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A four-variable automorphic kernel function

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Abstract

Let F be a number field, let $\mathbb{A}_F$ be its ring of adeles, and let $g_1, g_2, h_1, h_2 \in \mathrm{GL}_2(\mathbb{A}_F)$. We provide an absolutely convergent geometric expression for

$$\sum_{\pi} K_\pi(g_1, g_2) K_{\pi^\vee}(h_1, h_2) \mathrm{Res}_{s=1} L^S\left(s, \pi \times \pi^\vee\right),$$

where the sum is over isomorphism classes of cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$. Here K_π is the typical kernel function representing the action of a test function on the space of π.

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Contents

1 Introduction

Let F be a number field, let G be a connected reductive group over F, and let $A_G \leq G(F_\infty)$ be the connected component in the real topology of the $\mathbb{R}$-points of the greatest $\mathbb{Q}$-split torus in the center of $\mathrm{Res}_{F/\mathbb{Q}} G$. For $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$, let

$$R(f) : L^2(A_G G(F) \backslash G(\mathbb{A}_F)) \longrightarrow L^2\left(A_G G(F) \backslash G(\mathbb{A}_F)\right)$$

be the usual operation induced by the action of $G(\mathbb{A}_F)$ on the right. We let

$$K_f^{\mathrm{cusp}}(x, y) := \sum_{\pi} K_{\pi(f)}(x, y)$$

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be the kernel of $R(f)$ restricted to the cuspidal subspace. Here the sum is over isomorphism classes of cuspidal automorphic representations π of $A_G\backslash G(\mathbb{A}_F)$ (i.e., cuspidal automorphic representations of $G(\mathbb{A}_F)$ trivial on A_G). Moreover, $K_{\pi(f)}(x,y)$ is the unique smooth function with L^2-expansion

$$K_{\pi(f)}(x,y) = \sum_{\varphi\in\mathcal{B}(\pi)} \pi(f)\varphi(x)\overline{\varphi}(y),$$

where $\mathcal{B}(\pi)$ is an orthonormal basis of the space of π.

The starting point of any trace formula is a geometric expansion for $K_f^{\mathrm{cusp}}(x,y)$, namely

$$K_f^{\mathrm{cusp}}(x,y) = K_f(x,y) - K_f^{\mathrm{Eis}}(x,y),$$

where $K_f(x,y) := \sum_{\gamma\in G(F)} f(x^{-1}\gamma y)$ is the kernel function for $R(f)$ and $K_f^{\mathrm{Eis}}(x,y)$ is the contribution from Eisenstein series (which we will not explicate). Integrating $K_f^{\mathrm{cusp}}(x,y)$ along various subgroups of $(A_G G(F)\backslash G(\mathbb{A}_F))^{\times 2}$ yields various trace formulae. The canonical example is integration along the diagonal copy of $A_G G(F)\backslash G(\mathbb{A}_F)$; this leads to the usual trace formula.

Other subgroups can be used, and this leads to trace formulae that isolate representations having particular properties. For example, integrating along a twisted diagonal isolates representations isomorphic to their conjugates under an automorphism, and by Jacquet's philosophy that has been made more precise in work of Sakellaridis and Venkatesh [14], integration along spherical reductive subgroups ought to isolate representations that are functorial lifts from smaller groups.[1]

There are natural limits to what sort of representations can be isolated via these methods. As mentioned above, integration along a twisted diagonal isolates representations whose isomorphism class is invariant under a cyclic subgroup of the group $\mathrm{Out}_F(G)$ of outer automorphisms of G, and integrating along a pair of spherical subgroups seems usually to detect representations whose isomorphism class is invariant under a pair of involutory automorphisms, which can at most generate a dihedral subgroup of $\mathrm{Out}_F(G)$. As the author has advocated in [5,8], for applications to nonsolvable base change, it would be useful to develop trace formulae that isolate representations whose isomorphism class is invariant under nonsolvable subgroups of $\mathrm{Out}_F(G)$.

Let $f_1, f_2 \in C_c^\infty(A_G\backslash G(\mathbb{A}_F))$ and let $g_1, g_2, h_1, h_2 \in G(\mathbb{A}_F)$. One way of approaching the problem of isolating representations invariant under more automorphisms is to build geometric expressions for

$$\sum_{\pi} K_{\pi(f_1)}(g_1,g_2)K_{\pi^\vee(f_2)}(h_1,h_2)w(\pi), \tag{1.1}$$

where the sum is over isomorphism classes of cuspidal automorphic representations π of $A_G\backslash G(\mathbb{A}_F)$ and $w(\pi)\in\mathbb{C}$ is some weight factor. One could then integrate this kernel over two twisted diagonals and isolate representations whose isomorphism class is invariant under a subgroup of $\mathrm{Out}_F(G)$ generated by two elements. We recall that any finite simple nonabelian group is generated by two elements (see [9, Theorem 1.6] and the paragraph after it), so this is a quite general setup.

[1] Many important cases have been worked out, some by Jacquet himself, but the literature is too extensive to adequately cite here.

In this paper, we develop a geometric expression for (1.1) for a particular weight $w(\pi)$ in the special case where $G = \mathrm{GL}_2$.

1.1 The case at hand

We now let $G := \mathrm{GL}_2$ and $A := A_G$. Let S be a set of places of F including the infinite places such that $F/\mathbb{Q}$ is unramified outside of S and $\mathcal{O}_F^S$ has class number 1.

Let $f_1, f_2 \in C_c^\infty(A\backslash \mathrm{GL}_2(F_S))$, let $M_\ell, M_r \leq \mathrm{GL}_2$ be the split tori whose points in a $\mathbb{Z}$-algebra R are given by

$$M_\ell(R) := \left\{\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right) : x \in R^\times\right\},$$
$$M_r(R) := \left\{\left(\begin{smallmatrix} 1 & \\ & x\end{smallmatrix}\right) : x \in R^\times\right\},$$

let

$$g = (g_1, g_2, h_1, h_2) \in \mathrm{GL}_2(\mathbb{A}_F)^{\times 2} \times M_\ell(\mathbb{A}_F) \times M_r(\mathbb{A}_F)$$

and let

$$\Sigma^{\mathrm{cusp}}(g) := \sum_\pi K_{\pi(f_1 \mathbb{1}_{\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)})}(g_1, g_2) K_{\pi^\vee(f_2 \mathbb{1}_{\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)})}(h_1, h_2) \mathrm{Res}_{s=1} L(s, \pi \times \pi^{\vee S}).$$

where the sum is over isomorphism classes of cuspidal automorphic representations of $A\backslash G(\mathbb{A}_F)$. Let $\mathfrak{gl}_n$ denote the affine $\mathbb{Z}$-scheme of $n \times n$ matrices and let

$$\begin{aligned} \mathcal{V} &:= \mathfrak{gl}_2 \times \mathbb{G}_a \times \mathbb{G}_a \\ \mathcal{V}' &:= \mathfrak{gl}_2 \times \mathbb{G}_m \times \mathbb{G}_m, \end{aligned} \tag{1.2}$$

viewed as affine schemes over $\mathbb{Z}$. We also let $\mathcal{W} \subset \mathbb{G}_m \times \mathcal{V}'$ denote the closed subscheme whose points in a $\mathbb{Z}$-algebra are given by

$$\mathcal{W}(R) := \left\{(b, T, t_1, t_2) \in R^\times \times \mathcal{V}'(R) : b^{-1} \det T = t_1 t_2\right\}.$$

We note that there is an action

$$\mathrm{GL}_2^{\times 2} \times M_\ell \times M_r \times \mathbb{G}_m \times \mathcal{V} \longrightarrow \mathbb{G}_m \times \mathcal{V} \tag{1.3}$$

given on points by

$$(g_1, g_2, h_3, h_4).(b, T, t_1, t_2) = (b \det g_1 g_2^{-1} h_3^{-1} h_4, g_2^{-1} T g_1, t_1 \det h_3, t_2 \det h_4^{-1}).$$

This action preserves $\mathcal{W}$. By forgetting the $\mathbb{G}_m$ factor, we also obtain an action of $\mathrm{GL}_2^{\times 2} \times M_\ell \times M_r$ on $\mathcal{V}$ that preserves $\mathcal{V}'$.

Define

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{V}(R) \times \mathcal{V}(R) &\longrightarrow R \\ ((\delta, a_1, a_2), (T, t_1, t_2)) &\longmapsto \mathrm{tr}(\delta T) + a_1 t_1 + a_2 t_2. \end{aligned} \tag{1.4}$$

Writing $T = (t_{ij})$, we give $\mathcal{V}(\mathbb{A}_F)$ the additive Haar measure

$$\mathrm{d}v := \mathrm{d}t_{11} \mathrm{d}t_{12} \mathrm{d}t_{21} \mathrm{d}t_{22} \mathrm{d}t_1 \mathrm{d}t_2$$

where $\mathrm{d}x = \prod_v \mathrm{d}x_v$ is the Haar measure on $\mathbb{A}_F$ such that

$$\begin{cases} \mathrm{d}x_v \text{ is the Lebesgue measure if } v \text{ is real,} \\ \mathrm{d}x_v \text{ is twice the Lebesgue measure if } v \text{ is complex,} \\ \mathrm{d}x_v(\mathcal{O}_{F_v}) = 1 \text{ if } v \text{ is nonarchimedian.} \end{cases}$$

In general, all Haar measures in this paper are normalized as in [8, §2, (3.1.1)]. We let $\psi := \psi_{\mathbb{Q}} \circ \mathrm{tr}_{F/\mathbb{Q}}$ where $\psi_{\mathbb{Q}} = \psi_\infty \prod_p \psi_p$ is the unique additive character of $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ that is trivial on $\widehat{\mathbb{Z}}$ and satisfies $\psi_{\mathbb{R}}(x) = e^{-2\pi i x}$ (it is given explicitly in loc. cit.). Our choice of measure is not self-dual with respect to ψ, which leads to the appearance of powers of $d_F^{1/2}$ in our formulae.

For $\beta = (b, \alpha) \in \mathcal{W}(F_S)$, $f_1, f_2 \in C_c^\infty(A\backslash \mathrm{GL}_2(F_S))$, and $V \in C_c^\infty((0,\infty))$, we define an integral transform

$$I_S(f, \beta) := \frac{1}{\zeta_{FS}^\infty(1)} \int_{F_S} \left(\int_{\mathcal{V}(F_S)} V(|\det T|_S) f_1(T) f_2 \begin{pmatrix} -t_1 & \frac{b \det T - t_1 t_2}{t} \\ t & t_2 \end{pmatrix} \right.$$
$$\left. \times\, \psi\left(\frac{\langle \alpha, v\rangle}{t}\right) \frac{\mathrm{d}v}{|\det T|_S} \right) \frac{\mathrm{d}t}{|t|^3}. \tag{1.5}$$

The convergence of this integral is proven in Propositions 3.1 and 3.2 below.

Remark The function V has no relation to the scheme $\mathcal{V}$.

Let

$$g \in \mathrm{GL}_2(\mathbb{A}_F)^{\times 2} \times M_\ell(\mathbb{A}_F) \times M_r(\mathbb{A}_F).$$

The following is the main theorem of this paper:

Theorem 1.1 *If $R(f_1)$ and $R(f_2)$ have cuspidal image, then*

$$\Sigma^{\mathrm{cusp}}(g) = \frac{|\det h_1 h_2^{-1}|}{d_F^{7/2}} \sum_c \sum_{\beta=(b,\alpha)\in\mathcal{W}(F)} |c|_S I_S(f, g.(b, c\alpha)) \mathbb{1}_{\widehat{\mathcal{O}}_F^{S\times} \times \mathcal{V}(\widehat{\mathcal{O}}_F^S)}(g.(b,\alpha)),$$

where the sum on c is over a set of representatives for the nonzero ideals of $\mathcal{O}_F^S$. This sum converges absolutely.

Remark The assumption that $R(f_1)$ and $R(f_2)$ have cuspidal image is only invoked to simplify the spectral side of our expression. In fact, the only place in the paper where the assumption is used is in the assertion (1.7). In principle, this assumption should be no loss of generality spectrally [13].

Originally, we hoped to integrate this kernel over an appropriate subgroup of $A\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)^{\times 4}$ to isolate representations whose isomorphism class is invariant under a simple nonabelian subgroup of $\mathrm{Aut}_{\mathbb{Q}}(F)$. This would necessarily involve some truncation and some sort of version of the Rankin–Selberg method. Later we found an alternate expression for the kernel that will make this process easier [6]. However, Theorem 1.1 can be put to other use immediately. As a concrete example, let

$\chi_1, \chi_2, \chi_3, \chi_4 : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ be a quadruple of characters. By integrating over appropriate split tori and twisting over characters, the formula could be used to study the asymptotics of sums of products of L-functions of the form

$$L(\tfrac{1}{2}, \pi \otimes \chi_1)L(\tfrac{1}{2}, \pi \otimes \chi_2)\overline{L(\tfrac{1}{2}, \pi \otimes \chi_3)L(\tfrac{1}{2}, \pi \otimes \chi_4)}$$

as the analytic conductor of π increases. W. Zhang has also pointed out to the author the possibility of using the main theorem to prove a new Waldspurger-type formula (compare [15, §4.2]) involving products of L-functions as above.

Remark In [8], the authors provided an absolutely convergent geometric expansion of a trace formula that isolates representations whose isomorphism class is invariant under a simple nonabelian group. However, it is not clear how to write the resulting trace formula as a sum of terms that factor along places of $G(\mathbb{A}_F)$. We hope that the present approach and its refinement in [6] will allow us to work around this difficulty.

1.2 Outline of the proof

For $m \in \mathcal{O}_F^S$ let

$$\begin{aligned}\mathbb{1}_m &:= \mathbb{1}_{\{g\in\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S):(\det g)\widehat{\mathcal{O}}_F^S=m\widehat{\mathcal{O}}_F^S\}},\\ \mathbb{1}_{m,m} &:= \mathbb{1}_{m\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)}.\end{aligned}$$

be the usual unramified Hecke operators. For $? \in \{\emptyset, \mathrm{cusp}\}$ let

$$\begin{aligned}\Sigma^?(X) := \Sigma^?(X; g_1, g_2, h_1, h_2) &:= \sum_{a,m}\sum_{y_1,y_2\in F^\times}\frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S}K^?_{f_1\mathbb{1}_{a,a}*\mathbb{1}_m}(g_1, g_2)\\ &\times\int_{(F\backslash\mathbb{A}_F)^{\oplus 2}}K^?_{f_2\mathbb{1}_{a,a}*\mathbb{1}_m}\left(\left(\begin{smallmatrix}1 & t_1\\ & 1\end{smallmatrix}\right)h_1, \left(\begin{smallmatrix}1 & t_2\\ & 1\end{smallmatrix}\right)h_2\right)\psi(y_1t_1 + y_2t_2)\mathrm{d}t_1\,\mathrm{d}t_2, \qquad (1.6)\end{aligned}$$

where the sums on a and m are over a set of representatives for the (principal) nonzero ideals of $\mathcal{O}_F^S$.

The sums over a and m are finite for each X, and sum over $y_1, y_2 \in F^\times$ of the integrals over $F\backslash\mathbb{A}_F$ is part of the Fourier expansion of the smooth function

$$\begin{aligned}(F\backslash\mathbb{A}_F)^{\oplus 2} &\longrightarrow \mathbb{C}\\ (t_1, t_2) &\longmapsto d_F K_{f_2\mathbb{1}_{a,a}*\mathbb{1}_m}\left(\left(\begin{smallmatrix}1 & t_1\\ & 1\end{smallmatrix}\right)h_1, \left(\begin{smallmatrix}1 & t_2\\ & 1\end{smallmatrix}\right)h_2\right)\end{aligned}$$

evaluated at $(t_1, t_2) = 0$; hence, the sums over y_1, y_2 are rapidly decreasing.

Remark The motivation for introducing this partial Fourier expansion is that it has no effect on the cuspidal part of the kernel (compare Proposition 5.1), but eliminates the nongeneric spectrum. If the nongeneric spectrum were included, its contribution would be of size $O(X)$, whereas the cuspidal terms in which we are interested are of size $O(1)$ (since we are dividing by X). In principle, this should be unnecessary, as our assumption that $R(f_1)$ and $R(f_2)$ have cuspidal image also eliminates the nongeneric spectrum, but we do not know how to use the assumption that $R(f_1)$ and $R(f_2)$ have cuspidal image when working with the geometric side of the formula.

By our assumption that $R(f_1), R(f_2)$ have cuspidal image, we have

$$\Sigma^{\text{cusp}}(X) = \Sigma(X), \tag{1.7}$$

and the proof of Theorem 1.1 boils down to computing the limit as $X \to \infty$ of both sides of this expression. Regarding $\Sigma^{\text{cusp}}(X)$, the function of taking a sum over m and taking a limit as $X \to \infty$ is to isolate the pairs of representations π_1, π_2 occurring in the product

$$K_{f_1}^{\text{cusp}}(g_1, g_2)K_{f_2}^{\text{cusp}}(h_1, h_2) = \sum_{\pi_1,\pi_2} K_{\pi_1(f_1)}(g_1, g_2)K_{\pi_2(f_2)}(h_1, h_2)$$

such that $\pi_2 \cong \pi_1^\vee$. We use Rankin–Selberg theory to make the precise and compute $\lim_{X\to\infty} \Sigma^{\text{cusp}}(X)$ in Sect. 5.

The majority of this paper is devoted to computing $\lim_{X\to\infty} \Sigma(X)$ geometrically. The sum $\Sigma(X)$ is divided into two contributions in Sect. 2, namely $\Sigma_1(X)$, corresponding to the first Bruhat cell, and $\Sigma_2(X)$, corresponding the second Bruhat cell (both in the second kernel). We analyze $\Sigma_2(X)$ in Sect. 3. The main result is Theorem 3.6. The limit $\lim_{X\to\infty} \Sigma_1(X)$ turns out to be zero, as proven in Sect. 4. We note that the reason we can execute the limit of the geometric side is that the relevant exponential sums have the same length as the relevant modulus, just as in the beyond endoscopy approach to Rankin–Selberg L-functions exposed in [11] [compare the remark after (3.2) below]. In fact, one could probably use the geometric estimates of $\Sigma(X)$ we give in this paper to prove, independently of Rankin–Selberg theory, that a certain sum of Rankin–Selberg L-functions has an analytic continuation to $\mathrm{Re}(s) > 1 - \delta$ for some $\delta > 0$, but we do not pursue this as it is not our purpose here.

2 First manipulations with the geometric side

We consider, for $X \in \mathbb{R}_{>0}$,

$$\Sigma(X) = \sum_{a,m} \sum_{y_1,y_2\in F^\times} \frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S} K_{f_1\mathbb{1}_{a,a}*\mathbb{1}_m}(g_1, g_2) \times \int_{(F\backslash\mathbb{A}_F)^{\oplus 2}} K_{f_2\mathbb{1}_{a,a}*\mathbb{1}_m}\left(\begin{pmatrix}1 & t_1\\ & 1\end{pmatrix} h_1, \begin{pmatrix}1 & t_2\\ & 1\end{pmatrix} h_2\right) \psi(y_1t_1 + y_2t_2)\mathrm{d}t_1\,\mathrm{d}t_2.$$

Let $B = MN \leq \mathrm{GL}_2$ be the standard Borel subgroup of upper triangular matrices, with M the diagonal matrices and N the unipotent radical. The Bruhat decomposition is

$$\mathrm{GL}_2(F) = B(F) \amalg N(F)w_0B(F), \tag{2.1}$$

where $w_0 = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$. Here $B(F)$ (resp. $N(F)w_0B(F)$) is referred to as the first (resp. second) Bruhat cell. We apply this to write $\Sigma(X) = \Sigma_1(X) + \Sigma_2(X)$ as the sum of the term corresponding to the first Bruhat cell:

$$\Sigma_1(X) := \sum_{a,m} \sum_{y_1,y_2\in F^\times} \frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S} K_{f_1\mathbb{1}_{a,a}*\mathbb{1}_m}(g_1, g_2) \times \int_{(F\backslash\mathbb{A}_F)^{\oplus 2}} \sum_{\delta\in B(F)} f_2\mathbb{1}_m\left(a_S h_1^{-1}\begin{pmatrix}1 & -t_1\\ & 1\end{pmatrix}\delta\begin{pmatrix}1 & t_2\\ & 1\end{pmatrix} h_2\right) \psi(y_1t_1 + y_2t_2)\mathrm{d}t_1\,\mathrm{d}t_2$$

and the second Bruhat cell:

$$\Sigma_2(X) := \sum_{a,m} \sum_{y_1,y_2 \in F^\times} \frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S} K_{f_1 \mathbb{1}_{a,a} * \mathbb{1}_m}(g_1, g_2)$$
$$\times \int_{(F\backslash \mathbb{A}_F)^{\oplus 2}} \sum_{\delta \in N(F)w_0B(F)} f_2\mathbb{1}_m \left(a_S h_1^{-1} \begin{pmatrix} 1 & -t_1 \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix} h_2\right)$$
$$\times \psi(y_1t_1 + y_2t_2)\mathrm{d}t_1\,\mathrm{d}t_2. \tag{2.2}$$

There is a somewhat confusing point hidden in these formulae which we now elucidate. First

$$\mathbb{1}_{a,a} * \mathbb{1}_m(y) = \int_{\mathrm{GL}_2(\mathbb{A}_F^S)} \mathbb{1}_{(a,a)}(x)\mathbb{1}_m(x^{-1}y)\mathrm{d}x = \mathbb{1}_m\left(\left(a^S\right)^{-1} y\right).$$

To arrive at the expressions for $\Sigma_1(X)$ and $\Sigma_2(X)$ above one uses this and then a change of variables $\delta \mapsto a\delta$.

We will compute the limit of these expressions as $X \to \infty$ in the following sections, starting with (2.2). Throughout the remainder of this paper, any unspecified constants are allowed to depend on the quantities $F, S, f_1, f_2, V, g_1, g_2, h_1, h_2$.

3 The second Bruhat cell

We study the contribution (2.2) of the second Bruhat cell. Under the action of $N \times N$ on GL_2 via $(n_1, n_2) \cdot g := n_1^{-1}gn_2$, the stabilizers of elements in the second Bruhat cell are trivial, and each is in the orbit of a unique element of $w_0M(F)$. We therefore have that

$$\Sigma_2(X) = \sum_{a,m} \sum_{y_1,y_2 \in F^\times} \frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S} K_{f_1 \mathbb{1}_{a,a} * \mathbb{1}_m}(g_1, g_2)$$
$$\times \sum_{\delta \in w_0M(F)} \int_{\mathbb{A}_F^{\oplus 2}} f_2\mathbb{1}_m \left(a_S h_1^{-1} \begin{pmatrix} 1 & -t_1 \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix} h_2\right) \psi(y_1t_1 + y_2t_2)\mathrm{d}t_1\,\mathrm{d}t_2. \tag{3.1}$$

We write $\delta = \begin{pmatrix} & b/c \\ c & \end{pmatrix}$ and take a change of variables $(t_1, t_2) \mapsto (c^{-1}t_1, c^{-1}t_2)$ for each c to obtain

$$\sum_{y_1,y_2 \in F^\times} \sum_{\delta \in w_0M(F)} \int_{\mathbb{A}_F^{\oplus 2}} f_2\mathbb{1}_m \left(a_S h_1^{-1} \begin{pmatrix} 1 & -t_1 \\ & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix} h_2\right) \psi(y_1t_1 + y_2t_2)\mathrm{d}t_1\,\mathrm{d}t_2$$
$$= \sum_{b,c,y_1,y_2 \in F^\times} \int_{\mathbb{A}_F^{\oplus 2}} f_2\mathbb{1}_m \left(a_S h_1^{-1} \begin{pmatrix} -t_1 & \frac{b-t_1t_2}{c} \\ c & t_2 \end{pmatrix} h_2\right) \psi\left(\frac{y_1t_1 + y_2t_2}{c}\right) \mathrm{d}t_1\,\mathrm{d}t_2.$$

Substituting this into (3.1) and solving for m, we have

$$\Sigma_2(X) = \sum_{a,m} \frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S} \sum_{B \in \mathrm{GL}_2(F)} \sum_{b,c,y_1,y_2 \in F^\times} f_1\mathbb{1}_m(a_Sg_1^{-1}Bg_2)$$
$$\times \int_{\mathbb{A}_F^{\oplus 2}} f_2\mathbb{1}_m \left(a_S h_1^{-1} \begin{pmatrix} -t_1 & \frac{b-t_1t_2}{c} \\ c & t_2 \end{pmatrix} h_2\right) \psi\left(\frac{y_1t_1 + y_2t_2}{c}\right) \mathrm{d}t_1\,\mathrm{d}t_2$$
$$= \sum_{a} \sum_{B \in \mathrm{GL}_2(F)} \sum_{b,c,y_1,y_2 \in F^\times} \frac{V\left(\frac{|a^2 \det B|_S| \det g_1g_2^{-1}|^S}{X}\right)}{X| \det B|_S| \det g_1g_2^{-1}|^S} f_1\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_Sg_1^{-1}Bg_2)$$

$$\times\, \mathbb{1}_{\det g_1^{-1}g_2h_1h_2^{-1}\widehat{\mathcal{O}}_F^{S\times}}(b)\int_{\mathbb{A}_F^{\oplus 2}} f_2\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}\left(a_S h_1^{-1}\begin{pmatrix} -t_1 & \frac{b\det B-t_1t_2}{c} \\ c & t_2\end{pmatrix} h_2\right)$$
$$\times\, \psi\left(\frac{y_1t_1+y_2t_2}{c}\right)\mathrm{d}t_1\,\mathrm{d}t_2. \tag{3.2}$$

We hope the use of the symbol B for an element of $\mathfrak{gl}_2(F)$ and for the Borel subgroup of GL_2 does not cause confusion. Here we have extended the domain of $\frac{V(t)}{t}$ from $(0,\infty)$ to $\mathbb{R}$ by taking it to be zero outside of $(0,\infty)$ (it remains smooth upon extension because $V\in C_c^\infty((0,\infty))$.

Remark Assume for simplicity that $g_1=g_2=h_1=h_2=I$. The moduli in the sum above are the c. Considering the support of f_2, we see that $|c|_S\ll\sqrt{X}$ and the sum on $B\in\mathfrak{gl}_2(\mathcal{O}_F^S)$ is over matrices whose entries b_{ij} satisfy $|b_{ij}|_S\ll\sqrt{X}$. Thus, Poisson summation in B has a chance of being profitable, and indeed it is, as we will see in the next subsection.

3.1 Poisson summation in *B*

For $n\geq 1$ and $\Psi\in C_c^\infty(\mathfrak{gl}_n(\mathbb{A}_F))$ let

$$\widehat{\Psi}(X):=\int_{\mathfrak{gl}_n(\mathbb{A}_F)}\Psi(Y)\psi(\mathrm{tr}(XY))\mathrm{d}Y \tag{3.3}$$

be the Fourier transform of Ψ. We use the analogous notation in the local setting. Note in particular that

$$\widehat{\mathbb{1}}_{\mathfrak{gl}_n(\widehat{\mathcal{O}}_F^S)}=\mathbb{1}_{\mathfrak{gl}_n(\widehat{\mathcal{O}}_F^S)}.$$

For $g_1,g_2\in\mathrm{GL}_n(\mathbb{A}_F)$ and $a\in\mathbb{A}_F^\times$, the Fourier transform of

$$x\mapsto\Psi(ag_1xg_2)$$

is

$$|a|^{-n^2}|\det g_1g_2|^{-n}\widehat{\Psi}(a^{-1}g_2^{-1}xg_1^{-1}).$$

We also note that the Poisson summation formula holds in the form

$$\sum_{B\in\mathfrak{gl}_n(F)}\Psi(B)=d_F^{-n^2/2}\sum_{B\in\mathfrak{gl}_n(F)}\widehat{\Psi}(B).$$

We apply Poisson summation in $B\in\mathfrak{gl}_2(F)$ to (3.2) to see that $\Sigma_2(X)$ is equal to

$$d_F^{-2}\sum_a\sum_{b,c\in F^\times}\sum_{\alpha\in\mathcal{V}'(F)}\int_{\mathcal{V}(\mathbb{A}_F)}\frac{V\left(\frac{|a^2\det T|_S|\det g_1g_2^{-1}|^S}{X}\right)}{X|\det T|_S|\det g_1g_2^{-1}|^S}f_1\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_Sg_1^{-1}Tg_2)$$
$$\times\,\mathbb{1}_{\det g_1^{-1}g_2h_1h_2^{-1}\widehat{\mathcal{O}}_F^{S\times}}(b)f_2\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}\left(a_Sh_1^{-1}\begin{pmatrix}-t_1 & \frac{b\det T-t_1t_2}{c}\\ c & t_2\end{pmatrix}h_2\right)\psi\left(\frac{\langle\alpha,\nu\rangle}{c}\right)\mathrm{d}\nu. \tag{3.4}$$

Here $\mathcal{V}$ is defined as in (1.2) and $\langle\cdot,\cdot\rangle$ is defined as in (1.4).

Taking a change of variables

$$(T,t_1,t_2)\longmapsto(g_1Tg_2^{-1},t_1\det h_1,t_2\det h_2^{-1})$$

the above becomes

$$\frac{|\det g_1g_2^{-1}h_1h_2^{-1}|}{d_F^2}\sum_a\sum_{b,c\in F^\times}\sum_{\alpha\in\mathcal{V}'(F)}\int_{\mathcal{V}(\mathbb{A}_F)}\frac{V\left(\frac{|a^2\det T|_S|\det g_1g_2^{-1}|}{X}\right)}{X|\det T|_S}f_1\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_ST)$$
$$\times\,\mathbb{1}_{\det g_1^{-1}g_2h_1h_2^{-1}\widehat{\mathcal{O}}_F^{S\times}}(b)f_2\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}\left(a_S\begin{pmatrix}-t_1 & \frac{b\det(g_1g_2^{-1}h_1^{-1}h_2T)-t_1t_2}{c}\\ c & t_2\end{pmatrix}\right)\psi\left(\frac{\langle g.\alpha,\nu\rangle}{c}\right)\mathrm{d}\nu. \tag{3.5}$$

where we have set $g=(g_1,g_2,h_1,h_2)$ and (as in the introduction)

$$g.(T,t_1,t_2)=(g_2^{-1}Tg_1,t_1\det h_1,t_2\det h_2^{-1}).$$

3.2 Bounds on archimedean orbital integrals

In Sect. 3.5, we will apply Poisson summation in $c\in F^\times$ to $\Sigma_2(X)$. In order to work with the resulting sum, we require some bounds that are collected in this section. The archimedean bounds in Proposition 3.1 and the nonarchimedean bounds in Proposition 3.2 are obtained via the stationary phase method.

If w is a place of F, let

$$\left|\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\right|_w:=\max(|b_{ij}|_w). \tag{3.6}$$

Moreover, if $a\in\mathfrak{gl}_2(F_\infty)$ or $a\in F_\infty$ let

$$||a||_\infty=\max_{w|\infty}(|a|_w). \tag{3.7}$$

Proposition 3.1 *Let $f_1\in C_c^\infty(\mathrm{GL}_2(F_\infty))$ and $f_2\in C_c^\infty(A\backslash\mathrm{GL}_2(F_\infty))$. Assume $b\in F_\infty^\times$ satisfies $|b|_\infty\asymp 1$. The function*

$$F_\infty^\times\times\mathcal{V}'(F_\infty)\longrightarrow\mathbb{C}$$
$$(t,\alpha)\longmapsto\int_{\mathcal{V}(F_\infty)}f_1(T)f_2\begin{pmatrix}-t_1 & \frac{b\det T-t_1t_2}{t}\\ t & t_2\end{pmatrix}\psi_\infty\left(\frac{\langle\alpha,\nu\rangle}{t}\right)\mathrm{d}\nu$$

vanishes if b lies outside a compact subset of $F_\infty^\times$ depending only on f_1,f_2 and the bound on $|b|_\infty$.

Moreover, for any $N\in\mathbb{Z}_{\geq0}$, $\beta\in\mathbb{R}_{>0}$, (unitary) character $\chi:F_\infty^\times\to\mathbb{C}^\times$, and $s\in\mathbb{C}$ with $\beta>\mathrm{Re}(s)>-3$ the integral

$$\int_{F_\infty^\times}\left(\int_{\mathcal{V}(F_\infty)}f_1(T)f_2\begin{pmatrix}-t_1 & \frac{b\det T-t_1t_2}{t}\\ t & t_2\end{pmatrix}\psi_\infty\left(\frac{\langle\alpha,\nu\rangle}{t}\right)\mathrm{d}\nu\right)\chi(t)|t|^s\mathrm{d}t^\times$$

is bounded by a constant depending on f_1,f_2,N,β times

$$\max(||y_1||_\infty,||y_2||_\infty,||B||_\infty,C(\chi,\mathrm{Im}(s)))^{-N}\prod_{w|\infty}\min(|y_1|_w,1)^{-2},$$

where $\alpha=(B,y_1,y_2)\in V'(F_\infty)$.

In the proposition, $C(\chi,t)$ is the analytic conductor of χ normalized as in [3, §1].

Remark To clarify the assumptions in the proposition, note that if F has more than one infinite place then the set of $b \in F_\infty^\times$ with $|b|_\infty \asymp 1$ is noncompact.

Proof Choose $\widetilde{f}_2 \in C_c^\infty(\mathrm{GL}_2(F_\infty))$ such that $\int_A \widetilde{f}_2(zg)\mathrm{d}z^\times = f_2(g)$. Then taking a change of variables $(t_1, t_2) \mapsto z^{-1}(t_1, t_2)$, we see that the function in the proposition is equal to

$$\int_{A\times\mathcal{V}(F_\infty)} f_1(T)\widetilde{f}_2\begin{pmatrix} -t_1 & \frac{z^2 b \det T - t_1 t_2}{t} \\ t & t_2 \end{pmatrix} \psi_\infty\left(\frac{y_1 t_1 + y_2 t_2 + z\mathrm{tr}(BT)}{t}\right)\frac{\mathrm{d}\nu\,\mathrm{d}z^\times}{z^2} \tag{3.8}$$

when evaluated at $(z^{-1}t, y_1, y_2, B)$. The function $\widetilde{f}_2$ is evaluated on an element of determinant $-z^2 b \det T$, and hence for the integral to be nonzero b and z must lie in a compact set depending only on $f_1, \widetilde{f}_2$. We may therefore fix b and z and drop the integral over A in the ensuing argument and consider instead of (3.8) the integral

$$\int_{\mathcal{V}(F_\infty)} f_1(T)\widetilde{f}_2\begin{pmatrix} -t_1 & \frac{z^2 b \det T - t_1 t_2}{t} \\ t & t_2 \end{pmatrix} \psi_\infty\left(\frac{\langle \alpha_z, \nu\rangle}{t}\right)\mathrm{d}\nu, \tag{3.9}$$

where $\alpha_z := (zB, y_1, y_2)$.

We now employ an idea from [12] to rewrite this as an integral to which we can apply the stationary phase method. Write

$$f_3\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \int_{F_\infty} \widetilde{f}_2\begin{pmatrix} a_{11} & t \\ a_{21} & a_{22} \end{pmatrix}\psi_\infty(ta_{12})\mathrm{d}t;$$

it is a partial Fourier transform of $\widetilde{f}_2$. Let

$$\mathcal{F}_{x_3}(\nu) := \langle \alpha_z, \nu\rangle - z^2 x_3(b\det T - x_1 x_2).$$

By Fourier inversion, we have that (3.9) is equal to

$$\int_{F_\infty\times\mathcal{V}(F_\infty)} f_1(T)f_3\begin{pmatrix} -x_1 & x_3 \\ t & x_2 \end{pmatrix}\psi_\infty\left(\frac{\mathcal{F}_{x_3}(\nu)}{t}\right)\mathrm{d}\nu\,\mathrm{d}x_3.$$

Here we have renamed variables (so $\mathrm{d}\nu = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}T$). Thus, we are tasked with bounding, for each $w|\infty$ and each character $\chi : F_w^\times \to \mathbb{C}^\times$, the integral

$$\int_{F_w^\times}\left(\int_{F_w\times\mathcal{V}(F_w)} f_1(T)f_3\begin{pmatrix} -x_1 & x_3 \\ t & x_2 \end{pmatrix}\psi_w\left(\frac{\mathcal{F}_{x_3}(\nu)}{t}\right)\mathrm{d}\nu\,\mathrm{d}x_3\right)\chi(t)|t|^s\mathrm{d}t^\times. \tag{3.10}$$

Let $D := t\frac{\partial}{\partial t}$ and, if w is complex, $\overline{D} := \overline{t}\frac{\partial}{\partial \overline{t}}$. We view these as differential operators on $F_w^\times$. Let $f_4 \in C_c^\infty(\mathfrak{gl}_2(F_w))$. Suppose that for all $N \geq 0$, $i \geq 0$ (and if w is complex $j \geq 0$) one has

$$\begin{aligned}&\int_{F_w\times\mathcal{V}(F_w)} f_1(T)D^i\overline{D}^j\left(\frac{f_4\begin{pmatrix} -x_1 & x_3 \\ t & x_2 \end{pmatrix}}{|t|_w^3}\psi_w\left(\frac{\mathcal{F}_{x_3}(\nu)}{t}\right)\right)\mathrm{d}\nu\,\mathrm{d}x_3 \\ &\quad\ll_{f_1,f_4,i,j,N} |z^2 b|^{-6}\max(|y_1|_w, |y_2|_w, |(bz)^{-1}B|_w, 1)^{-N}\min(|y_1|_w, 1)^{-2}.\end{aligned} \tag{3.11}$$

for all t in the support of $f_4\begin{pmatrix} -x_1 & x_3 \\ t & x_2 \end{pmatrix}$ (this is a compact subset of F_w). Here we take $j = 0$ if w is real. Assuming this is the case a repeated application of integration by parts in t (and

$\overline{t}$ when w is complex) implies that (3.10) is convergent for $\mathrm{Re}(s) > -3$ and moreover that (3.10) is bounded by

$$O_{f_1,f_4,N,\beta}(\max(|y_1|_w, |y_2|_w, |(bz)^{-1}B|_w, C(\chi, \mathrm{Im}(s)))^{-N}) \min(|y_1|_w, 1)^{-2}$$

for $\beta > \mathrm{Re}(s) > -3$, and this implies the proposition. But, since f_1 and f_4 were arbitrary, it is not hard to see that the estimate (3.11) follows in general from the special case when $i = j = 0$. In other words, we have reduced the proposition to proving that for all $N \geq 0$ one has

$$\int_{F_w \times \mathcal{V}(F_w)} f_1(T) f_4\left(\begin{smallmatrix} -x_1 & x_3 \\ t & x_2 \end{smallmatrix}\right) \psi_w\left(\frac{\mathcal{F}_{x_3}(v)}{t}\right) \mathrm{d}v\, \mathrm{d}x_3 \ll_{f_1,f_4,N} \left|\frac{t}{b^2z^4}\right|^3 \max(|y_1|_w, |y_2|_w, |(bz)^{-1}B|_w, 1)^{-N} \min(|y_1|_w, 1)^{-2} \tag{3.12}$$

for each $t \in F_w^\times$.

We now apply the stationary phase method to estimate this sum. We view it as a family of phase integrals indexed by x_3. We will estimate, for each $x_3 \in F_w$, the integral

$$\int_{\mathcal{V}(F_w)} f_1(T) f_4\left(\begin{smallmatrix} -x_1 & x_3 \\ t & x_2 \end{smallmatrix}\right) \psi_w\left(\frac{\mathcal{F}_{x_3}(v)}{t}\right) \mathrm{d}v. \tag{3.13}$$

Let $D_{x_3} \subset \mathcal{V}(F_w)$ be the singular locus of $\mathcal{F}_{x_3}(v)$. We have

$$\nabla \mathcal{F}_{x_3}(v) = \begin{pmatrix} zb_{11} - z^2bx_3t_{22} \\ zb_{12} + z^2bx_3t_{12} \\ zb_{21} + z^2bx_3t_{21} \\ zb_{22} - z^2bx_3t_{11} \\ y_1 + x_3x_2 \\ y_2 + x_3x_1 \end{pmatrix}. \tag{3.14}$$

So D_{x_3} is empty if $x_3 = 0$ and otherwise D_{x_3} consists of the single point

$$\left(\frac{b_{22}}{zbx_3}, -\frac{b_{21}}{zbx_3}, -\frac{b_{12}}{zbx_3}, \frac{b_{11}}{zbx_3}, -\frac{y_1}{x_3}, -\frac{y_2}{x_3}\right)$$

and the determinant of the Hessian matrix of $\mathcal{F}_{x_3}(v)$, evaluated at the only point in D_{x_3}, is $\pm x_3^6 z^8 b^4$.

Now if D_{x_3} is not in the support of $f_1(T) f_4\left(\begin{smallmatrix} -x_1 & x_3 \\ t & x_2 \end{smallmatrix}\right)$, then (3.13) can be estimated using the Riemann–Lebesgue lemma. On the other hand, if D_{x_3} is in the support, we obtain a bound on (3.13) of the form $O_{f_1,f_4}\left(|\frac{t^3}{x_3^3 z^4 b^2}|_w\right)$ by the stationary phase method. Thus, we obtain a bound

$$\int_{F_w \times \mathcal{V}(F_w)} f_1(T) f_4\left(\begin{smallmatrix} -x_1 & x_3 \\ t & x_2 \end{smallmatrix}\right) \psi_w\left(\frac{\mathcal{F}_{x_3}(v)}{t}\right) \mathrm{d}v\, \mathrm{d}x_3 \ll_{f_1,f_4} \left|\frac{t}{b^2z^4}\right|_w^3 \int_{|x_3| \gg_{f_1,f_4} \max(|y_1|_w, |y_2|_w, |(bz)^{-1}B|_w)} \frac{f_5(x_3)\mathrm{d}x_3}{|x_3|_w^3}, \tag{3.15}$$

where f_5 is a Schwartz function on F_w. This in turn is bounded by

$$O_{N,f_5}\left(\left|\frac{t}{b^2z^4}\right|_w^3 \max(|y_1|_w, |y_2|_w, |(bz)^{-1}B|_w, 1)^{-N} \max(|y_1|_w, 1)^{-2}\right).$$

□

3.3 Bounds in the ramified nonarchimedean case

For this subsection, let w be a finite place of F. We omit it from notation, writing $F := F_w$. We write ϖ for a uniformizer of $\mathcal{O}_F$ and set $q := |\varpi|^{-1}$. We let $\delta \in \mathcal{O}_F$ be a generator of the absolute different $\mathcal{D}_F$ of $\mathcal{O}_F$. Finally for ideals $\mathfrak{m} \subsetneq \mathcal{O}_F$ we write

$$\mathcal{O}_F^\times(\mathfrak{m}) := 1 + \mathfrak{m}\mathcal{O}_F$$

and $\mathcal{O}_F^\times(\mathcal{O}_F) = \mathcal{O}_F^\times$. For $m \in \mathcal{O}_F - 0$ we also write $\mathcal{O}_F^\times(m) := \mathcal{O}_F^\times(m\mathcal{O}_F)$. Write

$$\alpha = (B, y_1, y_2) \quad \text{and} \quad B = (b_{ij}).$$

In this subsection, we prove the following proposition:

Proposition 3.2 *Let $\chi : F^\times \to \mathbb{C}^\times$ be a (unitary) character, let $f_1, f_2 \in C_c^\infty(\mathfrak{gl}_2(F))$ and assume that $b \in F^\times$. The integral*

$$\int_{F^\times \times \mathcal{V}(F)} f_1(T) f_2\begin{pmatrix} -t_1 & \frac{b \det T - t_1 t_2}{t} \\ t & t_2 \end{pmatrix} \psi\left(\frac{\langle \alpha, \nu\rangle}{t}\right) d\nu \chi(t)|t|^s dt^\times,$$

is absolutely convergent for $\mathrm{Re}(s) > 0$. *For $\alpha \in \mathcal{V}'(F)$ in the expression*

$$\int_{F^\times}\left(\int_{\mathcal{V}(F)} f_1(T) f_2\begin{pmatrix} -t_1 & \frac{b \det T - t_1 t_2}{t} \\ t & t_2 \end{pmatrix} \psi\left(\frac{\langle \alpha, \nu\rangle}{t}\right) d\nu\right) \chi(t)|t|^s dt^\times, \tag{3.16}$$

the integral over $F^\times$ is bounded in absolute value by a constant depending on f_1, f_2, $|b|$, w times $1+q^{6\min(\nu(b_{ij}),\nu(y_i))}\sum_{n=2}^\infty q^{-n(3+s)}$ for $\mathrm{Re}(s) > -3$. *Moreover, it vanishes if $|y_1|$, $|y_2|$, $|B|$ or the absolute norm of the conductor of χ is sufficiently large in a sense depending only on f_1, f_2, and $|b|$.*

Proof It is not hard to see that the integral over $F^\times \times \mathcal{V}(F)$ in (3.16) is absolutely convergent for $\mathrm{Re}(s) > 0$. We therefore assume that $\mathrm{Re}(s) > 0$ until otherwise stated to justify the ensuing manipulations.

Consider

$$\int_{\mathcal{V}(F)} f_1(T) f_2\begin{pmatrix} -t_1 & \frac{b \det T - t_1 t_2}{t} \\ t & t_2 \end{pmatrix} \psi\left(\frac{\langle \alpha, \nu\rangle}{t}\right) d\nu. \tag{3.17}$$

We claim that if $u \in \mathcal{O}_F^\times(\varpi^k)$ and $k \geq 1$ is large enough in a sense depending only on f_1, f_2 then

$$\int_{\mathcal{V}(F)} f_1(T) f_2\begin{pmatrix} -t_1 & \frac{b \det T - t_1 t_2}{ut} \\ ut & t_2 \end{pmatrix} \psi\left(\frac{\langle \alpha, \nu\rangle}{ut}\right) d\nu$$

is equal to (3.17). Indeed, this follows from a change of variables $\nu \mapsto u\nu$. We conclude that the integral (3.16) vanishes if $\chi|_{\mathcal{O}_F^\times(\varpi^k)}$ is nontrivial for k sufficiently large in a sense depending only on f_1 and f_2, in other words, if the conductor of χ is sufficiently large in a sense depending only on f_1, f_2.

After a change of variables $v \mapsto tv$ in (3.17), we see that it is equal to

$$\int_{\mathcal{V}(F)} f_1(tT) f_2 \left(\begin{smallmatrix} -tt_1 & t(b\det T - t_1t_2) \\ t & tt_2 \end{smallmatrix} \right) \psi\left(\langle \alpha, v\rangle\right) |t|^6 \mathrm{d}v.$$

Notice that

$$f_1(tT) f_2 \left(\begin{smallmatrix} -tt_1 & t(b\det T - t_1t_2) \\ t & tt_2 \end{smallmatrix} \right)$$

is invariant under $v \mapsto v + \varpi^k v'$ for any $v' \in \mathcal{V}(\mathcal{O}_F)$, provided that k is sufficiently large in a sense depending only on f_1, f_2 and $|b|$. Thus, (3.17) (and hence (3.16)) vanishes if $|y_1|$, $|y_2|$ or $|B|$ is sufficiently large in a sense depending only on f_1, f_2 and $|b|$.

Thus, we are left with proving the bound claimed in the proposition. As in the proof of Proposition 3.1, we employ a partial Fourier transform as in [12], writing

$$f_3 \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix} \right) := \int_F f_2 \left(\begin{smallmatrix} a_{11} & t \\ a_{21} & a_{22} \end{smallmatrix} \right) \psi(ta_{12}) \mathrm{d}t.$$

Let $d_F \in \mathbb{Z}_{>0}$ be the absolute discriminant of F. By Fourier inversion, we have that (3.16) is equal to d_F^{-1} times

$$\int_{F^\times} \left(\int_{F \times \mathcal{V}(F)} f_1(T) f_3 \left(\begin{smallmatrix} -x_1 & x_3 \\ t & x_2 \end{smallmatrix} \right) \psi\left(\frac{\langle \alpha, v\rangle - x_3(b \det T - x_1x_2)}{t} \right) \mathrm{d}x_3\, \mathrm{d}v \right) \chi(t)|t|^s \mathrm{d}t^\times.$$

We can assume that $f_1 = \mathbb{1}_{\gamma\varpi^{-m} + \varpi^k \mathfrak{gl}_2(\mathcal{O}_F)}$ and $f_2 = \mathbb{1}_{\beta\varpi^{-m} + \varpi^k \mathfrak{gl}_2(\mathcal{O}_F)}$ for some $\gamma, \beta \in \mathfrak{gl}_2(\mathcal{O}_F)$ and $m, k \geq 0$. Thus, the above becomes

$$\int_{F^\times} \left(\int_{F \times \mathcal{V}(F)} \mathbb{1}_{\varpi^k \mathfrak{gl}_2(\mathcal{O}_F)}(T - \gamma\varpi^{-m}) \mathbb{1}_{\varpi^k \mathfrak{gl}_2(\mathcal{O}_F)} \left(\left(\begin{smallmatrix} -x_1 & x_3 \\ t & x_2 \end{smallmatrix} \right) - \beta\varpi^{-m} \right) \right.$$
$$\left. \times\, \psi\left(\frac{\langle \alpha, v\rangle - x_3(b \det T - x_1x_2)}{t} \right) \mathrm{d}x_3\, \mathrm{d}v \right) \chi(t)|t|^s \mathrm{d}t^\times.$$

Applying a change of variables $(v, x_3, t) \mapsto \varpi^{-m}(v, x_3, t)$, we arrive at

$$\chi(\varpi^{-m}) q^{(7+s)m} \int_{F^\times} \left(\int_{F \times \mathcal{V}(F)} \mathbb{1}_{\varpi^{k+m} \mathfrak{gl}_2(\mathcal{O}_F)}(T - \gamma) \mathbb{1}_{\varpi^{k+m} \mathfrak{gl}_2(\mathcal{O}_F)} \left(\left(\begin{smallmatrix} -x_1 & x_3 \\ t & x_2 \end{smallmatrix} \right) - \beta \right) \right.$$
$$\left. \times\, \psi\left(\frac{\varpi^{2m}\langle \alpha, v\rangle - x_3(b \det T - x_1x_2)}{\varpi^{2m} t} \right) \mathrm{d}x_3\, \mathrm{d}v \right) \chi(t)|t|^s \mathrm{d}t^\times.$$

The factor $\chi(\varpi^{-m}) q^{(7+s)m}$ is inessential for our purposes so we drop it.

Now let $\ell \geq 0$ be the smallest integer such that $\varpi^\ell b \in \mathcal{O}_F$. We can then write the above as

$$\int_{F^\times} \left(\int_{F \times \mathcal{V}(F)} \mathbb{1}_{\varpi^{k+m} \mathfrak{gl}_2(\mathcal{O}_F)}(T - \gamma) \mathbb{1}_{\varpi^{k+m} \mathfrak{gl}_2(\mathcal{O}_F)} \left(\left(\begin{smallmatrix} -x_1 & x_3 \\ t & x_2 \end{smallmatrix} \right) - \beta \right) \right.$$
$$\left. \times\, \psi\left(\frac{\varpi^{2m+\ell}\langle \alpha, v\rangle - x_3(b\varpi^\ell \det T - \varpi^\ell x_1x_2)}{\varpi^{2m+\ell} t} \right) \mathrm{d}x_3\, \mathrm{d}v \right) \chi(t)|t|^s \mathrm{d}t^\times. \tag{3.18}$$

To bound this integral, we can and do assume $\chi = 1$. Write $\gamma = (\gamma_{ij})$, $\beta = (\beta_{ij})$. To ease notation, let

$$\mathcal{F}_{x_3}(v) := \delta\varpi^{2m+\ell}\langle \alpha, v\rangle - \delta x_3(b\varpi^{\ell}\det T - \varpi^{\ell}x_1x_2)$$

where $\delta \in \mathcal{O}_F$ is a generator for the absolute different of F. Then, (3.18), in the special case $\chi = 1$, is

$$\int_{F^\times}\left(\int_{F\times\mathcal{V}(F)} \mathbb{1}_{\varpi^{k+m}\mathfrak{gl}_2(\mathcal{O}_F)}(T-\gamma)\mathbb{1}_{\varpi^{k+m}\mathfrak{gl}_2(\mathcal{O}_F)}\begin{pmatrix} -x_1-\beta_{11} & x_3-\beta_{12} \\ t-\beta_{21} & x_2-\beta_{22}\end{pmatrix} \times \psi\left(\frac{\mathcal{F}_{x_3}(v)}{\delta\varpi^{2m+\ell}t}\right) \mathrm{d}x_3\,\mathrm{d}v\right)|t|^s \mathrm{d}t^\times.$$

We first observe that if $\varpi^{k+m} \nmid \beta_{21}$ then $|t| = |\beta_{21}|$ for all t in the support of the integrand, and it is easy to obtain the bound asserted in the lemma in this case. If $\varpi^{k+m} \mid \beta_{21}$, then the integral above is equal to

$$\int_{F^\times}\left(\int_{F\times\mathcal{V}(F)} \mathbb{1}_{\varpi^{k+m}\mathfrak{gl}_2(\mathcal{O}_F)}(T-\gamma)\mathbb{1}_{\varpi^{k+m}\mathfrak{gl}_2(\mathcal{O}_F)}\begin{pmatrix} -x_1-\beta_{11} & x_3-\beta_{12} \\ t & x_2-\beta_{22}\end{pmatrix} \times \psi\left(\frac{\mathcal{F}_{x_3}(v)}{\delta\varpi^{2m+\ell}t}\right) \mathrm{d}x_3\,\mathrm{d}v\right)|t|^s \mathrm{d}t^\times.$$

We claim that is bounded by a constant depending on $k, m, \ell, |b|, \beta, \gamma$ times

$$1 + q^{6\min(w(y_i),w(b_{ij}))}\sum_{n=2}^{\infty} q^{-n(s+3)}$$

for $\mathrm{Re}(s) > -3$; establishing this claim will complete the proof of the proposition. To prove the claim, it suffices to show that

$$\int_{F\times\mathcal{V}(F)} \mathbb{1}_{\varpi^{k+m}\mathfrak{gl}_2(\mathcal{O}_F)}(T-\gamma)\mathbb{1}_{\varpi^{k+m}\mathfrak{gl}_2(\mathcal{O}_F)}\begin{pmatrix} -x_1-\beta_{11} & x_3-\beta_{12} \\ t & x_2-\beta_{22}\end{pmatrix} \psi\left(\frac{\mathcal{F}_{x_3}(v)}{\delta\varpi^{2m+\ell}t}\right) \mathrm{d}x_3\,\mathrm{d}v \tag{3.19}$$

is bounded by a constant depending on $k, m, \ell, |b|, \beta, \gamma$ times $|t|^3$, provided that $w(t) \geq \max(2k, 2)$ (for $0 \leq w(t) \leq \max(2k, 2)$ we can just bound the integral trivially).

For this, we can apply the stationary phase method in the nonarchimedean setting as developed in [4]. In more detail, let p be the rational prime lying below w and let

$$D_{x_3} := \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}\mathrm{Spec}\left(\mathcal{O}_F[x_1,x_2,T]/\left(\nabla\mathcal{F}_{x_3}\right)\right) \subset \mathbb{A}^{6[F:\mathbb{Q}_p]}_{\mathbb{Z}_p} \quad (\text{affine } 6[F:\mathbb{Q}_p]\text{-space})$$

where $(\nabla\mathcal{F}_{x_3})$ is the ideal generated by the entries of the gradient $\nabla\mathcal{F}_{x_3}$. This gradient is

$$\nabla\mathcal{F}_{x_3}(x_1,x_2,T) = \delta\varpi^{\ell}\begin{pmatrix} \varpi^{2m}b_{11} - bx_3t_{22} \\ \varpi^{2m}b_{12} + bx_3t_{12} \\ \varpi^{2m}b_{21} + bx_3t_{21} \\ \varpi^{2m}b_{22} - bx_3t_{11} \\ \varpi^{2m}y_1 + x_3x_2 \\ \varpi^{2m}y_2 + x_3x_1 \end{pmatrix}.$$

Since $w(t) > \max(2k, m)$, we have

$$w(t) + 2m + \ell \geq \max(2(k+m), 2).$$

Therefore, by [4, Theorem 1.8(a)] we have that (3.19) is bounded by a constant times

$$q^{6a} \max_{x_3 \in \mathcal{O}_F} |D_{x_3}(\mathbb{Z}_p/p^a)|,$$

where $a = \lfloor \frac{w(t)+2m+\ell}{2} \rfloor$. But

$$\max_{x_3 \in \mathcal{O}_F} |D_{x_3}(\mathbb{Z}_p/p^a)| \ll_{m,k} q^{6\min(w(y_i), w(b_{ij}))},$$

so we deduce the proposition. □

3.4 The unramified computation

Fix a (finite) place $w \notin S$ of F. As in the previous subsection, in this subsection we omit w from notation, write $F := F_w$, let ϖ be a uniformizer of F and set $q := |\mathcal{O}_F/\varpi|$. We fix $b \in \mathcal{O}_F^\times$ for the section. Let $\chi : F^\times \to \mathbb{C}^\times$ be a (unitary) character. Moreover, let

$$P(b, v) := b \det T - t_1 t_2.$$

In this section, we prove the following proposition:

Proposition 3.3 *The integral*

$$\int_{\mathcal{O}_F} \int_{\mathcal{V}(\mathcal{O}_F)} \mathbb{1}_{t\mathcal{O}_F}(P(b, v)) \psi\left(\frac{\langle \alpha, v \rangle}{t}\right) \mathrm{d}v \chi(t) |t|^s \mathrm{d}t^\times$$

is absolutely convergent for $\mathrm{Re}(s) > -3$. *It vanishes if* χ *is ramified. If* χ *is unramified, it is equal to*

$$L(4+s, \chi)^{-1} \sum_{k=0}^{\infty} \frac{\chi(\varpi^k)}{q^{k(1+s)}} \mathbb{1}_{\mathcal{V}(\mathcal{O}_F)}\left(\varpi^{-k}\alpha\right) \int_{\mathcal{O}_F} \mathbb{1}_t(P(b^{-1}, \varpi^{-k}\alpha)) \chi(t) |t|^{s+3} \mathrm{d}t^\times.$$

We start with two preparatory lemmas:

Lemma 3.4 *For* $t \in \mathcal{O}_F$ *one has*

$$\int_{\mathcal{V}(\mathcal{O}_F)} \psi\left(\frac{P(b, v)}{t}\right) \mathrm{d}v = |t|^3.$$

Proof Assume first that $w(t) = 1$. Then writing $v = \left(\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right), z_1, z_2\right)$,

$$\int_{\mathcal{V}(\mathcal{O}_F)} \psi\left(\frac{P(b, v)}{t}\right)$$
$$= |t|^6 \sum_{x_1, x_2, x_3, x_4, z_1, z_2 \in \mathcal{O}_F/\varpi} \psi\left(\frac{b(x_1x_4 - x_2x_3) - z_1z_2}{t}\right)$$

Let $\mathbb{P}\mathcal{V}$ be the projectivization of the $\mathcal{O}_F$-module $\mathcal{V}$. Then grouping elements of $\mathcal{V}(\mathcal{O}_F/\varpi)\setminus 0$ according to their image in $\mathbb{P}\mathcal{V}(\mathcal{O}_F/\varpi)$ and evaluating the resulting Ramanujan sums, we see that the above is equal to

$$q^{-6}\left(1+\sum_{\substack{(x_1,x_2,x_3,x_4,z_1,z_2)\in\mathcal{P}\mathcal{V}(\mathcal{O}_F/\varpi)\\ b(x_1x_4-x_2x_3)=z_1z_2}} q-\sum_{(x_1,x_2,x_3,x_4,z_1,z_2)\in\mathcal{P}\mathcal{V}(\mathcal{O}_F/\varpi)} 1\right)$$

$$=q^{-6}\left(1-\frac{q^6-1}{q-1}+\sum_{\substack{(x_1,x_2,x_3,x_4,z_1,z_2)\in\mathbb{P}\mathcal{V}(\mathcal{O}_F/\varpi)\\ b(x_1x_4-x_2x_3)=z_1z_2}} q\right) \tag{3.20}$$

To count the points

$$(x_1,x_2,x_3,x_4,z_1,z_2)\in\mathbb{P}\mathcal{V}(\mathcal{O}_F/\varpi).$$

satisfying

$$b(x_1x_4-x_2x_3)=z_1z_2,$$

we observe that there are q^4 points with $z_1\neq 0$, q^3 points with $z_1=0, x_1\neq 0$, q^2 points with $z_1=x_1=0$, $x_2\neq 0$, and $\frac{q^3-1}{q-1}$ points with $z_1=x_1=x_2=0$. Thus, we end up with

$$q^4+q^3+2q^2+q+1$$

points. We deduce that (3.20) is equal to q^{-3}.

We now consider the case $w(t)>1$. Let

$$\nabla P(b,v)=\begin{pmatrix} bx_4\\ -bx_3\\ -bx_2\\ bx_1\\ -z_2\\ -z_1\end{pmatrix}$$

be the gradient of $P(b,v)$ and let

$$H(b,v):=\begin{pmatrix} &&&b&&\\ &&-b&&&\\ &-b&&&&\\ b&&&&&\\ &&&&&-1\\ &&&&-1& \end{pmatrix}$$

be the Hessian matrix of $P(b,v)$. Let p be the rational prime below w and let

$$D:=\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}(\mathcal{O}_F^6[v]/(\nabla P(b,v)))\subset\mathbb{A}^{6[F:\mathbb{Q}_p]},$$

where $(\nabla P(b,v))$ is the ideal generated by the entries of $\nabla P(b,v)$. Thus, D is a closed affine subscheme of $\mathbb{A}^{6[F:\mathbb{Q}_p]}$ that is étale over $\mathbb{Z}_p$ since $H(b,v)\in\mathrm{GL}_6(\mathcal{O}_F)$. Applying [4, Theorem 1.4] (a result which the authors attribute to Katz), one has

$$\int_{\mathcal{V}(\mathcal{O}_F)}\psi\left(\frac{P(b,v)}{t}\right)\mathrm{d}v=|t|^3\sum_{v\in D(\mathbb{Z}_p)}\psi\left(\frac{P(b,v)}{t}\right)G_t(H(b,v)), \tag{3.21}$$

where

$$G_t(H(b,v)) := \begin{cases} 1 & \text{if } 2|w(t), \\ q^{-3}\sum_{X\in(\mathcal{O}_F/\varpi)^6}\psi\left(\frac{X^tH(b,v)X}{2\varpi}\right) & \text{if } 2\nmid w(t). \end{cases} \tag{3.22}$$

Now $D(\mathbb{Z}_p)=0$, and for $2\nmid w(t)$, one has

$$G_t(H(b,v)) = q^3\int_{\mathcal{V}(\mathcal{O}_F)}\psi\left(\frac{P(b,v)}{\varpi}\right)\mathrm{d}v = 1.$$

Thus, altogether we deduce that (3.21) is equal to $|t|^3$ as claimed. □

Lemma 3.5 *Assume that $x\in\mathcal{O}_F^\times$. The integral*

$$\int_{\mathcal{V}(\mathcal{O}_F)}\psi\left(\frac{xP(b,v)+\langle\alpha,v\rangle}{t}\right)\mathrm{d}v$$

vanishes unless $\alpha\in\mathcal{V}(\mathcal{O}_F)$ and χ is unramified, in which case it is equal to

$$|t|^3\psi\left(\frac{-P(b^{-1},\alpha)}{xt}\right). \tag{3.23}$$

Proof It is easy to see that the integral vanishes unless $\alpha\in\mathcal{V}(\mathcal{O}_F)$. We henceforth assume $\alpha\in\mathcal{V}(\mathcal{O}_F)$. We observe that

$$P(b,v+v') = P(b,v)+P(b,v')+\langle f(v'),v\rangle, \tag{3.24}$$

where f is the $\mathcal{O}_F$-linear isomorphism

$$\begin{aligned} f:\mathcal{V}(\mathcal{O}_F) &\longrightarrow \mathcal{V}(\mathcal{O}_F) \\ \left(\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4\end{smallmatrix}\right),t_1,t_2\right) &\longmapsto \left(b\left(\begin{smallmatrix} x_4 & -x_2 \\ -x_3 & x_1\end{smallmatrix}\right),-t_2,-t_1\right). \end{aligned}$$

Thus,

$$\begin{aligned} &\int_{\mathcal{V}(\mathcal{O}_F)}\psi\left(\frac{xP(b,v)+\langle\alpha,v\rangle}{t}\right)\mathrm{d}v \\ &= \int_{\mathcal{V}(\mathcal{O}_F)}\psi\left(\frac{xP(b,v-x^{-1}f^{-1}(\alpha))+\langle\alpha,v\rangle-\langle\alpha,x^{-1}f^{-1}(\alpha)\rangle}{t}\right)\mathrm{d}v \\ &= \int_{\mathcal{V}(\mathcal{O}_F)}\psi\left(\frac{xP(b,v)+xP(b,-x^{-1}f^{-1}(\alpha))-\langle\alpha,x^{-1}f^{-1}(\alpha)\rangle}{t}\right)\mathrm{d}v \\ &= \psi\left(\frac{-P(b^{-1},\alpha)}{xt}\right)\int_{\mathcal{V}(\mathcal{O}_F)}\psi\left(\frac{xP(b,v)}{t}\right)\mathrm{d}v. \end{aligned}$$

Invoking Lemma 3.4, we see that this is equal to

$$|t|^3\psi\left(\frac{-P(b^{-1},\alpha)}{xt}\right).$$

□

Proof of Proposition 3.3 One has

$$\begin{aligned}&\int_{\mathcal{O}_F}\int_{\mathcal{V}(\mathcal{O}_F)}\mathbb{1}_{t\mathcal{O}_F}(P(b,v))\psi\left(\frac{\langle\alpha,v\rangle}{t}\right)\mathrm{d}v\chi(t)|t|^s\mathrm{d}t^{\times}\\&\quad=\int_{\mathcal{O}_F}\int_{\mathcal{V}(\mathcal{O}_F)}\int_{\mathcal{O}_F}\psi\left(\frac{xP(b,v)+\langle\alpha,v\rangle}{t}\right)\mathrm{d}x\,\mathrm{d}v\chi(t)|t|^s\,\mathrm{d}t^{\times}\\&\quad=\int_{\mathcal{O}_F}\sum_{k=0}^{v(t)}\int_{\mathcal{V}(\mathcal{O}_F)}\sum_{x\in(\mathcal{O}_F/t\varpi^{-k})^{\times}}\psi\left(\frac{x\varpi^kP(b,v)+\langle\alpha,v\rangle}{t}\right)\mathrm{d}v\chi(t)|t|^{s+1}\,\mathrm{d}t^{\times}\end{aligned}$$

We take a change of variables $t\mapsto\varpi^kt$ to arrive at

$$\sum_{k=0}^{\infty}\frac{\chi(\varpi^k)}{q^{k(s+1)}}\int_{\mathcal{O}_F}\int_{\mathcal{V}(\mathcal{O}_F)}\sum_{x\in(\mathcal{O}_F/t)^{\times}}\psi\left(\frac{xP(b,v)+\langle\varpi^{-k}\alpha,v\rangle}{t}\right)\mathrm{d}v\chi(t)|t|^{s+1}\mathrm{d}t^{\times}.$$

We now invoke Lemma 3.5 to see that this vanishes if χ is ramified and otherwise it is equal to

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\chi(\varpi^k)}{q^{k(s+1)}}\int_{\mathcal{O}_F}\sum_{x\in(\mathcal{O}_F/t)^{\times}}\mathbb{1}_{\mathcal{V}(\mathcal{O}_F)}\left(\varpi^{-k}\alpha\right)\psi\left(\frac{-P(b^{-1},\varpi^{-k}\alpha)}{xt}\right)\chi(t)|t|^{s+4}\,\mathrm{d}t^{\times}\\&\quad=\sum_{k=0}^{\infty}\frac{\chi(\varpi^k)}{q^{k(s+1)}}\mathbb{1}_{\mathcal{V}(\mathcal{O}_F)}\left(\varpi^{-k}\alpha\right)\Bigg(\int_{\mathcal{O}_F}\mathbb{1}_{t\mathcal{O}_F}(P(b^{-1},\varpi^{-k}\alpha)\chi(t)|t|^{s+3}\mathrm{d}t^{\times}\\&\quad-q^{-1}\int_{\varpi\mathcal{O}_F}\mathbb{1}_{t\varpi^{-1}}(P(b^{-1},\varpi^{-k}\alpha))\chi(t)|t|^{s+3}\mathrm{d}t^{\times}\Bigg)\\&\quad=\sum_{k=0}^{\infty}\frac{\chi(\varpi^k)}{q^{k(1+s)}}\mathbb{1}_{\mathcal{V}(\mathcal{O}_F)}\left(\varpi^{-k}\alpha\right)\left(1-\chi(\varpi)q^{-4-s}\right)\\&\times\int_{\mathcal{O}_F}\mathbb{1}_t(P(b^{-1},\varpi^{-k}\alpha))\chi(t)|t|^{s+3}\mathrm{d}t^{\times}.\end{aligned}$$

□

3.5 Poisson summation in *c*

Recall that (3.5) gives us the following equality:

$$\begin{aligned}\Sigma_2(X)=\frac{|\det g_1g_2^{-1}h_1h_2^{-1}|}{d_F^2}&\sum_a\sum_{b,c\in F^{\times}}\sum_{\alpha\in\mathcal{V}'(F)}\int_{\mathcal{V}(\mathbb{A}_F)}\frac{V\left(\frac{|a^2\det T|_S|\det g_1g_2^{-1}|}{X}\right)}{X|\det T|_S}\\&\times f_1\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_ST)\\&\times\mathbb{1}_{\det g_1^{-1}g_2h_1h_2^{-1}\widehat{\mathcal{O}}_F^{S\times}}(b)f_2\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}\left(a_S\begin{pmatrix}-t_1&\frac{\frac{b\det(g_1g_2^{-1}h_1^{-1}h_2T)-t_1t_2}{c}}{t_2}\\c&\end{pmatrix}\right)\\&\times\psi\left(\frac{\langle g.\alpha,v\rangle}{c}\right)\mathrm{d}v.\end{aligned}\tag{3.25}$$

where we have set $g=(g_1,g_2,h_1,h_2)$. In this subsection, we apply Poisson summation in c to this expression and asymptotically evaluate the resulting sum. The main result follows:

Theorem 3.6 *The limit* $\lim_{X\to\infty}\Sigma_2(X)$ *exists and is equal to the absolutely convergent sum*

$$\frac{|\det h_1h_2^{-1}|}{d^{5/2}}\sum_{c}\sum_{(b,\alpha)\in\mathcal{W}(F)}|c|_S I_S(f,g.(b,c\alpha))\mathbb{1}_{\widehat{\mathcal{O}}_F^{S\times}\times\mathcal{V}(\widehat{\mathcal{O}}_F^S)}(g.(b,\alpha)).$$

Here $I_S(f,\beta)$ is defined as in (1.5).

Proof We apply Poisson summation in $c\in F^\times$ to (3.25) to arrive at

$$\begin{aligned}\Sigma_2(X)=&\frac{|\det g_1g_2^{-1}h_1h_2^{-1}|}{d_F^{5/2}2\pi i\mathrm{Res}_{s=1}\zeta_F^\infty(s)}\\&\times\sum_a\sum_{(b,\alpha)\in F^\times\times\mathcal{V}'(F)}\sum_\chi\int_{\mathrm{Re}(s)=2}\int_{\mathbb{A}_F^\times}\left(\int_{\mathcal{V}(\mathbb{A}_F)}\frac{V\left(\frac{|a^2\det T|_S|\det g_1g_2^{-1}|}{X}\right)}{X|\det T|_S}\right.\\&\times f_1\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_ST)\\&\times\mathbb{1}_{\det g_1^{-1}g_2h_1h_2^{-1}\widehat{\mathcal{O}}_F^{S\times}}(b)f_2\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}\left(a_S\begin{pmatrix}-t_1&\frac{b\det g_1g_2^{-1}h_1^{-1}h_2T-t_1t_2}{t}\\t&t_2\end{pmatrix}\right)\\&\times\left.\psi\left(\frac{\langle g.\alpha,v\rangle}{t}\right)\mathrm{d}v\right)\chi(t)|t|^s\mathrm{d}t^\times\mathrm{d}s\end{aligned}\tag{3.26}$$

Here the sum on χ is over $(AF^\times\backslash\mathbb{A}_F^\times)^\wedge$. A convenient reference for this multiplicative version of Poisson summation is [1, §2]. We warn the reader that there is a difference of measure; if $\mathrm{d}x_{BB}^\times$ is the measure used in [1, §2], then

$$\mathrm{d}x_{BB}^\times=\zeta_{F\infty}(1)d_F^{-1/2}\mathrm{d}x^\times$$

where $\mathrm{d}x^\times$ is our measure. We will discuss justifying this application of Poisson summation in just a moment. The nonarchimedian integral

$$\begin{aligned}&\int_{\mathbb{A}_F^{S\times}}\int_{\mathcal{V}(\mathbb{A}_F^S)}\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(T)\mathbb{1}_{\det g_1^{-1}g_2h_1h_2^{-1}\widehat{\mathcal{O}}_F^{S\times}}(b)\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}\begin{pmatrix}-t_1&\frac{bg_1g_2^{-1}h_1^{-1}h_2\det T-t_1t_2}{t}\\t&t_2\end{pmatrix}\\&\times\psi\left(\frac{\langle g.\alpha,v\rangle}{t}\right)\mathrm{d}v\Big)\chi(t)|t|^s\mathrm{d}t^\times\end{aligned}$$

was computed in Proposition 3.3; it vanishes unless χ is unramified outside of S, in which case it is equal to

$$\begin{aligned}&\sum_c\frac{\chi^S(c)}{|c|_S^{1+s}}\mathbb{1}_{\widehat{\mathcal{O}}_F^{S\times}\times\mathcal{V}(\widehat{\mathcal{O}}_F^S)}\left(g.(b,c^{-1}\alpha)\right)L^S(s+4,\chi)^{-1}\\&\times\int_{\widehat{\mathcal{O}}_F^S}\mathbb{1}_t(P^\vee(g.(b,c^{-1}\alpha))\chi^S(t)(|t|^S)^{s+3}\mathrm{d}t^\times\end{aligned}$$

where the sum on c is over the nonzero ideals of $\mathcal{O}_F^S$ and $P^\vee(b,\alpha):=P(b^{-1},\alpha)$.

For $\alpha \in \mathcal{V}'(F_S)$ let

$$I_S(f,(b,\alpha),\chi,s) = \frac{1}{\zeta_{FS}^{\infty}(1)}\int_{F_S^\times}\left(\int_{\mathcal{V}(F_S)}\frac{V(|\det T|_S)}{|\det T|_S}f_1(T)f_2\begin{pmatrix} t_1 & \frac{b\det T - t_1t_2}{t} \\ t & t_2\end{pmatrix} \times\psi\left(\frac{\langle\alpha,\nu\rangle}{t}\right)\mathrm{d}\nu\right)\chi(t)|t|_S^s\mathrm{d}t^\times$$

and

$$D_{\chi,b,\alpha}(s) := L^S(s+4,\chi)^{-1}\int_{\widehat{\mathcal{O}}_F^S}\mathbb{1}_t(P^\vee(g.(b,\alpha)))\chi^S(t)|t|^{s+3}\mathrm{d}t^\times. \tag{3.27}$$

We note that for $\mathrm{Re}(s) > -3$ the transform $I_S(f,(b,\alpha),\chi,s)$ is rapidly decreasing as a function of α, χ and the analytic conductor of $\chi|\cdot|^s$ in a sense made precise by propositions 3.1 and 3.2.

Let

$$\begin{aligned}\Delta : \mathbb{R}_{>0} &\longrightarrow F_\infty^\times \\ z &\longmapsto z^{[F:\mathbb{Q}]^{-1}}\end{aligned} \tag{3.28}$$

where $z^{[F:\mathbb{Q}]^{-1}}$ is embedded diagonally. Taking a change of variables

$$((t,\nu),\alpha) \longmapsto \left(\Delta\left(\sqrt{\frac{X}{|\det g_1g_2^{-1}|}}\right)a_S^{-1}(t,\nu), c\alpha\right),$$

using the A-invariance of f_1, f_2, and bearing in mind the nonarchimedian computation just mentioned, we see that

$$\Sigma_2(X) = \frac{d_F^{-5/2}|\det h_1h_2^{-1}|}{2\pi i \mathrm{Res}_{s=1}\zeta_F^S(s)}\sum_{a,c}\sum_{(b,\alpha)\in F^\times\times\mathcal{V}'(F)}\sum_{\chi}\int_{\mathrm{Re}(s)=\sigma}\frac{\chi^S(a)\chi^S(c)}{|a|_S^{4+s}|c|_S^{1+s}} \times I_S(f, g.(b,c\alpha),\chi,s)\left(\frac{X}{|\det g_1g_2^{-1}|}\right)^{1+s/2}D_{\chi,b,\alpha}(s)\mathbb{1}_{\widehat{\mathcal{O}}_F^{S\times}\times\mathcal{V}(\widehat{\mathcal{O}}_F^S)}(g.(b,\alpha))\,\mathrm{d}s$$

where the sum on χ is over characters of $F^\times\backslash\mathbb{A}_F^\times/\widehat{\mathcal{O}}_F^{S\times}$. We note that we have used the fact that $\chi_S(a_S^{-1}) = \chi^S(a)$ to simplify the expression above. We now can justify our application of Poisson summation in c by noting that

$$|D_{\chi,b,\alpha}(s)| \le \zeta^S(s+4)\zeta^S(s+3)$$

for χ, b, α contributing to the sum and applying the estimates of Propositions 3.1 and 3.2.

We now move the contour of the integral over s to the line $\mathrm{Re}(s) = -\frac{5}{2}$. The integral $D_{\chi,b,\alpha}(s)$ is absolutely convergent in this range unless $P(b^{-1},\alpha) = 0$ (which occurs if and only if $P(b^{-1}\det g_1^{-1}g_2h_1h_2^{-1}, g.\alpha) = 0$). On the other hand, if $P(b^{-1},\alpha) = 0$, which is to say that $(b,\alpha) \in \mathcal{W}(F)$, then

$$D_{\chi,b,\alpha}(s) = \frac{L^S(s+3,\chi)}{L^S(s+4,\chi)}.$$

which is meromorphic in $\mathrm{Re}(s) > -3$, in fact holomorphic apart from a possible simple pole at $s = -2$. The simple pole only occurs if χ is trivial, in which case it has residue

$$\frac{\mathrm{Res}_{s=1}\zeta_F^S(s)}{\zeta_F^S(2)}.$$

Thus, we $\Sigma_2(X)$ is equal to the contribution of the residues plus a remainder term. The contribution of the residues is

$$\frac{d_F^{-5/2}|\det h_1h_2^{-1}|}{\mathrm{Res}_{s=1}\zeta_F^S(s)}\sum_{a,c}\frac{1}{|a|_S^2}\sum_{\beta=(b,\alpha)\in\mathcal{W}(F)}I_S(f,g.(b,c\alpha),1,-2)|c|_S\frac{\mathrm{Res}_{s=1}\zeta_F^S(s)}{\zeta^S(2)}$$
$$\times\mathbb{1}_{\widehat{\mathcal{O}}_F^{S\times}\times\mathcal{V}(\widehat{\mathcal{O}}_F^S)}(g.\alpha)$$
$$=\frac{|\det h_1h_2^{-1}|}{d_F^{5/2}}\sum_c\sum_{(b,\alpha)\in\mathcal{W}(F)}|c|_SI_S(f,g.(b,c\alpha))\mathbb{1}_{\widehat{\mathcal{O}}_F^{S\times}\times\mathcal{V}(\widehat{\mathcal{O}}_F^S)}(g.(b,\alpha)),$$

where we have set

$$I_S(f,(b,\alpha))=I_S(f,(b,\alpha),1,-2) \tag{3.29}$$

as in the introduction. Here we are using the fact that $dt_{S\setminus\infty}=\zeta_{FS\setminus\infty}(1)^{-1}dt^\times_{S\setminus\infty}$.

This is precisely the expression for $\lim_{X\to\infty}\Sigma_2(X)$ asserted in Theorem 3.6. The sum on c,b,α is absolutely convergent by Propositions 3.1 and 3.2. Thus, to complete the proof of Theorem 3.6 it suffices to prove that the remainder term mentioned above is indeed a remainder term. This remainder term is $\frac{d_F^{-5/2}|\det h_1h_2^{-1}|}{2\pi i\mathrm{Res}_{s=1}\zeta_F^\infty(s)}$ times

$$\sum_{a,c}\sum_{(b,\alpha)\in F^\times\times\mathcal{V}'(F)}\sum_\chi$$
$$\times\int_{\mathrm{Re}(s)=-\frac{5}{2}}\frac{\chi^S(ac)}{|a|_S^{4+s}|c|_S^{1+s}}I_S(f,g.(b,c\alpha),\chi,s)\left(\frac{X}{|\det g_1g_2^{-1}|}\right)^{1+s/2}$$
$$\times D_{\chi,b,\alpha}(s)\mathbb{1}_{\widehat{\mathcal{O}}_F^{S\times}\times\mathcal{V}(\widehat{\mathcal{O}}_F^S)}(g.(b,\alpha))\,ds. \tag{3.30}$$

Notice that the sum over a causes no problems since it only appears via the factor $\frac{\chi^S(a)}{|a|_S^{4+s}}$ and $\sum_a|a|_S^{-4-s}$ converges absolutely for $\mathrm{Re}(s)=-\frac{5}{2}$. Moreover, as mentioned above,

$$I_S(f,g.(b,c\alpha),\chi,s) \tag{3.31}$$

is rapidly decreasing as a function of b,α,c and the analytic conductor of $\chi|\cdot|^s$ is a sense made precise in Propositions 3.1 and 3.2. As for the factor $D_{\chi,b,\alpha}$, we observe that in view of (3.27) there is an $A>0$ such that if b,α,χ contribute to the sum above and $P(b^{-1},\alpha)\neq0$ and $\mathrm{Re}(s)=-\frac{5}{2}$ then

$$|D_{\chi,b,\alpha}(s)|\leq|P(b^{-1},\alpha)|_S^A.$$

If $P(b,\alpha)=0$, one has $D_{\chi,b,\alpha}(s)=L^S(s+4,\chi)^{-1}L(s+3,\chi)$ as mentioned above. Combining these observations, we easily deduce that (3.30) is $O_{f,V,\varepsilon}(X^{\varepsilon-1/4})$ for any $\varepsilon>0$. This completes the proof of Theorem 3.6. □

4 The first Bruhat cell

In this section, we study the contribution of the first Bruhat cell:

$$\Sigma_1(X):=\sum_{a,m}\sum_{y_1,y_2\in F^\times}\frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S}K_{f_1\mathbb{1}_{a,a}*\mathbb{1}_m}(g_1,g_2)$$
$$\times\int_{(F\setminus\mathbb{A}_F)^{\oplus2}}\sum_{\delta\in B(F)}f_2\mathbb{1}_m\left(a_Sh_1^{-1}\begin{pmatrix}1&-t_1\\&1\end{pmatrix}\delta\begin{pmatrix}1&t_2\\&1\end{pmatrix}h_2\right)\psi(y_1t_1+y_2t_2)dt_1\,dt_2. \tag{4.1}$$

The main result of this section, Lemma 4.1, asserts that $\lim_{X\to\infty}\Sigma_1(X)=0$.

Under the action of $N(F)\times N(F)$, every element of $B(F)$ is in the orbit of a unique element of the form

$$\delta=\begin{pmatrix} b & \\ & c\end{pmatrix}$$

and the stabilizer of such an element is

$$N_\gamma(\mathbb{A}_F)=\left\{\left(\begin{pmatrix}1 & t\\ & 1\end{pmatrix},\begin{pmatrix}1 & \frac{ct}{b}\\ & 1\end{pmatrix}\right)\in N(\mathbb{A}_F)\times N(\mathbb{A}_F): t\in\mathbb{A}_F\right\}.$$

We give $N_\gamma(\mathbb{A}_F)$ a Haar measure via the isomorphism

$$\begin{aligned}\mathbb{A}_F &\longrightarrow N_\gamma(\mathbb{A}_F)\\ t &\longmapsto \left(t,\frac{ct}{b}\right).\end{aligned}$$

One says that δ is **relevant** if the character

$$\begin{aligned}N(\mathbb{A}_F)\times N(\mathbb{A}_F)&\longrightarrow\mathbb{C}^\times\\ \left(\begin{pmatrix}1 & t_1\\ & 1\end{pmatrix},\begin{pmatrix}1 & t_2\\ & 1\end{pmatrix}\right)&\longmapsto\psi(y_1t_1+y_2t_2)\end{aligned}$$

is trivial on this stabilizer; thus, δ is relevant if and only if $-bc^{-1}y_1=y_2$. Only relevant elements can contribute to $\Sigma_1(X)$. Thus,

$$\begin{aligned}\Sigma_1(X)=&\sqrt{d_F}\sum_{a,m}\sum_{y\in F^\times}\frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S}K_{f_1\mathbb{1}_{a,a}*\mathbb{1}_m}(g_1,g_2)\\ &\times\sum_{b,c\in F^\times}\int_{\{(t,cb^{-1}t):t\in\mathbb{A}_F\}\backslash\mathbb{A}_F^{\oplus 2}}f_2\mathbb{1}_m\left(a_Sh_1^{-1}\begin{pmatrix}b & bt_2-ct_1\\ & c\end{pmatrix}h_2\right)\\ &\times\psi(yt_1-bc^{-1}yt_2)\frac{dt_1\,dt_2}{dt}.\end{aligned}\tag{4.2}$$

We compute

$$\begin{aligned}&\int_{\{(t,cb^{-1}t):t\in\mathbb{A}_F\}\backslash\mathbb{A}_F^{\oplus 2}}f_2\mathbb{1}_m\left(a_Sh_1^{-1}\begin{pmatrix}b & bt_2-ct_1\\ & c\end{pmatrix}h_2\right)\psi(yt_1-bc^{-1}yt_2)\frac{dt_1\,dt_2}{dt}\\ &=\int_{\{(t,t):t\in\mathbb{A}_F\}\backslash\mathbb{A}_F^{\oplus 2}}f_2\mathbb{1}_m\left(a_Sh_1^{-1}\begin{pmatrix}b & t_2-t_1\\ & c\end{pmatrix}h_2\right)\psi\left(\frac{y(t_1-t_2)}{c}\right)\frac{dt_1\,dt_2}{dt}\\ &=\int_{\mathbb{A}_F}f_2\mathbb{1}_m\left(a_Sh_1^{-1}\begin{pmatrix}b & t\\ & c\end{pmatrix}h_2\right)\psi\left(-\frac{yt}{c}\right)dt.\end{aligned}$$

Substituting this into the expression (4.2) for $\Sigma_1(X)$ and simplifying, we arrive at

$$\begin{aligned}\Sigma_1(X)=&\sqrt{d_F}\sum_a\frac{V\left(\frac{|a^2\det B|_S|\det g_1g_2^{-1}|^S}{X}\right)}{X|\det B|_S|\det g_1g_2^{-1}|^S}\times\sum_{B\in\mathfrak{gl}_2(F)}f_1\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_Sg_1^{-1}Bg_2)\\ &\times\sum_{b,c\in F^\times}\mathbb{1}_{\det g_1^{-1}g_2h_1h_2^{-1}\widehat{\mathcal{O}}_F^{S\times}}(b)\sum_{y\in F^\times}\int_{\mathbb{A}_F}f_2\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}\left(a_Sh_1^{-1}\begin{pmatrix}bc^{-1}\det B & t\\ & c\end{pmatrix}h_2\right)\psi\left(\frac{yt}{c}\right)dt\end{aligned}$$

$$= \sqrt{d_F} \sum_a \frac{V\left(\frac{|a^2 \det B|_S |\det g_1 g_2^{-1}|^S}{X}\right)}{X|\det B|_S |\det g_1 g_2^{-1}|^S} \sum_{B \in \mathfrak{gl}_2(F)} f_1 \mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_S g_1^{-1} B g_2)$$
$$\times \sum_{b,c \in F^\times} \mathbb{1}_{\det g_1^{-1} g_2 h_1 h_2^{-1} \widehat{\mathcal{O}}_F^{S\times}}(b) \sum_{y \in F^\times} \int_{\mathbb{A}_F} f_2 \mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)} \left(a_S h_1^{-1} \begin{pmatrix} bc^{-1}\det B & t \\ & c \end{pmatrix} h_2\right) \psi(yt)\, dt.$$

Let $S_0 \supseteq S$ be a finite set of places such that $g_1^{S_0}, g_2^{S_0}, h_1^{S_0}, h_2^{S_0} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F^{S_0})$. Then the above is equal to

$$\sqrt{d_F} \sum_a \sum_{y \in \mathcal{O}_F^{S_0} - 0} \frac{V\left(\frac{|a^2 \det B|_S |\det g_1 g_2^{-1}|^S}{X}\right)}{X|\det B|_S |\det g_1 g_2^{-1}|^S} \sum_{B \in \mathfrak{gl}_2(F)} f_1 \mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_S g_1^{-1} B g_2)$$
$$\times \sum_{b \in F^\times} \sum_{\substack{c \in \mathcal{O}_F^{S_0} - 0 \\ c | \det B \mathcal{O}_F^{S_0}}} \mathbb{1}_{\det g_1^{-1} g_2 h_1 h_2^{-1} \widehat{\mathcal{O}}_F^{S\times}}(b) \int_{F_{S_0}} f_2 \mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_{FS_0}^S)} \left(a_S h_1^{-1} \begin{pmatrix} bc^{-1}\det B & t \\ & c \end{pmatrix} h_2\right) \psi_{S_0}(yt)\, dt.$$

Notice that multiplying our representative a for an ideal in $\mathcal{O}_F^S$ by an element of $\mathcal{O}_F^{S\times}$ does not affect the sum, as it simply permutes the c, b, y and B sums. Therefore, we can and do assume that our representatives a for ideals in $\mathcal{O}_F^S$ are chosen so that

$$|a|_w \asymp |a|_S^{1/[F:\mathbb{Q}]} \tag{4.3}$$

for $w|\infty$ and

$$|a|_w \asymp 1 \tag{4.4}$$

for $w \in S \backslash \infty$.

Lemma 4.1 *For any $\varepsilon > 0$ one has $\Sigma_1(X) \ll_\varepsilon X^{\varepsilon - \frac{1}{2}}$.*

Proof Choose $\widetilde{f}_2 \in C_c^\infty(\mathrm{GL}_2(F_S))$ such that $\int_A \widetilde{f}_2(zg) dz^\times = f_2(g)$. Then,

$$\int_{F_{S_0}} f_2 \mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_{FS_0}^S)} \left(a_S h_1^{-1} \begin{pmatrix} bc^{-1}\det B & t \\ & c \end{pmatrix} h_2\right) \psi_{S_0}(yt)\, dt$$
$$= \int_{F_{S_0}} \int_A \widetilde{f}_2 \mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_{FS_0}^S)} \left(z a_S h_1^{-1} \begin{pmatrix} bc^{-1}\det B & t \\ & c \end{pmatrix} h_2\right) \psi_{S_0}(yt)\, dt\, dz^\times$$
$$= \int_{F_{S_0}} \int_A \widetilde{f}_2 \mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_{FS_0}^S)} \left(z a_S h_1^{-1} \begin{pmatrix} \Delta(\sqrt{X}^{-1}) bc^{-1}\det B & \Delta(\sqrt{X}^{-1}) t \\ & \Delta(\sqrt{X}^{-1}) c \end{pmatrix} h_2\right) \psi_{S_0}(yt)\, dt\, dz^\times$$
$$= \int_{F_{S_0}} \frac{\sqrt{X}}{|a|_S} \int_A \widetilde{f}_2 \mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_{FS_0}^S)} \left(z h_1^{-1} \begin{pmatrix} a_S \Delta(\sqrt{X}^{-1}) bc^{-1}\det B & t \\ & a_S \Delta(\sqrt{X}^{-1}) c \end{pmatrix} h_2\right)$$
$$\times \psi_{S_0}\left(\frac{\Delta(\sqrt{X}) yt}{a_S}\right) dt\, dz^\times. \tag{4.5}$$

Here Δ is defined as in (3.28).

By considering determinants, we see that in order for the integrand to be nonzero we must have

$$|z^2 a^2 \det B|_S \asymp X.$$

For a, B to contribute to $\Sigma_1(X)$ we must have $|a^2 \det B|_S \asymp X$, and hence, for z in the support of the integrand in (4.5) one has

$$|z|_w \asymp 1$$

for $w|\infty$. Thus, we can essentially ignore the integral over z. It also follows similarly that

$$|b|_w \asymp 1$$

for all $w|S_0$, so we may fix b and ignore the sum over b.

We also observe that for c to contribute to $\Sigma_1(X)$ we must have

$$|c|_w \ll \left|\frac{\Delta(\sqrt{X})}{a_S}\right|_w \tag{4.6}$$

for all $w \in S_0$. There are at most $O\left(\frac{\sqrt{X}}{|a|_S}\right)$ such $c \in \mathcal{O}_F^{S_0} - 0$.

The integral (4.5) vanishes if $|y|_w \gg |a|_w$ for $w \in S_0 - \infty$. Applying integration by parts in t_w for $w|\infty$ to (4.5) implies that for all $N \geq 0$ it is bounded by a constant depending on N times

$$\frac{\sqrt{X}}{|a|_S}\left|\int_A (D\widetilde{f}_2)\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}^S_{FS_0})}\left(zh_1^{-1}\begin{pmatrix} a_S\Delta(\sqrt{X}^{-1})bc^{-1}\det B & t \\ & a_S\Delta(\sqrt{X}^{-1})c\end{pmatrix}h_2\right)\right|\left|\frac{\Delta(\sqrt{X})y}{a}\right|_w^{-N} \tag{4.7}$$

for an appropriate differential operator D.

Given our conventions (4.3) and (4.4), we deduce that if $\varepsilon > 0$ and $|a|_S \leq X^{1/2-\varepsilon}$ then for any $N > 0$ one has

$$\begin{aligned}
&\sum_{y \in \mathcal{O}_F^{S_0}-0} V\left(\frac{|a^2\det B|_S}{X}\right)\sum_{B \in \mathfrak{gl}_2(F)} f_1\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}_F^S)}(a_S g_1^{-1}Bg_2)\\
&\times \sum_{b \in F^\times}\sum_{\substack{c \in \mathcal{O}_F^{S_0}-0\\ c|\det B\mathcal{O}_F^{S_0}}} \mathbb{1}_{\det g_1^{-1}g_2h_1h_2^{-1}\widehat{\mathcal{O}}_F^{S\times}}(b)\\
&\times \int_{F_{S_0}} f_2\mathbb{1}_{\mathfrak{gl}_2(\widehat{\mathcal{O}}^S_{FS_0})}\left(a_S h_1^{-1}\begin{pmatrix} bc^{-1}\det B & t \\ & c\end{pmatrix}h_2\right)\psi_{S_0}(yt)\,dt \ll_{\varepsilon,N} X^{-N}.
\end{aligned}$$

Here we are using (4.6) to obtain a bound of $\sqrt{X}$ for the number of contributing c.

We are left with the case $|a|_S \geq X^{1/2-\varepsilon}$. For this case, we observe that any B contributing to the sum satisfies

$$|B|_w \ll \frac{X^{[F:\mathbb{Q}]^{-1}}}{|a|_w^2} \ll X^{2\varepsilon[F:\mathbb{Q}]^{-1}}$$

for $w|\infty$ and $|B|_w \ll 1$ for $w|S_0 - \infty$. Here $|B|_w$ is defined as in (3.6).

Let the box norm $||\cdot||_\infty$ be defined as in (3.7). Then, for a small enough nonzero ideal $\mathfrak{N} \subseteq \mathcal{O}_F$ the contribution of these a to $\Sigma_1(X)$ is bounded by a constant times

$$\begin{aligned}
&\frac{1}{X}\sum_{\sqrt{X}^{1-2\varepsilon} \leq |a|_S \ll \sqrt{X}} \frac{|a^2|_S}{X}\frac{X}{|a^2|_S}\sum_{\substack{B \in \mathfrak{N}^{-1}\mathfrak{gl}_2(\mathcal{O}_F)\\ ||B||_\infty \ll X^{2\varepsilon/[F:\mathbb{Q}]}}}\ \sum_{\substack{y \in \mathfrak{N}^{-1}\mathcal{O}_F\\ ||y||_\infty \ll X^{2\varepsilon/[F:\mathbb{Q}]}}} 1\\
&\ll \frac{1}{X}\sqrt{X}X^{8\varepsilon}X^{2\varepsilon} = X^{10\varepsilon-\frac{1}{2}},
\end{aligned}$$

where one factor of $\frac{\sqrt{X}}{|a|_S}$ comes from the sum over c and the other factor of $\frac{\sqrt{X}}{|a|_S}$ comes from the bound (4.7), which has also been used to bound the sum on y. □

5 The cuspidal contribution and the proof of Theorem 1.1

Theorem 1.1 follows immediately upon combining (1.7), Theorem 3.6, Lemma 4.1, and the following proposition:

Proposition 5.1 *The limit* $\lim_{X\to\infty}\Sigma^{\mathrm{cusp}}(X)$ *exists and is equal to the absolutely convergent sum*

$$d_F\widetilde{V}(1)\sum_{\pi}K_{\pi(f_1\mathbb{1}_{\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)})}(g_1,g_2)K_{\pi^\vee(f_2\mathbb{1}_{\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)})}(h_1,h_2)\mathrm{Res}_{s=1}L(s,\pi\times\pi^{\vee S}).$$

Here the sum on π *is over isomorphism classes of cuspidal automorphic representations of* $A\backslash\mathrm{GL}_2(\mathbb{A}_F)$.

Proof We first observe that since $K_{f_2}^{\mathrm{cusp}}(x,y)$ is cuspidal

$$\sum_{y_1,y_2\in F^\times}\int_{(F\backslash\mathbb{A}_F)^{\oplus 2}}K^{\mathrm{cusp}}_{f_2\mathbb{1}_{a,a}*\mathbb{1}_m}\left(\left(\begin{smallmatrix}1&t_1\\&1\end{smallmatrix}\right)h_1,\left(\begin{smallmatrix}1&t_2\\&1\end{smallmatrix}\right)h_2\right)\psi(y_1t_1+y_2t_2)\mathrm{d}t_1\,\mathrm{d}t_2$$

is the Fourier expansion of $d_FK^{\mathrm{cusp}}_{f_2\mathbb{1}_{a,a}*\mathbb{1}_m}\left(\left(\begin{smallmatrix}1&x_1\\&1\end{smallmatrix}\right)h_1,\left(\begin{smallmatrix}1&x_2\\&1\end{smallmatrix}\right)h_2\right)$ evaluated at $x_1=x_2=0$. Thus,

$$\begin{aligned}\Sigma^{\mathrm{cusp}}(X)&=d_F\sum_{a,m}\frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S}K^{\mathrm{cusp}}_{f_1\mathbb{1}_{a,a}*\mathbb{1}_m}(g_1,g_2)K^{\mathrm{cusp}}_{f_2\mathbb{1}_{a,a}*\mathbb{1}_m}(h_1,h_2)\\&=d_F\sum_{\pi_1,\pi_2}K_{\pi_1(f_1\mathbb{1}_{\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)})}(g_1,g_2)K_{\pi_2(f_2\mathbb{1}_{\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)})}(h_1,h_2)\\&\quad\times\sum_{a,m}\frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S}\mathrm{tr}\,\pi_1^S(\mathbb{1}_{a,a}*\mathbb{1}_m)\mathrm{tr}\,\pi_2^S(\mathbb{1}_{a,a}*\mathbb{1}_m).\end{aligned}$$

Here the sum on π_1,π_2 is over pairs of cuspidal automorphic representations of $A\backslash\mathrm{GL}_2(\mathbb{A}_F)$. By Mellin inversion and standard preconvex estimates [3, (10)], there is a $\delta_1>0$ such that

$$\begin{aligned}&\sum_{a,m}\frac{V\left(\frac{|ma^2|_S}{X}\right)}{X|m|_S}\mathrm{tr}\,\pi_1^S(\mathbb{1}_{a,a}*\mathbb{1}_m)\mathrm{tr}\,\pi_2^S(\mathbb{1}_{a,a}*\mathbb{1}_m)\\&\quad=\widetilde{V}(1)\mathrm{Res}_{s=1}L^S(s,\pi_1\times\pi_2)+O\left(\frac{C(\pi_1\times\pi_2)^{\delta_1}}{X}\right),\end{aligned}$$

where $C(\pi_1\times\pi_2):=C(\pi_1\times\pi_2,0)$ is the analytic conductor of $\pi_1\times\pi_2$ (compare, e.g., [2, §3]). By Rankin–Selberg theory, the residue is nonzero only if $\pi_1\cong\pi_2^\vee$, in which case it is bounded by $C(\pi\times\pi^\vee)^{\delta_2}$ for some $\delta_2>0$ [3, (10)]. We also recall that

$$C(\pi_1\times\pi_2)\ll C(\pi_1)^2C(\pi_2)^2$$

by [3, (8)].

Thus, by dominated convergence, to complete the proof it suffices to show that

$$\sum_{\pi_1,\pi_2}C(\pi_1)^NC(\pi_2)^N|K_{\pi_1(f_1\mathbb{1}_{\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)})}(g_1,g_2)K_{\pi_2(f_2\mathbb{1}_{\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S)})}(h_1,h_2)|$$

is bounded for any $N>0$. By a standard argument (compare the proof of [7, Theorem 3.1]), to prove this it suffices to show that for any $N>0$ and $f\in C_c^\infty(A\backslash\mathrm{GL}_2(\mathbb{A}_F))$ the sum

$$\sum_{\pi}C(\pi)^NK_{\pi(f*f^*)}(g,g)$$

is bounded. Here $f^*(g) := \overline{f}(g^{-1})$. But this is implied by [10, (15')] (stated in adelic language in [7, Theorem 3.5]) and the fact that $\mathrm{tr}\,\pi(f * f^*)$ is rapidly decreasing as a function of $C(\pi)$ [5, Lemma 4.4] (compare the proof of [7, Theorem 3.1]). □

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