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REVIEW

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A note on 'Some fixed point theorems for generalized contractive mappings in complete metric spaces'

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Abstract

Very recently, Hussain *et al.* (Fixed Point Theory Appl. 2015:185, 2015) introduced the concept of JS-contraction and established some fixed point theorems for such contractions. In this paper, we introduce a new method of proofs that allows us to prove fixed point theorems for JS-contraction in complete metric spaces by removing two conditions in theorems of Hussain *et al.* Thus, we prove that fixed point Theorems 2.3-2.8 and Corollary 2.9 of Hussain *et al.* actually are consequences, and not generalizations, of the corresponding theorems of Ćirić, Chatterjea, Kannan, and Reich.

MSC: 47H10; 54H25

Keywords: fixed point theorem; Ćirić contraction; JS-contraction

1 Introduction

The Banach contraction principle [2] is the first important result on fixed points for contractive-type mappings, which states that each Banach contraction $T: X \to X$ (*i.e.*, there exists $\lambda \in [0,1)$ such that $d(Tx, Ty) \le \lambda d(x,y)$ for each $x, y \in X$) has a unique fixed point, provided that (X, d) is a complete metric space. This well-known theorem, which is an essential tool in many branches of mathematical analysis, first appeared in an explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution of an integral equation. So far, according to its importance and simplicity, many authors have obtained interesting extensions and generalizations of the Banach contraction principle (see [1, 3–11]).

The concepts of Ćirić contraction and JS-contraction have been introduced, respectively, by Ćirić [6] and Hussain *et al.* [1] as follows.

Definition 1 Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be:

(i) a Ćirić contraction (see [6]) if there exist nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$ such that

$$d(Tx, Ty) \le qd(x,y) + rd(x, Tx) + sd(y, Ty) + t\big[d(x, Ty) + d(y, Tx)\big],$$
$$\forall x, y \in X; \tag{1}$$

(ii) a JS-contraction (see [1]) if there exist $\psi \in \Psi$ and nonnegative numbers q, r, s, t with $q + r + s + 2t < 1$ such that

$$\psi\left(d(Tx, Ty)\right) \leq \psi\left(d(x, y)\right)^q \psi\left(d(x, Tx)\right)^r \psi\left(d(y, Ty)\right)^s \psi\left(d(x, Ty) + d(y, Tx)\right)^t, \quad \forall x, y \in X, \tag{2}$$

where Ψ is the set of all functions $\psi : [0, +\infty) \to [1, +\infty)$ satisfying the following conditions:

(ψ_1) ψ is nondecreasing, and $\psi(t) = 1$ if and only if $t = 0$;
(ψ_2) for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(ψ_3) there exist $r \in (0, 1)$ and $l \in (0, +\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = l$;
(ψ_4) $\psi(a + b) \leq \psi(a)\psi(b)$ for all $a, b > 0$.

For convenience, we denote by Ψ_1 the set of all nondecreasing functions $\psi : (0, +\infty) \to (1, +\infty)$ satisfying (ψ_2) and (ψ_3) and by Ψ_2 the set of all functions $\psi : [0, +\infty) \to [1, +\infty)$ satisfying (ψ_1), (ψ_2), and (ψ_4).

Remark 1

(i) If $f(t) = e^{\sqrt{t}}$ for $t \geq 0$, then $f \in \Psi \cap \Psi_1 \cap \Psi_2$. If $g(t) = e^t$ for $t \geq 0$, then $g \in \Psi_2$, but $g \notin \Psi \cup \Psi_1$ since $\lim_{t\to 0^+} \frac{e^t-1}{t^r} = 0$ for each $r \in (0, 1)$, that is, (ψ_3) is not satisfied. If $h(t) = e^{\sqrt{te^t}}$ for $t \geq 0$, then $h \in \Psi_1$, but $h \notin \Psi \cup \Psi_2$ since $e^{\sqrt{(t_0+s_0)e^{(t_0+s_0)}}} = e^{\sqrt{2}e} > e^{2\sqrt{e}} = e^{\sqrt{t_0e^{t_0}}} e^{\sqrt{s_0e^{s_0}}}$ whenever $t_0 = s_0 = 1$, that is, (ψ_4) is not satisfied.
(ii) Clearly, $\Psi \subseteq \Psi_1$ and $\Psi \subseteq \Psi_2$. Moreover, from (i) it follows that $\Psi \subset \Psi_1$ and $\Psi \subset \Psi_2$.
(iii) From (i) we conclude that $\Psi_1 \not\subset \Psi_2$, $\Psi_2 \not\subset \Psi_1$, and $\Psi_1 \cap \Psi_2 \neq \emptyset$.

In 1971, Ćirić [6] established the following fixed point theorem.

Theorem 1 ([6]) *Let (X, d) be a complete metric space, and $T : X \to X$ be a Ćirić contraction. Then T has a unique fixed point in X.*

Recently, Jleli and Samet [8] proved the following fixed point theorem, which is a real generalization of the Banach contraction principle.

Theorem 2 ([8], Corollary 2.1) *Let (X, d) be a complete metric space, and $T : X \to X$. Assume that there exist $\psi \in \Psi_1$ and $k \in (0, 1)$ such that*

$$\forall x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \psi\left(d(Tx, Ty)\right) \leq \psi\left(d(x, y)\right)^k. \tag{3}$$

Then T has a unique fixed point in X.

The Banach contraction principle follows immediately from Theorem 2. Indeed, let $T : X \to X$ and $k \in (0, 1)$ be such that (3) holds. Then, if we choose $\psi(t) = e^{\sqrt{t}} \in \Psi_1$ and $k = \sqrt{\lambda}$ in (3), then we get $\sqrt{d(Tx, Ty)} \leq \sqrt{\lambda}\sqrt{d(x, y)}$, that is,

$$d(Tx, Ty) \leq \lambda d(x, y), \quad \forall x, y \in X,$$

which means that T is a Banach contraction. Note that Theorem 2 is a real generalization of the Banach contraction principle (see Example in [8]), but the Banach contraction principle is not a particular case of Theorem 2 with $\psi(t) = e^t$ since $e^t \notin \Psi_1$.

Very recently, Hussain *et al.* [1] presented the following extension of Theorem 2.

Theorem 3 ([1], Theorem 2.3) *Let (X,d) be a complete metric space, and $T : X \to X$ a continuous JS-contraction. Then T has a unique fixed point in X.*

Remark 2 It is clear that Theorem 1 is not a particular case of Theorem 3 since in Theorem 1 the mapping T does not have to be continuous. In addition, even letting $\psi(t) = e^{\sqrt{t}}$ in (2), we only obtain

$$\sqrt{d(Tx,Ty)} \le q\sqrt{d(x,y)} + r\sqrt{d(x,Tx)} + s\sqrt{d(y,Ty)} + t\sqrt{d(x,Ty) + d(y,Tx)}, \quad \forall x,y \in X,$$

which does not imply (1) whenever $qr + rs + st \neq 0$, and hence Theorem 1 cannot be derived from Theorem 3 by using the method used in [8]. Therefore, Theorem 3 may not be a real generalization of Theorem 1.

The main purpose of this paper is to show that the results concerned in metric spaces with JS-contractions in [1] are immediate consequences of Theorem 1. Note that in [1] b-complete b-metric spaces are also considered.

In this paper, we first introduce a new metric D in a given metric space (X,d) induced by the metric d, and then we prove that (X,D) is complete if and only if (X,d) is complete. Then we show that each JS-contraction with $\psi \in \Psi_2$ in (X,d) is certainly a Ćirić contraction in (X,D). By using a new method we prove that Theorem 3 remains valid without assumption (ψ_3) and the continuity of T, which appear in Theorem 3. Therefore, Theorem 3 and Theorems 2.3-2.8 and Corollary 2.9 in [1] are not generalizations of Ćirić, Chatterjea, Kannan, and Reich theorems, as asserted in [1].

2 Main results

For $\psi \in \Psi_2$ and $t \in [0,+\infty)$, set $\eta(t) = \ln(\psi(t))$. Then it is easy to check that $\eta : [0,+\infty) \to [0,+\infty)$ has the following properties:

(η_1) η is nondecreasing, and $\eta(t) = 0$ if and only if $t = 0$;
(η_2) for each sequence $\{t_n\} \subset (0,+\infty)$, $\lim_{n\to\infty} \eta(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$;
(η_3) $\eta(a+b) \le \eta(a) + \eta(b)$ for all $a,b > 0$.

Since (η_1) and (η_2) are clear, we only show (η_3). We have

$$\eta(a+b) = \ln\bigl(\psi(a+b)\bigr) \le \ln\bigl(\psi(a)\psi(b)\bigr) = \ln\bigl(\psi(a)\bigr) + \ln\bigl(\psi(b)\bigr) = \eta(a) + \eta(b).$$

Lemma 1 *Let (X,d) be a metric space, and $\psi \in \Psi_2$. Then (X,D) is a metric space, where $D(x,y) = \eta(d(x,y)) = \ln(\psi(d(x,y)))$.*

Proof For each $x \in X$, we have $D(x,x) = \eta(d(x,x)) = 0$ by (η_1). For all $x,y \in X$ with $D(x,y) = 0$, we have $\eta(d(x,y)) = 0$ and hence $d(x,y) = 0$ by (η_1). Hence, for all $x,y \in X$, $D(x,y) = 0$ if and only if $x = y$.

For all $x,y \in X$, we have $D(x,y) = \eta(d(x,y)) = \eta(d(y,x)) = D(y,x)$.

For all $x, y, z \in X$ with $z \neq x$ and $z \neq y$, by (η_1) and (η_3) we have $D(x,y) = \eta(d(x,y)) \leq \eta(d(x,z) + d(z,y)) \leq \eta(d(x,z)) + \eta(d(z,y)) = D(x,z) + D(z,y)$. For all $x \in X$ and $y = z \in X$, we have $D(x,y) = D(x,z) = D(x,z) + D(y,z)$ by (η_1). For all $x = z \in X$ and $y \in X$, we have $D(x,y) = D(z,y) = D(x,z) + D(z,y)$ by (η_1). For all $x = y = z \in X$, we have $D(x,y) = 0 = D(x,z) + D(y,z)$ by (η_1). Hence, for all $x, y, z \in X$, we always have $D(x,y) \leq D(x,z) + D(z,y)$. This shows that (X, D) is a metric space. The proof is complete. □

Lemma 2 *Let (X,d) be a metric space, and $\psi \in \Psi_2$. Then (X,D) is complete if and only if (X,d) is complete, where $D(x,y) = \eta(d(x,y)) = \ln(\psi(d(x,y)))$.*

Proof Suppose that (X,d) is complete and $\{x_n\}$ is a Cauchy sequence of (X,D), that is, $\lim_{m,n\to\infty} D(x_n,x_m) = 0$. Then we have $\lim_{m,n\to\infty} \eta(d(x_n,x_m)) = 0$, and hence $\lim_{m,n\to\infty} d(x_n, x_m) = 0$ by (η_2). Moreover, by the completeness of (X,d) there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n,x) = 0$, and so $\lim_{n\to\infty} D(x_n,x) = \lim_{n\to\infty} \eta(d(x_n,x)) = 0$ by (η_2). Hence, (X,D) is complete. Similarly, we can show that if (X,D) is complete, then (X,d) is complete. The proof is complete. □

Lemma 3 *Let (X,d) be a metric space, and $T : X \to X$ be a JS-contraction with $\psi \in \Psi_2$. Then T is a Ćirić contraction in (X,D), where $D(x,y) = \eta(d(x,y)) = \ln(\psi(d(x,y)))$.*

Proof It follows from (2) that, for all $x, y \in X$,

$$\begin{aligned} D(Tx,Ty) &= \eta\bigl(d(Tx,Ty)\bigr) = \ln\bigl(\psi\bigl(d(Tx,Ty)\bigr)\bigr) \\ &\leq \ln\bigl(\psi\bigl(d(x,y)\bigr)^q \psi\bigl(d(x,Tx)\bigr)^r \psi\bigl(d(y,Ty)\bigr)^s \psi\bigl(d(x,Ty)+d(y,Tx)\bigr)^t\bigr) \\ &= q\ln\bigl(\psi\bigl(d(x,y)\bigr)\bigr) + r\ln\bigl(\psi\bigl(d(x,Tx)\bigr)\bigr) + s\ln\bigl(\psi\bigl(d(y,Ty)\bigr)\bigr) \\ &\quad + t\bigl[\ln\bigl(\psi\bigl(d(x,Ty)\bigr)\bigr) + \ln\bigl(\psi\bigl(d(y,Tx)\bigr)\bigr)\bigr] \\ &= qD(x,y) + rD(x,Tx) + sD(y,Ty) + t\bigl[D(x,Ty) + D(y,Tx)\bigr], \end{aligned}$$

that is, (1) is satisfied with respect to the metric D, and hence T is a Ćirić contraction in (X,D). The proof is complete. □

Theorem 4 *Let (X,d) be a complete metric space, and $T : X \to X$ be a JS-contraction with $\psi \in \Psi_2$. Then T has a unique fixed point in X.*

Proof Since (X,d) is a complete metric space, (X,D) is also a complete metric space by Lemma 2. Note that T is a Ćirić contraction in (X,D) by Lemma 3. Therefore, T has a unique fixed point in X by Theorem 1. The proof is complete. □

Remark 3 In comparison with Theorem 3, assumption (ψ_3) and the continuity of T have been removed from Theorem 4. Hence, Theorem 4 indeed improves Theorem 3.

Theorem 5 *Theorem* 4 *implies Theorem* 1.

Proof Let $\psi(t) = e^t$ for $t \geq 0$. Clearly, $e^t \in \Psi_2$ by Remark 1. By (2) we have

$$d(Tx,Ty) \leq qd(x,y) + rd(x,Tx) + sd(y,Ty) + t\bigl[d(x,Ty) + d(y,Tx)\bigr], \quad \forall x,y \in X,$$

which implies that a Ćirić contraction $T : X \to X$ is certainly a JS-contraction with $\psi(t) = e^t$. Thus, Theorem 1 immediately follows from Theorem 4. The proof is complete. □

Remark 4 It follows from Theorem 5 and the proof of Theorem 4 that Theorem 1 is equivalent to Theorem 4.

Remark 5 It is clear that Theorems 2.3-2.8 and Corollary 2.9 are immediate consequences of Theorem 1 but the converse is not true by Remark 2, and hence they are not real generalizations of Theorem 1. Note that Hussain *et al.* [1] also considered sufficient conditions for the existence of a fixed point of a JS-contraction in b-complete b-metric spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have contributed in this work on an equal basis. All authors read and approved the final manuscript.

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References
1. Hussain, N, Parvaneh, V, Samet, B, Vetro, C: Some fixed point theorems for generalized contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2015**, 185 (2015)
2. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 133-181 (1922) (in French)
3. Kannan, R: Some results on fixed points. Bull. Calcutta Math. Soc. **60**, 71-76 (1968)
4. Chatterjea, SK: Fixed point theorems. C. R. Acad. Bulgare Sci. **25**, 727-730 (1972)
5. Reich, S: Some remarks concerning contraction mappings. Can. Math. Bull. **14**, 121-124 (1971)
6. Ćirić, L: Generalized contractions and fixed-point theorems. Publ. Inst. Math. (Belgr.) **12**(26), 19-26 (1971)
7. Ćirić, L: A generalization of Banach's contraction principle. Proc. Am. Math. Soc. **45**(2), 267-273 (1974)
8. Jleli, M, Samet, B: A new generalization of the Banach contraction principle. J. Inequal. Appl. **2014**, 38 (2014)
9. Aydi, H, Karapinar, E, Samet, B: Remarks on some recent fixed point theorems. Fixed Point Theory Appl. **2012**, 76 (2012)
10. Aydi, H, Abbas, M, Vetro, C: Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces. Topol. Appl. **159**, 3234-3242 (2012)
11. Aydi, H, Karapinar, E, Samet, B: Fixed points for generalized (α, ψ)-contractions on generalized metric spaces. J. Inequal. Appl. **2014**, 229 (2014)