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Fixed point results in C^*-algebra-valued metric spaces are direct consequences of their standard metric counterparts

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Abstract

Very recently, Ma *et al.* (Fixed Point Theory Appl. 2014:206, 2014) introduced C^*-algebra-valued metric spaces as a new concept. Also, Ma and Jiang (Fixed Point Theory Appl. 2015:222, 2015), generalizing this concept, introduced C^*-algebra-valued b-metric spaces. In both frameworks, these and other authors proved some fixed point results. We show in this paper that all these results (as well as many others) can be directly obtained as consequences of their standard metric or b-metric counterparts.

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1 Introduction

One of the main directions in obtaining possible generalizations of fixed point results in metric spaces is introducing new types of spaces. One of such attempts was made by Ma *et al.* in [1], where they introduced C^*-algebra-valued metric spaces as a new concept and proved some related fixed point results. This line of research was continued in [2–6], where several other fixed point results were obtained in the framework of C^*-algebra-valued metric, as well as (more general) C^*-algebra-valued b-metric spaces.

However, it was observed by Alsulami *et al.* in [7] that these results are in fact not new. Namely, they showed that, using the Banach-Alaoglu theorem and the Gelfand representation, the basic result of [1–5] can be reduced to the existing corresponding fixed point theorems in the setting of standard metric spaces.

In this paper, we will show the same for several more results of this type, including those from [6] and those in C^*-algebra-valued b-metric spaces. Moreover, our method is easier since it uses just basic properties of C^*-algebras. Even several other known theorems can be adapted to hold true in these new kinds of spaces.

In a way, the paper can be considered as written along the lines of well-known papers [8, 9] where, in some other situations, it was also shown that a noncritical approach to 'generalizations' of fixed point results can lead to results which have no real meaning.

2 Preliminaries

Recall that a Banach algebra $\mathbb{A}$ (over the field $\mathbb{C}$ of complex numbers) is said to be a C^*-algebra if there is an involution $*$ in $\mathbb{A}$ (*i.e.*, a mapping $* : \mathbb{A} \to \mathbb{A}$ satisfying $a^{**} = a$ for each $a \in \mathbb{A}$) such that, for all $a, b \in \mathbb{A}$ and $\lambda, \mu \in \mathbb{C}$, the following holds:

(i) $(\lambda a + \mu b)^* = \overline{\lambda} a + \overline{\mu} b$;
(ii) $(ab)^* = b^* a^*$;
(iii) $\|a^* a\| = \|a\|^2$.

Note that, from (iii), it easily follows that $\|a\| = \|a^*\|$ for each $a \in \mathbb{A}$.

In the rest of this paper, $\mathbb{A}$ will always be a unital C^*-algebra with the unit i and the zero element θ. $\mathbb{A}_h$ will denote the set of all self-adjoint elements a (*i.e.*, satisfying $a^* = a$), and $\mathbb{A}^+$ will be the set of positive elements of $\mathbb{A}$, *i.e.*, the elements $a \in \mathbb{A}_h$ having the spectrum $\sigma(a)$ contained in $[0, +\infty)$. It is easy to see that $\mathbb{A}^+$ is a (closed) cone in the normed space $\mathbb{A}$ (see, *e.g.*, [10], Lemma 2.2.3), thus inducing a partial order $\preceq$ on $\mathbb{A}_h$ by $a \preceq b$ if and only if $b - a \in \mathbb{A}^+$.

For further terminology and basic results in C^*-algebras we will refer to [10] (as we have already mentioned). In particular, we will use the following simple result.

Lemma 1 ([10], Theorem 2.2.5)

(1) $\mathbb{A}^+ = \{a^* a : a \in \mathbb{A}\}$;
(2) *if* $a, b \in \mathbb{A}_h$, $a \preceq b$, *and* $c \in \mathbb{A}$, *then* $c^* a c \preceq c^* b c$;
(3) *for all* $a, b \in \mathbb{A}_h$, *if* $\theta \preceq a \preceq b$ *then* $\|a\| \leq \|b\|$.

In the standard terminology used for cones in normed spaces (see, *e.g.*, [11], Ch. 6), the property (2) of the previous lemma means that the cone $\mathbb{A}^+$ in $\mathbb{A}_h$ is normal with normal constant equal to 1 (in this case the norm is called monotone). This simple fact will be crucial for obtaining our results.

3 Main results

3.1 Fixed point results in C^*-algebra-valued metric spaces

In [1], Ma *et al.* introduced the following concept, being, in fact, a special case of previously known concepts of cone metric spaces [12] and cone metric spaces over Banach algebras [13, 14].

Definition 1 ([1], Definition 2.1) Let X be a nonempty set and let $d : X \times X \to \mathbb{A}$ satisfy

(i) $d(x, y) \succeq \theta$ for all $x, y \in X$ and $d(x, y) = \theta \iff x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

As the main result, they proved the following.

Theorem 1 ([1], Theorem 2.1) *Suppose that* $(X, \mathbb{A}, d)$ *is a* C^**-algebra-valued metric space and let for a mapping* $T : X \to X$ *there exists* $a \in \mathbb{A}$ *with* $\|a\| < 1$ *such that*

$$d(Tx, Ty) \preceq a^* d(x, y) a, \quad \text{for all } x, y \in X. \tag{3.1}$$

Then T *has a unique fixed point in* X.

As our first contribution, we prove that Theorem 1 is not a new result.

Theorem 2 *Theorem* 1 *is equivalent to the Banach contraction principle* (*BCP*).

Proof First of all, obviously, taking $\mathbb{A} = \mathbb{R}$ (with standard operations, absolute value as the norm and involution given by $a^* = a$) in Theorem 1, the condition (3.1) reduces to

$$d(Tx, Ty) \le a^2 d(x, y), \quad \text{for all } x, y \in X,$$

with $a^2 \in [0, 1)$, hence Theorem 1 reduces to BCP.

Conversely, let the conditions of Theorem 1 be satisfied. Denote

$$D(x, y) = \|d(x, y)\| \quad \text{for all } x, y \in X.$$

Then it is easy to see that (X, D) is a complete (standard) metric space. In particular, the triangular inequality follows from $\theta \preceq d(x, y) \preceq d(x, z) + d(z, y)$ and Lemma 1(3):

$$\begin{aligned} D(x, y) &= \|d(x, y)\| \le \|d(x, z) + d(z, y)\| \\ &\le \|d(x, z)\| + \|d(z, y)\| = D(x, z) + D(z, y). \end{aligned}$$

Moreover, $T : X \to X$ is a (Banach-type) contraction in (X, D) since (3.1) and Lemma 1(3) imply that, for all $x, y \in X$,

$$\begin{aligned} D(Tx, Ty) &\le \|a^* d(x, y) a\| \le \|a^*\| \|d(x, y)\| \|a\| \\ &= \|a\|^2 D(x, y), \end{aligned}$$

where $\|a\|^2 \in [0, 1)$. Hence, BCP implies that T has a unique fixed point. □

Remark 1 In a similar way, it is easy to show that the following results from [1, 4] can be directly reduced to their well-known standard metric counterparts:

(1) the fixed point result for expansion mappings [1], Theorem 2.2;

(2) the Chatterjea fixed point result [1], Theorem 2.3 (with the contractive condition in the form $d(Tx, Ty) \preceq a^*(d(x, Ty) + d(y, Tx))a$, $a \in \mathbb{A}$, $\|a\| < 1/\sqrt{2}$);

(3) a fixed point result for contractions in C^*-algebra-valued spaces endowed with a graph [4], Theorem 2.5 (this reduces to [15], Theorem 3.1).

In fact, the same is true for several more general results, *e.g.*, for most of the fixed point results contained in the well-known paper [16]. As an example, we prove the following.

Theorem 3 *Let* $(X, \mathbb{A}, d)$ *be a* C^**-algebra-valued metric space and* $T : X \to X$ *be a mapping. Suppose that there exists* $a \in \mathbb{A}$ *with* $\|a\| < 1$ *and that for all* $x, y \in X$ *there exists*

$$u(x, y) \in \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{3.2}$$

such that

$$d(Tx, Ty) \preceq a^* u(x, y) a. \tag{3.3}$$

Then T *has a unique fixed point in* X.

Proof As in the proof of Theorem 2, denote $D(x,y) = \|d(x,y)\|$ for $x,y \in X$. Then (X,D) is a complete (standard) metric space. For arbitrary $x,y \in X$, choose $u(x,y)$ such that (3.2) and (3.3) hold. Then, by Lemma 1(3),

$$\begin{aligned} D(Tx,Ty) = \|d(Tx,Ty)\| &\leq \|a^*\| \cdot \|u(x,y)\| \cdot \|a\| \\ &\leq \|a\|^2 \max\{\|d(x,y)\|, \|d(x,Tx)\|, \|d(y,Ty)\|, \|d(x,Ty)\|, \|d(y,Tx)\|\} \\ &= \|a\|^2 \max\{D(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\}, \end{aligned}$$

where $\|a\|^2 \in [0,1)$. Hence, $T: X \to X$ is a quasicontraction (in the sense of [17]) and it follows that it has a unique fixed point in X. □

Of course, the results of Kannan, Zamfirescu, Hardy-Rogers (and many others; see [16]) follow as special cases.

Moreover, several known common fixed point results can be easily reformulated in the framework of C^*-algebra-valued metric spaces.

3.2 Fixed point results in C^*-algebra-valued *b*-metric spaces

In an attempt to extend further the obtained results, Ma and Jiang introduced in [2] the following concept (thus generalizing the concept of a *b*-metric space of Czerwik [18]).

Definition 2 ([2], Definition 2.1) Let X be a nonempty set. A mapping $d: X \times X \to \mathbb{A}$ is called a C^*-algebra-valued *b*-metric on X if there exists $b \in \mathbb{A}$ such that $b \succeq i$ and the following conditions are satisfied:

(i) $d(x,y) \succeq \theta$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,y) \preceq b[d(x,z) + d(z,y)]$ for all $x,y,z \in X$.

Then $(X,\mathbb{A},d,b)$ is called a C^*-algebra-valued *b*-metric space.

In [2], as well as in [5, 6], several fixed point results were obtained in C^*-algebra-valued *b*-metric spaces. However, we will show that neither of these results is in fact new - all of them can be simply reduced to their known *b*-metric counterparts. As an example, we prove this for the following result from [2].

Theorem 4 ([2], Theorem 2.1) *Suppose that $(X,\mathbb{A},d,b)$ is a C^*-algebra-valued b-metric space and that for a mapping $T: X \to X$ there exists $a \in \mathbb{A}$ with $\|a\| < 1$ such that*

$$d(Tx,Ty) \preceq a^* d(x,y) a \quad \text{for all } x,y \in X. \tag{3.4}$$

Then there exists a unique fixed point of T in X.

Recall the following *b*-metric version of BCP.

Theorem 5 ([19], Theorem 2.1) *Let (X,D,s) be a complete b-metric space and let $T: X \to X$ be a map such that, for some $\lambda \in [0,1)$ and for all $x,y \in X$,*

$$D(Tx,Ty) \leq \lambda D(x,y). \tag{3.5}$$

Then T has a unique fixed point in X.

Theorem 6 *Theorem* 4 *is equivalent to Theorem* 5.

Proof Again, it is obvious that Theorem 4 implies Theorem 5. In order to prove the opposite, it is enough to put $D(x,y) = \|d(x,y)\|$, $\|b\| = s$, and $\|a\| = \lambda \in [0,1)$, whence (X,D,s) becomes a complete b-metric space and the condition (3.4) reduces to the condition (3.5). This proves our claim. □

Remark 2 We note some other results from [2, 6] that can be reduced in the same way to well-known results in b-metric spaces:

(1) the Chatterjea-type fixed point result [2], Theorem 2.2 (with the contractive condition in the form $d(Tx,Ty) \preceq a^*(d(x,Ty)+d(y,Tx))a$, with $a \in \mathbb{A}$, $\|a\| < 1/\|b\|\sqrt{2}$);

(2) the Kannan-type fixed point result [2], Theorem 2.3 (with the contractive condition in the form $d(Tx,Ty) \preceq a^*(d(x,Tx)+d(y,Ty))a$, with $a \in \mathbb{A}$, $\|a\| < 1/\sqrt{2\|b\|}$);

(3) the Banach-type cyclic fixed point result [6], Theorem 4.1 (with the improved condition $\|\lambda\| < 1$ instead of $\|\lambda\| < 1/\|b\|$);

(4) the Banach-type fixed point result for expansive mappings [6], Theorem 4.4 (the same comment);

(5) the Kannan-type, resp. Chatterjea-type cyclic fixed point results [6], Theorem 4.5 and Theorem 4.7 (with contractive conditions as in (2), resp. (1)).

Naturally, the same applies to several other fixed and common fixed point results in b-metric spaces.

Remark 3 We note that the conclusions of this paper do not hold in cone metric spaces over Banach algebras treated in [13, 14] and several other articles. Namely, Lemma 1(3) does not necessarily hold in arbitrary Banach algebras. Also, in the fixed point results obtained in these spaces, usually the spectral radius $r(a)$ is used instead of the norm $\|a\|$. Since, in general, $r(a) < \|a\|$ (in Banach algebras which are not C^*-algebras), these results are more general and cannot be reduced (at least not directly) to their metric counterparts.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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