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Generalized α-nonexpansive mappings in Banach spaces

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Abstract

We consider a new type of monotone nonexpansive mappings in an ordered Banach space X with partial order $\preceq$. This new class of nonlinear mappings properly contains nonexpansive, firmly-nonexpansive and Suzuki-type generalized nonexpansive mappings and partially extends α-nonexpansive mappings. We obtain some existence theorems and weak and strong convergence theorems for the Mann iteration. Some useful examples are presented to illustrate the facts.

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1 Introduction

Throughout this paper, $(X, \|\cdot\|)$ denotes a real Banach space, $\mathbb{N}$ the set of natural numbers and $\mathbb{R}$ the set of real numbers. Let K be a subset of X and $T : K \to K$ be a self-mapping. A point $z \in X$ is said to be a fixed point of T if $T(z) = z$. The mapping $T : K \to K$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$ and quasinonexpansive [1] if $\|T(x) - y\| \leq \|x - y\|$ for all $x \in K$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T.

The study of the existence of fixed points of nonexpansive mappings was initiated in 1965 by Browder [2], Göhde [3] and Kirk [4] independently. Indeed, Browder [2] and Göhde [3] obtained an existence theorem for a nonexpansive mapping on a uniformly convex Banach space, while Kirk [4] obtained the same result in a reflexive Banach space using the normal structure property (see also [5–7]).

A number of extensions and generalizations of nonexpansive mappings have been considered by many mathematicians in recent years. In 2008, Suzuki [8] introduced an interesting generalization of nonexpansive mappings and obtained some existence and convergence results.

Definition 1.1 ([8]) A mapping $T : K \to K$ is said to satisfy condition (C) if for all $x, y \in K$

$$\frac{1}{2}\|x - T(x)\| \leq \|x - y\| \quad \text{implies} \quad \|T(x) - T(y)\| \leq \|x - y\|.$$

The mapping satisfying condition (C) is also known as a Suzuki-type generalized nonexpansive mapping.

Theorem 1.2 ([8]) *Let K be a nonempty convex subset of a Banach space X and $T : K \to K$ be a mapping satisfying condition* (C). *Assume also that either of the following holds*:

- *K is compact*;
- *K is weakly compact*, *and X has the Opial property.*

Then T has a fixed point.

Recently, Aoyama and Kohsaka [9] introduced a new class of nonexpansive mappings, namely *α-nonexpansive mappings*, and obtained a fixed point theorem for such mappings.

Definition 1.3 ([9]) Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be α-nonexpansive if for all $x, y \in K$ and $\alpha < 1$

$$\left\| T(x) - T(y) \right\|^2 \leq \alpha \left\| T(x) - y \right\|^2 + \alpha \left\| x - T(y) \right\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

Remark 1.4 In [10], Ariza-Ruiz *et al.* showed that the concept of α-nonexpansive mapping is trivial for $\alpha < 0$.

Theorem 1.5 ([9]) *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ be an α-nonexpansive mapping. Then $F(T)$ is nonempty if and only if there exists $x \in K$ such that $\{T^n(x)\}$ is bounded.*

Remark 1.6 It is interesting to note that nonexpansive mappings are continuous on their domains, but Suzuki-type generalized nonexpansive mappings and α-nonexpansive mappings need not be continuous (see [8], Example 1, and Example 3.3 below).

On the other hand, fixed point theory in partially ordered metric spaces has been initiated by Ran and Reurings [11] for finding applications to matrix equations. Nieto and López [12] extended their result for nondecreasing mappings and presented an application to differential equations. Recently Song *et al.* [13] extended the notion of α-nonexpansive mapping to monotone α-nonexpansive mapping in ordered Banach spaces and obtained some existence and convergence theorem for the Mann iteration (see also [14, 15] and the references therein).

Motivated by the works of Suzuki [8], Aoyama and Kohsaka [9], Bin Dehaish and Khamsi [14], Song *et al.* [13, 15] and others, we obtain some existence and convergence results in ordered Banach spaces for a wider class of nonexpansive mappings considered in [16]. Particularly, in Section 3, some auxiliary results are presented. In Section 4, we obtain some existence theorems in ordered Banach spaces. In Section 5, we establish some weak and strong convergence theorems for the Mann iteration. Some illustrative examples are also presented. Our results complement, extend and generalize a number of existence and convergence theorems including Theorems 1.2, 1.5 and certain results in [13, 15]. Proof techniques used herein are slightly different from [8, 9, 13, 15].

2 Preliminaries

Let $\mathcal{X}$ be an ordered Banach space with the norm $\|\cdot\|$ and the partial order $\preceq$.

Definition 2.1 A subset $\mathcal{C}$ of a real Banach space $\mathcal{X}$ is said to be a closed convex cone if the following assumptions hold:

- $\mathcal{C}$ is nonempty closed and $\mathcal{C} \neq \{0\}$;
- $ax + by \in \mathcal{C}$ for $x, y \in \mathcal{C}$ and $a, b \in \mathbb{R}$ with $a, b \geq 0$;
- if $x \in \mathcal{C}$ and $-x \in \mathcal{C}$ implies $x = 0$.

A partial order $\preceq$ in $\mathcal{X}$ with respect the closed convex cone $\mathcal{C}$ is defined as follows:

$$x \preceq y \ (x \prec y) \quad \Leftrightarrow \quad y - x \in \mathcal{C} \ (y - x \in \dot{\mathcal{C}})$$

for all $x, y \in \mathcal{X}$, where $\dot{\mathcal{C}}$ is an interior of $\mathcal{C}$.

A Banach space $\mathcal{X}$ is said to be uniformly convex in every direction (in short, UCED) if for each $\varepsilon \in (0, 2]$ and $z \in \mathcal{X}$ with $\|z\| = 1$, there exists $\delta(\varepsilon, z) > 0$ such that

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta(\varepsilon, z)$$

for all $x, y \in \mathcal{X}$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \in \{tz : t \in [-2, -\varepsilon] \cup [\varepsilon, 2]\}$. $\mathcal{X}$ is said to be uniformly convex if $\mathcal{X}$ is UCED and

$$\inf\{\delta(\varepsilon, z) : \|z\| = 1\} > 0.$$

The class of uniformly convex spaces is smaller than the class of UCED spaces.

A Banach space $\mathcal{X}$ is said to have the Opial property [17] if for every weakly convergent sequence $\{x_n\}$ in $\mathcal{X}$ with weak limit z,

$$\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in \mathcal{X}$ with $y \neq z$. All Hilbert spaces, finite dimensional Banach spaces and ℓ^p $(1 < p < \infty)$ have the Opial property. On the other hand, the uniformly convex spaces $L_p[0, 2\pi]$ $(p \neq 2)$ do not have the Opial property [7].

Definition 2.2 ([18]) Let $\mathcal{K}$ be a subset of a normed space $\mathcal{X}$. A mapping $T : \mathcal{K} \to \mathcal{K}$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - T(x)\| \geq f(d(x, F(T)))$ for all $x \in \mathcal{K}$, where $d(x, F(T))$ denotes the distance of x from $F(T)$.

Let $\mathcal{K}$ be a nonempty subset of a Banach space $\mathcal{X}$ and $\{x_n\}$ be a bounded sequence in $\mathcal{X}$. For each $x \in \mathcal{X}$, define:

(i) Asymptotic radius of $\{x_n\}$ at x by $r(x, \{x_n\}) := \limsup_{n\to\infty} \|x_n - x\|$.

(ii) Asymptotic radius of $\{x_n\}$ relative to $\mathcal{K}$ by $r(\mathcal{K}, \{x_n\}) := \inf\{r(x, \{x_n\}) : x \in \mathcal{K}\}$.

(iii) Asymptotic center of $\{x_n\}$ relative to $\mathcal{K}$ by $A(\mathcal{K}, \{x_n\}) := \{x \in \mathcal{K} : r(x, \{x_n\}) = r(\mathcal{K}, \{x_n\})\}$.

We note that $A(\mathcal{K}, \{x_n\})$ is nonempty. Further, if $\mathcal{X}$ is uniformly convex, then $A(\mathcal{K}, \{x_n\})$ has exactly one point [7].

Throughout, we will assume that order intervals are closed and convex subsets of an ordered Banach space $(\mathcal{X}, \preceq)$. We denote these as follows:

$$[a, \rightarrow) := \{x \in \mathcal{X}; a \preceq x\} \quad \text{and} \quad (\leftarrow, b] := \{x \in \mathcal{X}; x \preceq b\}$$

for any $a, b \in \mathcal{X}$ (cf. [14]).

Definition 2.3 ([13]) Let $(\mathcal{X} \preceq)$ be a partially ordered Banach space and $T : \mathcal{X} \to \mathcal{X}$ be a mapping. The mapping T is said to be monotone if for all $x, y \in \mathcal{X}$,

$$x \preceq y \quad \text{implies} \quad T(x) \preceq T(y).$$

The following iteration process is known as the Mann iteration process [19]:

$$\begin{cases} x_1 \in \mathcal{K}, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T(x_n), \quad n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $\{\beta_n\}$ is a sequence in $[0,1]$.

3 Monotone generalized α-nonexpansive mappings

Definition 3.1 Let $\mathcal{K}$ be a nonempty subset of an ordered Banach space $(\mathcal{X}, \preceq)$. A mapping $T : \mathcal{K} \to \mathcal{K}$ will be called a monotone generalized α-nonexpansive mapping if T is monotone and there exists $\alpha \in [0,1)$ such that

$$\begin{aligned} &\frac{1}{2}\left\|x - T(x)\right\| \leq \|x - y\| \quad \text{implies} \\ &\left\|T(x) - T(y)\right\| \leq \alpha\left\|T(x) - y\right\| + \alpha\left\|T(y) - x\right\| + (1 - 2\alpha)\|x - y\| \end{aligned} \tag{3.1}$$

for all $x, y \in \mathcal{K}$ with $x \preceq y$ (see [16], Definition 3.1).

Now we present some basic properties of generalized α-nonexpansive mappings.

Proposition 3.2 *Every monotone mapping satisfying condition* (C) *is a monotone generalized α-nonexpansive mapping but the converse is not true.*

When $\alpha = 0$, a generalized α-nonexpansive mapping reduces to a mapping satisfying condition (C). The following example shows that the reverse implication does not hold.

Example 3.3 ([16]) Let $\mathcal{K} = [0,4]$ be a subset of $\mathbb{R}$ endowed with the usual norm and usual order. Define $T : \mathcal{K} \to \mathcal{K}$ by

$$Tx = \begin{cases} 0, & \text{if } x \neq 4, \\ 2, & \text{if } x = 4. \end{cases}$$

Then, for $x \in (2, 8/3]$ and $y = 4$,

$$\frac{1}{2}\left\|x - T(x)\right\| \leq \|x - y\| \quad \text{and} \quad \left\|T(x) - T(y)\right\| = 2 > \|x - y\|,$$

and T does not satisfy condition (C). Again, for $x \in (2,3]$ and $y = 4$,

$$\frac{1}{2}\left\|y - T(y)\right\| \leq \|x - y\| \quad \text{and} \quad \left\|T(x) - T(y)\right\| > \|x - y\|,$$

and T does not satisfy condition (C). However, T is α-nonexpansive with $\alpha \geq \frac{1}{2}$ and a generalized α-nonexpansive mapping with $\alpha \geq \frac{1}{3}$.

Example 3.4 ([16]) Let $\mathcal{X} = \{(0,0),(2,0),(0,4),(4,0),(4,5),(5,4)\}$ be a subset of $\mathbb{R}^2$ with dictionary order. Define a norm $\|\cdot\|$ on $\mathcal{X}$ by $\|(x_1,x_2)\| = |x_1| + |x_2|$. Then $(X,\|\cdot\|)$ is a Banach space. Define a mapping $T : X \to X$ by

$$T : \begin{pmatrix} (0,0),(2,0),(0,4),(4,0),(4,5),(5,4) \\ (0,0),(0,0),(0,0),(2,0),(4,0),(0,4) \end{pmatrix}.$$

We note that for $\alpha \geq \frac{1}{5}$,

$$\|T(x) - T(y)\| \leq \alpha\|x - T(y)\| + \alpha\|T(x) - y\| + (1-2\alpha)\|x - y\|$$

if $(x,y) \neq ((4,5),(5,4))$. In the case $x = (4,5)$ and $y = (5,4)$, we have

$$\frac{1}{2}\|x - T(x)\| = \frac{1}{2}\|y - T(y)\| = \frac{5}{2} > 2 = \|x - y\|.$$

Therefore T is a generalized α-nonexpansive mapping.

However, for $x = (4,5)$ and $y = (5,4)$,

$$\begin{aligned} \|T(x) - T(y)\|^2 &= 64 > 42\alpha + 4 \\ &= 25\alpha + 25\alpha + (1-2\alpha)\cdot 4 \\ &= \alpha\|x - T(y)\|^2 + \alpha\|T(x) - y\|^2 + (1-2\alpha)\|x - y\|^2. \end{aligned}$$

Therefore T is not an α-nonexpansive mapping for any $\alpha < 1$. Further, for $x = (4,0)$ and $y = (5,4)$,

$$\frac{1}{2}\|x - T(x)\| = 1 < 5 = \|x - y\| \quad \text{but} \quad \|T(x) - T(y)\| = 6 > 5 = \|x - y\|.$$

Thus T is not a Suzuki-type generalized nonexpansive mapping as well.

Proposition 3.5 *Let $\mathcal{K}$ be a nonempty subset of an ordered Banach space $(\mathcal{X},\preceq)$ and $T : \mathcal{K} \to \mathcal{K}$ be a monotone generalized α-nonexpansive mapping with a fixed point $y \in \mathcal{K}$ with $x \preceq y$. Then T is monotone quasinonexpansive.*

Proof It may be completed following the proof of Proposition 2 [8]. □

Lemma 3.6 *Let $\mathcal{K}$ be a nonempty subset of an ordered Banach space $(\mathcal{X},\preceq)$ and $T : \mathcal{K} \to \mathcal{K}$ be a generalized α-nonexpansive mapping. Then $F(T)$ is closed. Moreover, if E is strictly convex and $\mathcal{K}$ is convex, then $F(T)$ is also convex.*

Proof It may be completed following the proof of Lemma 4 [8]. □

The following lemmas will be useful to prove our main results, which are modeled on the pattern of [8].

Lemma 3.7 *Let $\mathcal{K}$ be a nonempty subset of an ordered Banach space $(\mathcal{X},\preceq)$ and $T : \mathcal{K} \to \mathcal{K}$ be a generalized α-nonexpansive mapping. Then, for each $x, y \in \mathcal{K}$ with $x \preceq y$:*

(i) $\|T(x) - T^2(x)\| \leq \|x - T(x)\|$;
(ii) *Either* $\frac{1}{2}\|x - T(x)\| \leq \|x - y\|$ *or* $\frac{1}{2}\|T(x) - T^2(x)\| \leq \|T(x) - y\|$;
(iii) *Either* $\|T(x) - T(y)\| \leq \alpha\|T(x) - y\| + \alpha\|x - T(y)\| + (1 - 2\alpha)\|x - y\|$ *or* $\|T^2(x) - T(y)\| \leq \alpha\|T(x) - T(y)\| + \alpha\|T^2(x) - y\| + (1 - 2\alpha)\|T(x) - y\|$.

Proof It may be completed following the proof of [8], Lemma 5. □

Lemma 3.8 *Let $\mathcal{K}$ be a nonempty subset of an ordered Banach space $(\mathcal{X}, \preceq)$ and $T : \mathcal{K} \to \mathcal{K}$ be a generalized α-nonexpansive mapping. Then, for all $x, y \in \mathcal{K}$ with $x \preceq y$,*

$$\|x - T(y)\| \leq \frac{(3 + \alpha)}{(1 - \alpha)}\|x - T(x)\| + \|x - y\|.$$

Proof From Lemma 3.7, we have for all $x, y \in \mathcal{K}$ either

$$\|T(x) - T(y)\| \leq \alpha\|T(x) - y\| + \alpha\|x - T(y)\| + (1 - 2\alpha)\|x - y\|$$

or

$$\|T^2(x) - T(y)\| \leq \alpha\|T(x) - T(y)\| + \alpha\|T^2(x) - y\| + (1 - 2\alpha)\|T(x) - y\|.$$

In the first case, we have

$$\begin{aligned}
\|x - T(y)\| &\leq \|x - T(x)\| + \|T(x) - T(y)\| \\
&\leq \|x - T(x)\| + \alpha\|T(x) - y\| + \alpha\|T(y) - x\| + (1 - 2\alpha)\|x - y\| \\
&\leq \|x - T(x)\| + \alpha\|T(x) - x\| + \alpha\|x - y\| + \alpha\|T(y) - x\| \\
&\quad + (1 - 2\alpha)\|x - y\|.
\end{aligned}$$

This implies that

$$\|x - T(y)\| \leq \frac{(1 + \alpha)}{(1 - \alpha)}\|T(x) - x\| + \|x - y\|.$$

In the other case, we have

$$\begin{aligned}
\|x - T(y)\| &\leq \|x - T(x)\| + \|T(x) - T^2(x)\| + \|T^2(x) - T(y)\| \\
&\leq 2\|x - T(x)\| + \alpha\|T(x) - T(y)\| + \alpha\|T^2(x) - y\| \\
&\quad + (1 - 2\alpha)\|T(x) - y\| \\
&\leq 2\|x - T(x)\| + \alpha\|T(x) - x\| + \alpha\|T(y) - x\| \\
&\quad + \alpha\|T^2(x) - T(x)\| + \alpha\|T(x) - y\| + (1 - 2\alpha)\|T(x) - y\| \\
&\leq (2 + \alpha)\|x - T(x)\| + \alpha\|T(y) - x\| + \alpha\|x - T(x)\| \\
&\quad + (1 - \alpha)\|T(x) - y\| \\
&\leq (2 + \alpha)\|x - T(x)\| + \alpha\|T(y) - x\| + \alpha\|x - T(x)\| \\
&\quad + (1 - \alpha)\|T(x) - x\| + (1 - \alpha)\|x - y\|.
\end{aligned}$$

This implies

$$\left\|x - T(y)\right\| \leq \frac{(3+\alpha)}{(1-\alpha)}\left\|x - T(x)\right\| + \|x - y\|.$$

Therefore in both the cases we get the desired result. □

4 Existence results

In this section, we present some existence theorems for monotone generalized α-nonexpansive mappings.

Theorem 4.1 *Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex ordered Banach spaces$(\mathcal{X}, \preceq)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone generalized α-nonexpansive mapping. Then $F(T) \neq \emptyset$ if and only if $\{T^n(x)\}$ is a bounded sequence for some $x \in \mathcal{K}$, provided $T^n(x) \preceq y$ for some $y \in \mathcal{K}$ and $x \preceq T(x)$.*

Proof Suppose that $\{T^n(x)\}$ is a bounded sequence for some $x \in \mathcal{K}$. Since T is monotone and $x \preceq T(x)$, we get $T(x) \preceq T^2(x)$. Continuing in this way, we get

$$T(x) \preceq T^2(x) \preceq T^3(x) \preceq T^4(x) \cdots.$$

Define $x_n = T^n(x)$ for all $n \in \mathbb{N}$. Then the asymptotic center of $\{x_n\}$ with respect to $\mathcal{K}$ is $A(\mathcal{K}, \{x_n\}) = \{z\}$ such that $x_n \preceq z$ for all $n \in \mathbb{N}$, such z is unique. Now we claim that

$$\|x_{n+1} - x_{n+2}\| \leq \|x_n - x_{n+1}\|.$$

Since $\frac{1}{2}\|x_n - T(x_n)\| = \frac{1}{2}\|x_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$, by (3.1)

$$\begin{aligned}
\|x_{n+1} - x_{n+2}\| &= \left\|T(x_n) - T(x_{n+1})\right\| \\
&\leq \alpha\left\|T(x_n) - x_{n+1}\right\| + \alpha\left\|x_n - T(x_{n+1})\right\| + (1-2\alpha)\|x_n - x_{n+1}\| \\
&\leq \alpha\|x_n - x_{n+2}\| + (1-2\alpha)\|x_n - x_{n+1}\| \\
&\leq \alpha\|x_n - x_{n+1}\| + \alpha\|x_{n+1} - x_{n+2}\| + (1-2\alpha)\|x_n - x_{n+1}\|.
\end{aligned}$$

This implies that

$$\|x_{n+1} - x_{n+2}\| \leq \|x_n - x_{n+1}\|. \tag{4.1}$$

Now, for all $n \in \mathbb{N}$, we claim that either

$$\|x_n - x_{n+1}\| \leq 2\|x_n - z\| \quad \text{or} \quad \|x_{n+1} - x_{n+2}\| \leq 2\|x_{n+1} - z\|.$$

Arguing by contradiction, we suppose that for some $n \in \mathbb{N}$

$$2\|x_n - z\| < \|x_n - x_{n+1}\| \quad \text{and} \quad 2\|x_{n+1} - z\| < \|x_{n+1} - x_{n+2}\|.$$

By the triangle inequality and (4.1),

$$\begin{aligned}\|x_n - x_{n+1}\| &\leq \|x_n - z\| + \|x_{n+1} - z\| \\ &< \frac{1}{2}\|x_n - x_{n+1}\| + \frac{1}{2}\|x_{n+1} - x_{n+2}\| \\ &\leq \frac{1}{2}\big\{\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\|\big\} = \|x_n - x_{n+1}\|,\end{aligned}$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$, either

$$\frac{1}{2}\|x_n - x_{n+1}\| \leq \|x_n - z\| \quad \text{or} \quad \frac{1}{2}\|x_{n+1} - x_{n+2}\| \leq \|x_{n+1} - z\|.$$

In the first case, $\frac{1}{2}\|x_n - x_{n+1}\| = \frac{1}{2}\|x_n - T(x_n)\| \leq \|x_n - z\|$, and by (3.1) we have

$$\big\|T(x_n) - T(z)\big\| \leq \alpha\big\|T(x_n) - z\big\| + \alpha\big\|x_n - T(z)\big\| + (1 - 2\alpha)\|x_n - z\|.$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty}\big\|T(x_n) - T(z)\big\| &\leq \alpha\limsup_{n\to\infty}\big\|T(x_n) - z\big\| + \alpha\limsup_{n\to\infty}\big\|x_n - T(z)\big\| \\ &\quad + (1 - 2\alpha)\limsup_{n\to\infty}\|x_n - z\|.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\big\|x_n - T(z)\big\| \leq \limsup_{n\to\infty}\|x_n - z\|.$$

Consequently, $T(z) \in A(\mathcal{K}, \{x_n\})$, ensuring that $T(z) = z$. Similarly, in the second case we can deduce that $T(z) = z$. Conversely, suppose that $F(T) \neq \emptyset$. So there exist some $w \in F(T)$ and $T^n(w) = w$ for all $n \in \mathbb{N}$. Therefore, $\{T^n(w)\}$ is a constant sequence and $\{T^n(w)\}$ is bounded. This completes the proof. □

Now we present another existence theorem in a UCED ordered Banach space. The following lemma is quite useful in our result.

Lemma 4.2 ([14]) *Let $\mathcal{K}$ be a weakly compact nonempty convex subset of a UCED Banach space $\mathcal{X}$. Let $\tau : \mathcal{K} \to [0, \infty)$ be a type function. Then there exists a unique minimum point $z \in \mathcal{K}$ such that*

$$\tau(z) = \inf\big\{\tau(x) : x \in \mathcal{K}\big\}.$$

Theorem 4.3 *Let $\mathcal{K}$ be a weakly compact nonempty convex subset of a UCED ordered Banach space $(\mathcal{X}, \preceq)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone generalized α-nonexpansive mapping. Then $F(T) \neq \emptyset$, provided $x \preceq T(x)$.*

Proof Since T is monotone and $x \preceq T(x)$, we get $T(x) \preceq T^2(x)$. Continuing in this way, we get

$$T(x) \preceq T^2(x) \preceq T^3(x) \preceq T^4(x) \cdots$$

Define $x_n = T^n(x)$ for all $n \in \mathbb{N}$.

Since $\mathcal{K}$ is weakly compact, and by the construction of $\{x_n\}$, we have

$$\mathcal{K}_\infty = \bigcap_{n=1}^{\infty} [x_n, \rightarrow) \cap \mathcal{K} = \bigcap_{n=1}^{\infty} \{x \in \mathcal{K}; x_n \preceq x\} \neq \emptyset.$$

Let $x \in \mathcal{K}_\infty$. Then $x_n \preceq x$. Since T is monotone, we have

$$x_n \preceq T(x_n) \preceq T(x)$$

for all $n \in \mathbb{N}$. This implies that $T(\mathcal{K}_\infty) \subset \mathcal{K}_\infty$. Let $\tau : \mathcal{K}_\infty \to [0, \infty)$ be a type function generated by $\{x_n\}$, that is,

$$\tau(x) = \limsup_{n\to\infty} \|x_n - x\|.$$

From Lemma 4.2 there exists a unique element $z \in \mathcal{K}_\infty$ such that

$$\tau(z) = \inf\bigl\{\tau(x); x \in \mathcal{K}_\infty\bigr\}.$$

Now, for all $n \in \mathbb{N}$, if $x_n = x_{n+1}$, then $\|x_n - x_{n+1}\| \leq \|x_n - z\|$ for all $n \in \mathbb{N}$ again if $x_n \prec x_{n+1}$, then $x_n \prec x_{n+1} \preceq z$. Thus in both cases we have

$$\|x_n - x_{n+1}\| \leq \|x_n - z\|$$

for all $n \in \mathbb{N}$. Then we have $\frac{1}{2}\|x_n - x_{n+1}\| = \frac{1}{2}\|x_n - T(x_n)\| \leq \|x_n - z\|$, by (3.1), we have

$$\bigl\|T(x_n) - T(z)\bigr\| \leq \alpha\bigl\|T(x_n) - z\bigr\| + \alpha\bigl\|x_n - T(z)\bigr\| + (1 - 2\alpha)\|x_n - z\|.$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty}\bigl\|T(x_n) - T(z)\bigr\| &\leq \alpha\limsup_{n\to\infty}\bigl\|T(x_n) - z\bigr\| + \alpha\limsup_{n\to\infty}\bigl\|x_n - T(z)\bigr\| \\ &\quad + (1 - 2\alpha)\limsup_{n\to\infty}\|x_n - z\|.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\bigl\|x_n - T(z)\bigr\| \leq \limsup_{n\to\infty}\|x_n - z\|.$$

Since $\tau(z) = \inf\{\tau(x); x \in \mathcal{K}_\infty\}$, by the uniqueness of a minimum point, it follows that $T(z) = z$, that is, z is a fixed point of T. □

Corollary 4.4 (Compare Theorem 5 [8]) *Let $\mathcal{K}$ be a weakly compact nonempty convex subset of a UCED ordered Banach space $(\mathcal{X}, \preceq)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone mapping satisfying condition* (C). *Then $F(T) \neq \emptyset$, provided $x \preceq T(x)$.*

Theorem 4.5 *Let $\mathcal{X}$ be a uniformly convex Banach space with the partial order $\preceq$ with respect to a closed convex cone $\mathcal{C}$. Let $T : \mathcal{C} \to \mathcal{C}$ be a monotone generalized α-nonexpansive mapping. Suppose that the sequence $\{T^n(0)\}$ is bounded with $T^n(0) \preceq y$ for some $y \in \mathcal{C}$. Then $F(T) \neq \emptyset$.*

Proof Since $x = 0 \preceq T(0) = T(x)$. Then from Theorem 4.1 conclusion follows. □

Now onwards $\mathbb{R}_+^m := \{(r_1, r_2, \dots, r_m) : r_j \geq 0, j = 1, 2, \dots, m\}$, where $\mathbb{R}$ is the set of real numbers.

Theorem 4.6 *Let $T : \mathbb{R}_+^m \to \mathbb{R}_+^m$ be a monotone generalized α-nonexpansive mapping. If $\{T^n(0)\}$ is bounded, then $F(T) \neq \emptyset$.*

Proof Let $T^n(0) = \{r_1^n, r_2^n, \dots, r_m^n) \in \mathbb{R}_+^m$. By the boundedness of $T^n(0)$ there exists $r > 0$ such that $r_j^n \leq r$ for all $n \in \mathbb{N}$ and $j = 1, 2, \dots, m$. By taking $y = (r, r, \dots, r)$, conclusion follows from Theorem 4.5. □

Lemma 4.7 ([13]) *Let $\mathcal{K}$ be a nonempty closed convex subset of an ordered Banach space $(\mathcal{X}, \preceq)$. Let $T : \mathcal{K} \to \mathcal{K}$ be a monotone mapping. Fix $x_1 \in \mathcal{K}$ such that $x_1 \preceq T(x_1)$ (or $T(x_1) \preceq x_1$). Consider the Mann iteration sequence $\{x_n\}$ defined by* (2.1)*. Then we have*

$$x_n \preceq x_{n+1} \preceq T(x_n) \qquad \big(\text{or } T(x_n) \preceq x_{n+1} \preceq x_n\big) \tag{4.2}$$

for all $n \in \mathbb{N}$. Moreover, $\{x_n\}$ has at most one weak limit point. Hence if $\mathcal{K}$ is weakly compact, then $\{x_n\}$ is weakly convergent.

Theorem 4.8 *Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex ordered Banach space $(\mathcal{X}, \preceq)$ and $T : \mathcal{K} \to \mathcal{K}$ be a monotone generalized α-nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by* (2.1) *is bounded with $x_n \preceq y$ for some $y \in \mathcal{K}$ and* $\lim_{n\to\infty} \inf \|T(x_n) - x_n\| = 0$. *Then $F(T) \neq \emptyset$.*

Proof Suppose that $\{x_n\}$ is a bounded sequence and $\lim_{n\to\infty} \inf \|T(x_n) - x_n\| = 0$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \left\| T(x_{n_k}) - x_{n_k} \right\| = 0.$$

The asymptotic center of $\{x_{n_k}\}$ with respect to $\mathcal{K}$ is $A(\mathcal{K}, \{x_{n_k}\}) = \{z\}$ such that $x_{n_k} \preceq z$ for all $n \in \mathbb{N}$, such z is unique. By the definition of asymptotic radius,

$$r\big(T(z)\big) = \limsup_{k\to\infty} \left\| x_{n_k} - T(z) \right\|.$$

Using Lemma 3.8, we get

$$\begin{aligned} r\big(T(z)\big) &= \limsup_{k\to\infty} \left\| x_{n_k} - T(z) \right\| \\ &\leq \frac{(3+\alpha)}{(1-\alpha)} \limsup_{k\to\infty} \left\| T(x_{n_k}) - x_{n_k} \right\| + \limsup_{k\to\infty} \|x_{n_k} - z\| \\ &= r(z). \end{aligned}$$

The uniqueness of point z implies that $T(z) = z$. □

Now we give an existence result for the Mann iteration in a UCED ordered Banach space.

Theorem 4.9 *Let $\mathcal{K}$ be a weakly compact nonempty convex subset of a UCED ordered Banach space $(\mathcal{X}, \preceq)$ and $T : \mathcal{K} \to \mathcal{K}$ be a monotone generalized α-nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by* (2.1) *and* $\lim_{n\to\infty} \inf \| T(x_n) - x_n\| = 0$. *Then $F(T) \neq \emptyset$.*

Proof Suppose that $\{x_n\}$ is a bounded sequence and $\lim_{n\to\infty} \inf \| T(x_n) - x_n\| = 0$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \| T(x_{n_k}) - x_{n_k} \| = 0.$$

Since $\mathcal{K}$ is weakly compact, and by the construction of $\{x_{n_k}\}$, we have

$$\mathcal{K}_\infty = \bigcap_{k=1}^{\infty} [x_{n_k}, \to) \cap \mathcal{K} = \bigcap_{k=1}^{\infty} \{x \in \mathcal{K}; x_{n_k} \preceq x\} \neq \emptyset.$$

Let $x \in \mathcal{K}_\infty$. Then $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$. Since T is monotone, we have

$$x_{n_k} \preceq T(x_{n_k}) \preceq T(x)$$

for all $k \in \mathbb{N}$. This implies that $T(\mathcal{K}_\infty) \subset \mathcal{K}_\infty$. Let $\tau : \mathcal{K}_\infty \to [0, \infty)$ be a type function generated by $\{x_{n_k}\}$, that is,

$$\tau(x) = \limsup_{k\to\infty} \|x_{n_k} - x\|.$$

From Lemma 4.2 there exists a unique element $z \in \mathcal{K}_\infty$ such that

$$\tau(z) = \inf\{\tau(x); x \in \mathcal{K}_\infty\}.$$

By the definition of type function,

$$\tau\bigl(T(z)\bigr) = \limsup_{k\to\infty} \bigl\| x_{n_k} - T(z) \bigr\|.$$

Using Lemma 3.8, we get

$$\begin{aligned} \tau\bigl(T(z)\bigr) &= \limsup_{k\to\infty} \bigl\| x_{n_k} - T(z) \bigr\| \\ &\leq \frac{(3+\alpha)}{(1-\alpha)} \limsup_{k\to\infty} \bigl\| T(x_{n_k}) - x_{n_k} \bigr\| + \limsup_{k\to\infty} \|x_{n_k} - z\| \\ &= \tau(z). \end{aligned}$$

The uniqueness of a minimum point implies that $T(z) = z$. □

Theorem 4.10 *Let $\mathcal{X}$ be a uniformly convex Banach space with the partial order $\preceq$ with respect to a closed convex cone $\mathcal{C}$. Let $T : \mathcal{C} \to \mathcal{C}$ be a monotone generalized alpha-nonexpansive mapping. Suppose that $x_1 = 0$ and the sequence defined by* (2.1) *is bounded with $x_n \preceq y$ for some $y \in \mathcal{C}$ and* $\lim_{n\to\infty} \inf \| T(x_n) - x_n\| = 0$. *Then $F(T) \neq \emptyset$.*

Proof Since $x_1 = 0 \preceq T(0) = T(x_1)$, and conclusion follows from Theorem 4.8. □

5 Convergence results

In this section, we present some convergence results for monotone generalized α-nonexpansive mappings using the Mann iteration process. In the sequel we also need the following lemma from [20].

Lemma 5.1 *Let $r > 0$ be a fixed real number. If $\mathcal{X}$ is a uniformly convex Banach space, then there exists a continuous strictly increasing convex function $g : [0,+\infty) \to [0,+\infty)$ with $g(0) = 0$ such that*

$$\left\|\lambda x + (1-\lambda)y\right\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g\bigl(\|x-y\|\bigr)$$

for all $x, y \in B_r(0) = \{u \in E : \|u\| \le r\}$ and $\lambda \in [0,1]$.

Theorem 5.2 *Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex ordered Banach space $(\mathcal{X}, \preceq)$ and $T : \mathcal{K} \to \mathcal{K}$ be a monotone generalized α-nonexpansive mapping. Assume that there exists $x_1 \in \mathcal{K}$ such that $x_1 \preceq T(x_1)$ (or $T(x_1) \preceq x_1$). Suppose that $F(T)$ is nonempty and $x_1 \preceq z$ for every $z \in F(T)$. Let $\{x_n\}$ be defined by* (2.1)*. Then the following assertions hold*:

(1) *the sequence $\{x_n\}$ is bounded*;
(2) $\lim_{n\to\infty}\|x_n - z\|$ *and* $\lim_{n\to\infty} d(x_n, F(T))$ *exist, where $d(x, F(T))$ denotes the distance from x to $F(T)$*;
(3) $\liminf_{n\to\infty}\|T(x_n) - x_n\| = 0$, *when* $\limsup_{n\to\infty}\beta_n(1-\beta_n) > 0$;
(4) $\lim_{n\to\infty}\|T(x_n) - x_n\| = 0$, *when* $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$.

Proof Suppose that $F(T) \neq \emptyset$, and let $z \in F(T)$. Since $x_1 \preceq z$, the monotonicity of T implies $T(x_1) \preceq T(z) = z$. By (4.2), $x_2 \preceq T(x_1) \preceq z$. Continuing in this way, we get

$$x_n \preceq x_{n+1} \preceq T(x_n) \preceq z.$$

By (2.1) and Proposition 3.5, we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \left\|(1-\beta_n)x_n + \beta_n T(x_n) - z\right\| \\
&\le (1-\beta_n)\|x_n - z\| + \beta_n\left\|T(x_n) - z\right\| \\
&\le (1-\beta_n)\|x_n - z\| + \beta_n\|x_n - z\| \\
&= \|x_n - z\|.
\end{aligned}$$

Thus the sequence $\{\|x_n - z\|\}$ is nonincreasing and bounded. Thus, $\lim_{n\to\infty}\|x_n - z\|$ exists. Hence $\lim_{n\to\infty} d(x_n, F(T))$ exists. By (2.1), Proposition 3.5 and Lemma 5.1, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|(1-\beta_n)x_n + \beta_n T(x_n) - z\right\|^2 \\
&= \left\|(1-\beta_n)(x_n - z) + \beta_n\bigl(T(x_n) - z\bigr)\right\|^2 \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\left\|T(x_n) - z\right\|^2 - \beta_n(1-\beta_n)g\bigl(\left\|x_n - T(x_n)\right\|\bigr) \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\|x_n - z\|^2 - \beta_n(1-\beta_n)g\bigl(\left\|x_n - T(x_n)\right\|\bigr) \\
&= \|x_n - z\|^2 - \beta_n(1-\beta_n)g\bigl(\left\|x_n - T(x_n)\right\|\bigr).
\end{aligned}$$

Thus we have

$$\beta_n(1-\beta_n)g\big(\|x_n - T(x_n)\|\big) \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Letting $n \to \infty$, we get

$$\limsup_{n\to\infty} \beta_n(1-\beta_n)g\big(\|x_n - T(x_n)\|\big) = 0. \tag{5.1}$$

By assumption in (3), we have

$$\limsup_{n\to\infty} \beta_n(1-\beta_n) > 0$$

since

$$\Big(\limsup_{n\to\infty} \beta_n(1-\beta_n)\Big)\Big(\liminf_{n\to\infty} g\big(\|x_n - T(x_n)\|\big)\Big) \le \Big(\limsup_{n\to\infty} \beta_n(1-\beta_n)g\big(\|x_n - T(x_n)\|\big)\Big).$$

By (5.1), we get

$$\liminf_{n\to\infty} g\big(\|x_n - T(x_n)\|\big) = 0,$$

and by the property of function g

$$\liminf_{n\to\infty} \|x_n - T(x_n)\| = 0.$$

Further, by assumption in (4), we have

$$\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$$

since

$$\Big(\liminf_{n\to\infty} \beta_n(1-\beta_n)\Big)\Big(\limsup_{n\to\infty} g\big(\|x_n - T(x_n)\|\big)\Big) \le \limsup_{n\to\infty} \beta_n(1-\beta_n)g\big(\|x_n - T(x_n)\|\big).$$

By (5.1), we get

$$\limsup_{n\to\infty} g\big(\|x_n - T(x_n)\|\big) = 0.$$

Therefore

$$\lim_{n\to\infty} g\big(\|x_n - T(x_n)\|\big) = 0. \qquad \square$$

Theorem 5.3 *Let $(\mathcal{X}, \preceq)$ be a uniformly convex ordered Banach space with the Opial property, and $\mathcal{K}$, T and $\{x_n\}$ are the same as in Theorem* 5.2. *If $F(T) \neq \emptyset$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges weakly to a fixed point of T, provided $F(T)$ is a totally ordered set.*

Proof By Theorem 5.2, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$. Since $\mathcal{X}$ is uniformly convex, $\mathcal{X}$ is reflexive. By the reflexiveness of $\mathcal{X}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $p \in \mathcal{K}$. By using Lemma 4.7, we have

$$x_1 \preceq x_{n_j} \preceq p \qquad (\text{or } p \preceq x_{n_j} \preceq x_1) \quad \text{for all } j \in \mathbb{N}.$$

By Lemma 3.8, we have

$$\|x_{n_j} - T(p)\| \leq \frac{(3+\alpha)}{(1-\alpha)} \|x_{n_j} - T(x_{n_j})\| + \|x_{n_j} - p\|.$$

This implies

$$\liminf_{j\to\infty} \|x_{n_j} - T(p)\| \leq \liminf_{n\to\infty} \|x_{n_j} - p\|.$$

By the Opial property, we get $T(p) = p$. Then $p \in F(T)$. Now we show that $\{x_n\}$ converges weakly to the point p. Arguing by contradiction, suppose that $\{x_n\}$ has two subsequences $\{x_{n_j}\}$ and $\{x_{n_k}\}$ converging weakly to p and q, respectively. By a similar argument as for $p \in F(T)$, we have $q \in F(T)$. By Theorem 5.2 $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F(T)$.

Now, by the Opial property, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{j\to\infty} \|x_{n_j} - p\| < \lim_{j\to\infty} \|x_{n_j} - q\| \\ &= \lim_{n\to\infty} \|x_n - q\| = \lim_{k\to\infty} \|x_{n_k} - q\| \\ &< \lim_{k\to\infty} \|x_{n_k} - p\| = \lim_{n\to\infty} \|x_n - p\|, \end{aligned}$$

which is a contradiction. Thus $\{x_n\}$ converges weakly to $p \in F(T)$. □

Theorem 5.4 *Let $\mathcal{K}$ be a nonempty closed convex subset of an ordered Banach space $(\mathcal{X}, \preceq)$ and $T : \mathcal{K} \to \mathcal{K}$ be a monotone generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (2.1) *with $x_1 \preceq z$ for all $z \in F(T)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T))$ denotes the distance from x to $F(T)$, provided $F(T)$ is a totally ordered set.*

Proof Necessity is obvious. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Theorem 5.2, $\lim_{n\to\infty} d(x_n, F(T))$ exists, so

$$\lim_{n\to\infty} d\bigl(x_n, F(T)\bigr) = 0. \tag{5.2}$$

In view of (5.2), let $\{x_{n_j}\}$ be a subsequence of the sequence $\{x_n\}$ such that $\|x_{n_j} - z_j\| \leq \frac{1}{2^j}$ for all $j \geq 1$, where $\{z_j\}$ is a sequence in $F(T)$. By Lemma 5.2, we have

$$\|x_{n_{j+1}} - z_j\| \leq \|x_{n_j} - z_j\| \leq \frac{1}{2^j}. \tag{5.3}$$

By the triangle inequality and (5.3), we have

$$\begin{aligned}\|z_{j+1}-z_j\| &\le \|z_{j+1}-x_{n_{j+1}}\| + \|x_{n_{j+1}}-z_j\| \\ &\le \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}}.\end{aligned}$$

A standard argument shows that $\{z_j\}$ is a Cauchy sequence in $F(T)$. By Lemma 3.6, $F(T)$ is closed, so $\{z_j\}$ converges to some $z \in F(T)$.

Now, by the triangle inequality, we have

$$\|x_{n_j}-z\| \le \|x_{n_j}-z_j\| + \|z_j-z\|.$$

Letting $j \to \infty$ implies that $\{x_{n_j}\}$ converges strongly to z. Since by Theorem 5.2, $\lim_{n\to\infty}\|x_n-z\|$ exists, the sequence $\{x_n\}$ converges strongly to z. □

Now we present a strong convergence theorem for a mapping satisfying condition (I).

Theorem 5.5 *Let $(\mathcal{X},\preceq)$ be a uniformly convex ordered Banach space, and $\mathcal{K}$, T and $\{x_n\}$ are the same as in Theorem* 5.2. *Let T satisfy condition (I) with $F(T)\neq\emptyset$, $\limsup_{n\to\infty}\beta_n(1-\beta_n)>0$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof From Theorem 5.2, it follows that

$$\liminf_{n\to\infty}\big\|T(x_n)-x_n\big\| = 0. \tag{5.4}$$

Since T satisfies condition (I), we have

$$\big\|x_n - T(x_n)\big\| \ge f\big(d\big(x_n,F(T)\big)\big).$$

From (5.4) we get

$$\liminf_{n\to\infty} f\big(d\big(x_n,F(T)\big)\big) = 0.$$

Since $f:[0,\infty)\to[0,\infty)$ is a nondecreasing function with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$, we have

$$\liminf_{n\to\infty} d\big(x_n,F(T)\big) = 0.$$

Therefore all the assumptions of Theorem 5.4 are satisfied, and $\{x_n\}$ converges strongly to a fixed point of T. □

The following result is a slightly different version of Theorem 5.5.

Theorem 5.6 *Let $(\mathcal{X},\preceq)$ be a uniformly convex ordered Banach space, and $\mathcal{K}$, T and $\{x_n\}$ are the same as in Theorem* 5.2. *Let T satisfy condition (I) with $F(T)\neq\emptyset$, $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Competing interests
The authors declare that they do not have any competing interests.

Authors' contributions
Each author equally contributed to this paper, read and approved the final manuscript.

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