x}{\varepsilon} \right) \cdot (\vec{u}^{\varepsilon, \alpha}(x) - \mathcal{M}(\vec{u}^{\varepsilon, \alpha}(x))) \, dx \right| \leq \\ & \leq \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N |f(x_k)| \left\| \vec{F} \left(\frac{x}{\varepsilon} \right) \right\|_{L^2(Y_k^\varepsilon)} \|\vec{u}^{\varepsilon, \alpha}(x) - \mathcal{M}(\vec{u}^{\varepsilon, \alpha}(x))\|_{L^2(Y_k^\varepsilon)} \end{aligned} \quad (22)$$

In the last factor at the right-hand side of relation (22) we apply a Korn inequality for functions with zero mean value; it is known that in the Korn inequalities (as

in the Poincaré-Wirtinger inequality for the scalar case, see for instance [4] and [8]) the constant $c(\varepsilon) = c \cdot \varepsilon$ is related to the measure ε of the region Y_k^ε

$$\begin{aligned} & \|\vec{u}^{\varepsilon, \alpha}(x) - \mathcal{M}(\vec{u}^{\varepsilon, \alpha}(x))\|_{L^2(Y_k^\varepsilon)} \leq \\ & \leq c \cdot \varepsilon \left\| \sum_{i,j=1}^n e_{ij}(\vec{u}^{\varepsilon, \alpha}(x)) \right\|_{L^2(Y_k^\varepsilon)} \leq c \cdot \varepsilon \|\vec{u}^{\varepsilon, \alpha}(x)\|_{H^1(Y_k^\varepsilon)} \end{aligned} \quad (23)$$

The inequalities (21), (22) and (23) give us

$$\begin{aligned} & \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \left| \int_{Y_k^\varepsilon} f(x_k) \vec{F}\left(\frac{x}{\varepsilon}\right) \cdot \vec{u}^{\varepsilon, \alpha}(x) \, dx \right| \leq \\ & \leq \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N |f(x_k)| \left\| \vec{F}\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(Y_k^\varepsilon)} \cdot c \cdot \varepsilon \|\vec{u}^{\varepsilon, \alpha}\|_{H^1(Y_k^\varepsilon)} \leq \\ & \leq \varepsilon^{1-\alpha} C_3 \left\{ \max_{x \in \Omega} |f(x)| \right\} \sum_{k=1}^N \left\| \vec{F}\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(Y_k^\varepsilon)} \sum_{k=1}^N \|\vec{u}^{\varepsilon, \alpha}\|_{H^1(Y_k^\varepsilon)} \leq \\ & \leq \varepsilon^{1-\alpha} C_4 \left\{ \max_{x \in \Omega} |f(x)| \right\} \left\| \vec{F}\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} \|\vec{u}^{\varepsilon, \alpha}\|_{H^1(\Omega)} \leq \\ & \leq \varepsilon^{1-\alpha} C_5 \|\vec{u}^{\varepsilon, \alpha}(x)\|_{H^1(\Omega)} \end{aligned} \quad (24)$$

In view to obtain the estimate of the couple force, in the second term at the right-hand side of formula (19), we apply the Lipschitz inequality to the function $f(x) \in \mathcal{C}^1$ in the cell Y_k^ε

$$|f(x) - f(x_k)| \leq C_6 \cdot \varepsilon \quad x \in Y_k^\varepsilon$$

and then we have

$$\begin{aligned} & \frac{1}{\varepsilon^\alpha} \sum_{k=1}^N \int_{Y_k^\varepsilon} |f(x) - f(x_k)| \left| \vec{F}\left(\frac{x}{\varepsilon}\right) \right| |\vec{u}^{\varepsilon, \alpha}(x)| \, dx \leq \\ & \leq \frac{1}{\varepsilon^\alpha} \cdot C_6 \cdot \varepsilon \sum_{k=1}^N \int_{Y_k^\varepsilon} \left| \vec{F}\left(\frac{x}{\varepsilon}\right) \right| |\vec{u}^{\varepsilon, \alpha}(x)| \, dx. \end{aligned} \quad (25)$$

Adding on the cell from 1 to N and applying the Cauchy-Schwartz inequality, we have:

$$\frac{1}{\varepsilon^\alpha} \cdot C_6 \cdot \varepsilon \sum_{k=1}^N \int_{Y_k^\varepsilon} \left| \vec{F}\left(\frac{x}{\varepsilon}\right) \right| |\vec{u}^{\varepsilon, \alpha}(x)| \, dx \leq$$

$$\begin{aligned}
&\leq \varepsilon^{1-\alpha} C_6 \int_{\Omega} \left| \vec{F} \left(\frac{x}{\varepsilon} \right) \right| |\vec{u}^{\varepsilon, \alpha}(x)| \, dx \leq \\
&\leq \varepsilon^{1-\alpha} C_6 \left\| \vec{F} \left(\frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega)} \|\vec{u}^{\varepsilon, \alpha}(x)\|_{L^2(\Omega)} \leq \\
&\leq \varepsilon^{1-\alpha} C_6 \left\| \vec{F} \left(\frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega)} \|\vec{u}^{\varepsilon, \alpha}(x)\|_{H^1(\Omega)}. \quad (26)
\end{aligned}$$

The relations (24) and (26) give a uniform estimate for any $\varepsilon \in (0, 1]$ if $0 < \alpha \leq 1$. If $\alpha > 1$ the estimate is uniform only for $0 < k \leq \varepsilon \leq 1$, and we can not obtain a convergent subsequence of $\vec{u}^{\varepsilon, \alpha}(x)$ as ε goes to zero. The proof of the estimate of the couple force is then complete.

4. Two-scale convergence limit

A vector valued functions $\vec{u}^{\varepsilon, \alpha}(x)$ in $(L^2(\Omega))^3$ is said to two-scale converge to a limit $\vec{u}^{0, \alpha}(x, y)$ belonging to $(L^2(\Omega \times Y))^3$ if, for any function $\varphi(x, y)$ in $(\mathcal{D}(\Omega; C_{per}^\infty(Y)))^3$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \vec{u}^{\varepsilon, \alpha}(x) \varphi \left(x, \frac{x}{\varepsilon} \right) dx \longrightarrow \int_{\Omega} \int_Y \vec{u}^{0, \alpha}(x, y) \varphi(x, y) dx dy.$$

By estimate (13), the bounded sequence $\vec{u}^{\varepsilon, \alpha}(x)$ weakly converges in $(H_0^1(\Omega))^3$ to a limit $\vec{u}^{0, \alpha}(x)$. Then (see, for instance [1], [4] or [7]), if $\vec{u}^{\varepsilon, \alpha}(x)$ two-scale converges to $\vec{u}^{0, \alpha}(x)$, there exists a function $\vec{u}^{1, \alpha}(x, y)$ in $(L^2(\Omega, H_{per}^1(Y)/\mathbf{R}))^3$ such that, up to a subsequence, $e_{ijx}(\vec{u}^{\varepsilon, \alpha}(x))$ two-scale converges:

$$e_{ijx}(\vec{u}^{\varepsilon, \alpha}(x)) \rightharpoonup e_{ijx}(\vec{u}^{0, \alpha}(x)) + e_{ijy}(\vec{u}^{1, \alpha}(x, y)).$$

The weak formulation (16) of the problem (3), in the context of the two-scale convergence, gives the limit problem in the following variational form:

$$\text{find } (u^{0, \alpha}, u^{1, \alpha}) \in (H_0^1(\Omega))^3 \times (L^2(\Omega, H_{per}^1(Y)/\mathbf{R}))^3 = \mathbf{V}$$

such that

$$\begin{aligned}
&\int_{\Omega} \int_Y a_{ijkh}(y) \left[e_{khx}(\vec{u}^{0, \alpha}(x)) + e_{khy}(\vec{u}^{1, \alpha}(x, y)) \right] \cdot \\
&\quad \cdot \left[e_{khx}(\vec{v}^0(x)) + e_{khy}(\vec{v}^1(x, y)) \right] dx dy = \\
&= \delta_{\alpha 1} \int_{\Omega} \int_Y f(x) \vec{F}(y) \cdot \vec{v}^1(x, y) dx dy + \int_{\Omega} \int_Y \vec{G}(x, y) \cdot \vec{v}^0(x) dx dy, \quad (27)
\end{aligned}$$

$\forall (v_0, v_1) \in \mathbf{V}$ and where

$$\delta_{\alpha 1} = \begin{cases} 1 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \neq 1. \end{cases}$$

By the coercivity (9) of the bilinear form defined by the left-hand side of (27), we can apply the Lax-Milgram lemma and we have a well posed formulation of the limit two-scale problem. Integrating by parts the equation (27) in x and y and choosing $\vec{v}^0(x) = 0$ we get the microscopic equation

$$-\frac{\partial}{\partial y_j} \left\{ a_{ijkh}(y) \left[e_{khx} \left(\vec{u}^{0,\alpha}(x) \right) + e_{khy} \left(\vec{u}^{1,\alpha}(x, y) \right) \right] \right\} = f(x) \vec{F}_i(y) \quad (28)$$

By the linearity of the equation (28), we solve the following problem in the period Y

$$\begin{cases} -\frac{\partial}{\partial y_j} \left[a_{ijkh}(y) e_{khy} \left(\vec{w}^{lm,\alpha}(y) \right) \right] = \frac{\partial a_{ijlm}(y)}{\partial y_j} & \text{in } Y \\ \vec{w}^{lm,\alpha}(y) & Y - \text{periodic} \end{cases} \quad (29)$$

and, if $\alpha = 1$, we solve the further equation:

$$\begin{cases} -\frac{\partial}{\partial y_j} \left[a_{ijkh}(y) e_{khy} \left(\vec{v}^\alpha(y) \right) \right] = \vec{F}_i(y) & \text{in } Y \\ \vec{v}^\alpha(y) & Y - \text{periodic} \end{cases} \quad (30)$$

The solution of problems (29) and (30) gives the solution of the microscopic equation (28)

$$\vec{u}^{1,\alpha}(x, y) = e_{khx} \left(\vec{u}^{0,\alpha}(x) \right) \vec{w}^{kh,\alpha}(y) + \delta_{\alpha 1} f(x) \vec{v}^\alpha(y) \quad (31)$$

We remark that, at the microscopic level, the couple force has, if $\alpha = 1$, an important role.

As in classical homogenization we define

$$a_{ijkh}^{\text{hom},\alpha} = \int_Y a_{ijlm}(y) \left[\delta_{lk} \delta_{mh} + e_{lmj} \left(\vec{w}^{kh,\alpha}(y) \right) \right] dy \quad (32)$$

Moreover we define

$$\Phi_{ij}^\alpha = \int_Y a_{ijkh}(y) e_{khy} \left(\vec{v}^\alpha \right) dy \quad (33)$$

At this point we write the limit homogenized equation

$$\left\{ \begin{array}{l} -\frac{\partial}{\partial x_j} \left[a_{ijkh}^{\text{hom}, \alpha}(y) e_{k h x} \left(\bar{u}^{0, \alpha}(x) \right) \right] = \\ \quad = \delta_{\alpha 1} \Phi_{ij}^{\alpha} \frac{\partial}{\partial x_j} (f(x)) + \int_Y \vec{G}_i(x, y) \quad \text{in } \Omega \\ \bar{u}^{0, \alpha}(x) = 0 \quad \text{on } \partial\Omega \end{array} \right. \quad (34)$$

REMARK 2 We remark that if $a_{ijkh}(y) = \bar{a}_{ijkh}$ is constant in the period Y , then Φ_{ij}^{α} in formula (33) is zero. In same way if the intensity $f(x)$ of the couple force is constant, then Φ_{ij}^{α} in formula (33) is zero. Moreover if $0 < \alpha < 1$, then Φ_{ij}^{α} in formula (33) is zero. Then the effect of the force $(f(x)/\varepsilon^{\alpha})F(y)$ may work at the microscopic level, if the medium is non homogeneous and $f(x)$ is non constant and if $\alpha = 1$.

References

- [1] G. Allaire. Homogenization and two scale convergence. *SIAM J. Math. Anal.* 23:1482-1518, 1992.
- [2] A. Bensoussan, J.L. Lions, G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. North-Holland, 1978.
- [3] V. Chiadò Piat, M. Codegone. Scattering problem in a perforated domain. *Rev. R. Acad. Cien. Serie A Mat.* 97:447-454, 2003.
- [4] D. Cioranescu, P. Donato. *An Introduction to Homogenization*. Oxford University Press, Oxford, 1999.
- [5] M. Codegone. Problème d'homogénéisation en théorie de la diffraction. *C.R. Acad.Sc. Paris.* 288:387-389, 1979.
- [6] M. Codegone. G-Convergence and Scattering Problems. *Boll. U.M.I.* 6.1-A:367-375, 1982.
- [7] V.V. Jikov, S.M. Kozlov, O.A. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. Springer-Verlag, Berlin Heidelberg, 1994.
- [8] O.A. Oleinik, A.S. Shamaev, G.A. Yosifian. *Mathematical problems in elasticity and homogenization*. Studies in mathematics and its applications, Elsevier Science Publishers, Amsterdam, 1992.
- [9] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.* 20:608-623, 1989.
- [10] J. Sanchez-Hubert, E. Sanchez-Palencia. *Vibration and Coupling of Continuous Systems Asymptotic Methods*. Springer-Verlag, Berlin, 1989.
- [11] E. Sanchez-Palencia. Comportement local et macroscopique d'un type de milieux physiques hétérogènes. *Internat. J. Engrg. Sci.* 12:331-351, 1974.
- [12] E. Sanchez-Palencia. *Non Homogeneous Media and Vibration Theory*. Lecture Notes in Physics, Springer-Verlag, Berlin, 1980.