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Common fixed point theorems for three pairs of self-mappings satisfying the common $(E.A)$ property in Menger probabilistic G-metric spaces

Chuanxi Zhu, Qiang Tu* and Zhaoqi Wu

*Correspondence: qiangtu126@126.com
Department of Mathematics, Nanchang University, Nanchang, 330031, P.R. China

Abstract

In this paper, we generalize the algebraic sum $\oplus$ of Fang. Based on this concept, we prove some common fixed point theorems for three pairs of self-mappings satisfying the common (*E.A*) property in Menger *PGM*-spaces. Finally, an example is given to exemplify our main results.

MSC: Primary 47H10; secondary 46S10

Keywords: common fixed point; Menger *PGM*-space; common (*E.A*) property; weakly compatible mappings

1 Introduction

As a generalization of a metric space, the concept of a probabilistic metric space has been introduced by Menger [1, 2]. Fixed point theory in a probabilistic metric space is an important branch of probabilistic analysis, and many results on the existence of fixed points or solutions of nonlinear equations under various types of conditions in Menger PM-spaces have been extensively studied by many scholars (see *e.g.* [3, 4]). In 2006, Mustafa and Sims [5] introduced the concept of a generalized metric space, based on the notion of a generalized metric space, many authors obtained many fixed point theorems for mappings satisfying different contractive conditions in generalized metric spaces (see [6–12]). Moreover, Zhou *et al.* [13] defined the notion of a generalized probabilistic metric space or a PGM-space as a generalization of a PM-space and a G-metric space. After that, Zhu *et al.* [14] obtained some fixed point theorems.

In 2002, Aamri and Moutawakil [15] defined a property for a pair of mappings, *i.e.*, the so-called property $(E.A)$, which is a generalization of the concept of noncompatibility. In 2009, Fang and Gang [16] defined the property $(E.A)$ for two mappings in Menger PM-spaces and studied the existence of and common fixed points in such spaces. Recently, Wu *et al.* [17] defined a property for two hybrid pairs of mappings satisfying the common property $(E.A)$ in Menger PM-spaces. Gu and Yin [18] introduced the concept of common $(E.A)$ property and obtained some common fixed point theorems for three pairs of self-mappings satisfying the common $(E.A)$ property in generalized metric spaces.

The aim of this paper is to introduce the common $(E.A)$ property in Menger PGM-spaces, generalize the algebraic sum $\oplus$ in [16], and study the common fixed point theorems for three pairs of weakly compatible self-mappings under strict contractive conditions in Menger PGM-spaces. Our results do not rely on any commuting or continuity condition of the mappings.

2 Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, and $\mathbb{Z}^+$ be the set of all positive integers.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We shall denote by $\mathscr{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:

(1) $\Delta(a,1) = a$;

(2) $\Delta(a,b) = \Delta(b,a)$;

(3) $a \geq b, c \geq d \Rightarrow \Delta(a,c) \geq \Delta(b,d)$;

(4) $\Delta(a, \Delta(b,c)) = \Delta(\Delta(a,b), c)$.

A typical example of a t-norm is Δ_m, where $\Delta_m(a,b) = \min\{a,b\}$, for each $a,b \in [0,1]$.

Definition 2.1 [13] A Menger probabilistic G-metric space (for short, a PGM-space) is a triple (X, G^*, Δ), where X is a nonempty set, Δ is a continuous t-norm, and G^* is a mapping from $X \times X \times X$ into $\mathscr{D}$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x,y,z)) satisfying the following conditions:

(PGM-1) $G^*_{x,y,z}(t) = 1$ for all $x,y,z \in X$ and $t > 0$ if and only if $x = y = z$;

(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x,y,z \in X$ with $z \neq y$ and $t > 0$;

(PGM-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = \cdots$ (symmetry in all three variables);

(PGM-4) $G^*_{x,y,z}(t+s) \geq \Delta(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x,y,z,a \in X$ and $s,t \geq 0$.

Example 2.1 [13] Let (X, G) be a G-metric space, where $G(x,y,z) = |x-y| + |y-z| + |z-x|$. Define $G^*_{x,y,z}(t) = \frac{t}{t+G(x,y,z)}$ for all $x,y,z \in X$. Then (X, G^*, Δ_m) is a Menger PGM-space.

Definition 2.2 [13] Let (X, G^*, Δ) be a Menger PGM-space and x_0 be any point in X. For any $\epsilon > 0$ and δ with $0 < \delta < 1$, and (ϵ, δ)-neighborhood of x_0 is the set of all points y in X for which $G^*_{x_0,y,y}(\epsilon) > 1 - \delta$ and $G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta$. We write

$$N_{x_0}(\epsilon, \delta) = \{y \in X : G^*_{x_0,y,y}(\epsilon) > 1 - \delta, G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta\},$$

which means that $N_{x_0}(\epsilon, \delta)$ is the set of all points y in X for which the probability of the distance from x_0 to y being less than ϵ is greater than $1 - \delta$.

Definition 2.3 [13] Let (X, G^*, Δ) be a PGM-space, and $\{x_n\}$ is a sequence in X.

(1) $\{x_n\}$ is said to be convergent to a point $x \in X$ (write $x_n \to x$), if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $x_n \in N_{x_0}(\epsilon, \delta)$ whenever $n > M_{\epsilon,\delta}$;
(2) $\{x_n\}$ is called a *Cauchy* sequence, if for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\epsilon) > 1 - \delta$ whenever $n, m, l > M_{\epsilon,\delta}$;
(3) (X, G^*, Δ) is said to be complete, if every *Cauchy* sequence in X converges to a point in X.

Remark 2.1 Let (X, G^*, Δ) be a Menger PGM-space, $\{x_n\}$ is a sequence in X. Then the following are equivalent:
(1) $\{x_n\}$ is convergent to a point $x \in X$;
(2) $G^*_{x_n,x_n,x}(t) \to 1$ as $n \to \infty$, for all $t > 0$;
(3) $G^*_{x_n,x,x}(t) \to 1$ as $n \to \infty$, for all $t > 0$.

We can analogously prove the following lemma as in Menger PM-spaces.

Lemma 2.1 *Let (X, G^*, Δ) be a Menger PGM-space with Δ a continuous t-norm, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences in X and $x, y, z \in X$, if $\{x_n\} \to x$, $\{y_n\} \to y$, and $\{z_n\} \to z$ as $n \to \infty$. Then*
(1) $\liminf_{n\to\infty} G^*_{x_n,y_n,z_n}(t) \geq G^*_{x,y,z}(t)$ *for all* $t > 0$;
(2) $G^*_{x,y,z}(t + o) \geq \limsup_{n\to\infty} G^*_{x_n,y_n,z_n}(t)$ *for all* $t > 0$.

Particularly, if t_0 is a continuous point of $G_{x,y,z}(\cdot)$, then $\lim_{n\to\infty} G_{x_n,y_n,z_n}(t_0) = G_{x,y,z}(t_0)$.

Lemma 2.2 [14] *Let (X, G^*, Δ) be a Menger PGM-space. For each $\lambda \in (0, 1]$, define a function G^*_λ by*

$$G^*_\lambda(x, y, z) = \inf_t \bigl\{ t \geq 0 : G^*_{x,y,z}(t) > 1 - \lambda \bigr\},$$

for $x, y, z \in X$, then
(1) $G^*_\lambda(x, y, z) < t$ *if and only if* $G^*_{x,y,z}(t) > 1 - \lambda$;
(2) $G^*_\lambda(x, y, z) = 0$ *for all* $\lambda \in (0, 1]$ *if and only if* $x = y = z$;
(3) $G^*_\lambda(x, y, z) = G^*_\lambda(y, x, z) = G^*_\lambda(y, z, x) = \cdots$;
(4) *if* $\Delta = \Delta_m$, *then for every* $\lambda \in (0, 1]$, $G^*_\lambda(x, y, z) \leq G^*_\lambda(x, a, a) + G^*_\lambda(a, y, z)$.

Definition 2.4 [19] Let f and g be self-mappings of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called point of coincidence of f and g.

Definition 2.5 Let S and T be two self-mappings of a Menger PGM-space (X, G^*, Δ). S and T are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, *i.e.*, if $Tu = Su$ for some $u \in X$ implies that $TSu = STu$.

Definition 2.6 [18] Let (X, d) be a G-metric space and A, B, S, and T four self-mappings on X. The pairs (A, S) and (B, T) are said to satisfy the common $(E.A)$ property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$ for some $t \in X$.

Definition 2.7 Let (X, G^*, Δ) be a Menger *PGM*-space and A, B, S, and T four self-mappings on X. The pairs (A, S) and (B, T) are said to satisfy the common $(E.A)$ property if there exist two sequences x_n and y_n in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$ for some $t \in X$.

Definition 2.8 [16] Let $F_1, F_2 \in \mathscr{D}$. The algebraic sum $F_1 \oplus F_2$ of F_1 and F_2 is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\bigl\{F_1(t_1), F_2(t_2)\bigr\} \tag{2.1}$$

for all $t \in \mathbb{R}$.

As a generalization, we give the following definition.

Definition 2.9 Let $F_1, F_2, F_3 \in \mathscr{D}$. The algebraic sum $F_1 \oplus F_2 \oplus F_3$ of F_1, F_2, and F_3 is defined by

$$(F_1 \oplus F_2 \oplus F_3)(t) = \sup_{t_1+t_2+t_3=t} \min\bigl\{F_1(t_1), F_2(t_2), F_3(t_3)\bigr\} \tag{2.2}$$

for all $t \in \mathbb{R}$.

Remark 2.2 Let $F_3(t) = H(t)$, then (2.1) and (2.2) are equivalent.

Definition 2.10 [20] Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and $\phi^n(t)$ be the nth iteration of $\phi(t)$,

(i) ϕ is nondecreasing;
(ii) ϕ is upper semi-continuous from the right;
(iii) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$.

We define Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying conditions (i), (ii), and (iii).

Lemma 2.3 *Let (X, G^*, Δ) be a Menger PGM-space and $x, y, z \in X$. If there exists $\phi \in \Phi$, such that*

$$G^*_{x,y,z}\bigl(\phi(t) + o\bigr) \geq G^*_{x,y,z}(t), \tag{2.3}$$

for all $t > 0$. Then $x = y = z$.

Proof Let $\lambda \in (0, 1]$ and we put $a = G^*_\lambda(x, y, z)$. Since $\phi(\cdot)$ is upper semi-continuous from the right at the point a, for given $\epsilon > 0$, there exists $s > a$ such that $\phi(s) < \phi(a) + \varepsilon$. By Lemma 2.2, $s > G^*_\lambda(x, y, y)$ implies that $G^*_{x,y,z}(s) > 1 - \lambda$. So, it follows from (2.3) that

$$G^*_{x,y,z}\bigl(\phi(s) + \epsilon\bigr) \geq G^*_{x,y,z}\bigl(\phi(s) + o\bigr) \geq G^*_{x,y,z}(s) > 1 - \lambda,$$

which implies that $G^*_\lambda(x, y, z) < \phi(s) + \epsilon < \phi(a) + 2\epsilon$. By the arbitrariness of ϵ, we get $a = G^*_\lambda(x, y, z) \leq \phi(a)$, thus $a = 0$, *i.e.*, $G^*_\lambda(x, y, z) = 0$. By (2) of Lemma 2.2, we conclude that $x = y = z$. □

3 Main results

In this section, we will establish some new common fixed point theorems in Menger *PGM*-spaces.

Theorem 3.1 *Let* (X, G^*, Δ) *be a Menger PGM-space. Suppose the self-mappings* f, g, h, R, S, *and* $T : X \to X$ *satisfy the following conditions*:

$$\begin{aligned} G^*_{fx,gy,hz}\big(\phi(t)\big) \geq \min\big\{ & G^*_{Rx,Sy,Tz}(t), G^*_{fx,Rx,Rx}(t), G^*_{gy,Sy,Sy}(t), G^*_{hz,Tz,Tz}(t), \\ & \big[G^*_{fx,Sy,Tz} \oplus G^*_{Rx,gy,Tz} \oplus G^*_{Rx,Sy,hz}\big](3t), \\ & \big[G^*_{fx,gy,Tz} \oplus G^*_{fx,Sy,hz} \oplus G^*_{Rx,gy,hz}\big](3t)\big\} \end{aligned} \tag{3.1}$$

for all x, y, *and* $z \in X$, $t > 0$, *where* $\phi \in \Phi$. *If one of the following conditions is satisfied, then the pairs* (f, R), (g, S), *and* (h, T) *have a common fixed point of coincidence in* X:

(i) *the subspace* Rx *is closed in* X, $fx \subseteq Sx$, $gx \subseteq Tx$, *and the two pairs of* (f, R) *and* (g, S) *satisfy the common* $(E.A)$ *property*;

(ii) *the subspace* Sx *is closed in* X, $gx \subseteq Tx$, $hx \subseteq Rx$, *and the two pairs of* (g, S) *and* (h, T) *satisfy the common* $(E.A)$ *property*;

(iii) *the subspace* Tx *is closed in* X, $fx \subseteq Sx$, $hx \subseteq Rx$, *and the two pairs of* (f, R) *and* (h, T) *satisfy the common* $(E.A)$ *property.*

Moreover, if the pairs (f, R), (g, S), *and* (h, T) *are weakly compatible, then* f, g, h, R, S, *and* T *have a unique common fixed point in* X.

Proof First, we suppose that the subspace Rx is closed in X, $fx \subseteq Sx$, $gx \subseteq Tx$, and the two pairs of (f, R) and (g, S) satisfy the common $(E.A)$ property. Then by Definition 2.6 we know that there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Sy_n = t,$$

for some $t \in X$. Since $gx \subseteq Tx$, there exists a sequence $\{z_n\}$ in X such that $gy_n = Tz_n$. Hence $\lim_{n\to\infty} Tz_n = a$. Next, we will show $\lim_{n\to\infty} hz_n = a$. In fact, if $\lim_{n\to\infty} hz_n = z \neq a$, then from (3.1) we can get

$$\begin{aligned} G^*_{fx_n,gy_n,hz_n}\big(\phi(t)\big) \geq \min\big\{ & G^*_{Rx_n,Sy_n,Tz_n}(t), G^*_{fx_n,Rx_n,Rx_n}(t), G^*_{gy_n,Sy_n,Sy_n}(t), G^*_{hz_n,Tz_n,Tz_n}(t), \\ & \big[G^*_{fx_n,Sy_n,Tz_n} \oplus G^*_{Rx_n,gy_n,Tz_n} \oplus G^*_{Rx_n,Sy_n,hz_n}\big](3t), \\ & \big[G^*_{fx_n,gy_n,Tz_n} \oplus G^*_{fx_n,Sy_n,hz_n} \oplus G^*_{Rx_n,gy_n,hz_n}\big](3t)\big\}. \end{aligned}$$

On letting $n \to \infty$, and by (2) of Lemma 2.1, we can obtain

$$\begin{aligned} G^*_{a,a,z}\big(\phi(t) + o\big) & \geq \limsup_{n\to\infty} G^*_{fx_n,gy_n,hz_n}\big(\phi(t)\big) \\ & \geq \min\Big\{1, 1, 1, G^*_{z,a,a}(t), \\ & \quad \lim_{n\to\infty}\big[G^*_{fx_n,Sy_n,Tz_n} \oplus G^*_{Rx_n,gy_n,Tz_n} \oplus G^*_{Rx_n,Sy_n,hz_n}\big](3t), \\ & \quad \lim_{n\to\infty}\big[G^*_{fx_n,gy_n,Tz_n} \oplus G^*_{fx_n,Sy_n,hz_n} \oplus G^*_{Rx_n,gy_n,hz_n}\big](3t)\Big\}. \end{aligned} \tag{3.2}$$

In addition, by Definition 2.7, it is easy to verify that

$$\begin{aligned}
&\lim_{n\to\infty}\bigl[G^*_{fx_n,Sy_n,Tz_n}\oplus G^*_{Rx_n,gy_n,Tz_n}\oplus G^*_{Rx_n,Sy_n,hz_n}\bigr](3t)\\
&\quad\geq \lim_{n\to\infty}\min\bigl\{G^*_{fx_n,Sy_n,Tz_n}(t),G^*_{Rx_n,gy_n,Tz_n}(t),G^*_{Rx_n,Sy_n,hz_n}(t)\bigr\}\\
&\quad\geq \min\bigl\{G^*_{a,a,a}(t),G^*_{a,a,a}(t),G^*_{a,a,z}(t)\bigr\}=G^*_{a,a,z}(t).
\end{aligned}\tag{3.3}$$

Similarly, we also have

$$\lim_{n\to\infty}\bigl[G^*_{fx_n,gy_n,Tz_n}\oplus G^*_{fx_n,Sy_n,hz_n}\oplus G^*_{Rx_n,gy_n,hz_n}\bigr](3t)\geq G^*_{a,a,z}(t).$$

Then (3.2) is

$$G^*_{a,a,z}\bigl(\phi(t)+o\bigr)\geq G^*_{a,a,z}(t)$$

for all $t>0$. By Lemma 2.3, we have $a=z$. So, $\lim_{n\to\infty}hz_n=a$.

Since Rx is a closed subset of X and $\lim_{n\to\infty}Rx_n=a$, there exists $p\in X$ such that $a=Rp$, we claim that $fp=a$. Suppose not, then by using (3.1), we obtain

$$\begin{aligned}
G^*_{fp,gy_n,hz_n}\bigl(\phi(t)\bigr)\geq \min\bigl\{&G^*_{Rp,Sy_n,Tz_n}(t),G^*_{fp,Rp,Rp}(t),G^*_{gy_n,Sy_n,Sy_n}(t),G^*_{hz_n,Tz_n,Tz_n}(t),\\
&\bigl[G^*_{fp,Sy_n,Tz_n}\oplus G^*_{Rp,gy_n,Tz_n}\oplus G^*_{Rp,Sy_n,hz_n}\bigr](3t),\\
&\bigl[G^*_{fp,gy_n,Tz_n}\oplus G^*_{fp,Sy_n,hz_n}\oplus G^*_{Rp,gy_n,hz_n}\bigr](3t)\bigr\}.
\end{aligned}$$

Taking $n\to\infty$ on the two sides of the above inequality, similar to (3.3), we get

$$\begin{aligned}
G^*_{fp,a,a}\bigl(\phi(t)+o\bigr)\geq{}& \min\Bigl\{1,G^*_{fp,a,a}(t),1,1,\lim_{n\to\infty}\bigl[G^*_{fp,Sy_n,Tz_n}\oplus G^*_{Rp,gy_n,Tz_n}\oplus G^*_{Rp,Sy_n,hz_n}\bigr](3t),\\
&\lim_{n\to\infty}\bigl[G^*_{fp,gy_n,Tz_n}\oplus G^*_{fp,Sy_n,hz_n}\oplus G^*_{Rp,gy_n,hz_n}\bigr](3t)\Bigr\}\\
\geq{}& \min\bigl\{1,G^*_{fp,a,a}(t),1,1,G^*_{fp,a,a}(t),G^*_{fp,a,a}(t)\bigr\}=G^*_{fp,a,a}(t).
\end{aligned}$$

By Lemma 2.3, we have $fp=a=Rp$. Hence, p is the coincidence point of the pair (f,R).

By condition $fx\subseteq Sx$ and $fp=a$, there exists $u\in X$ such that $a=Su$. Now we claim that $gu=a$. In fact, if $gu\neq a$, then from (3.1), we have

$$\begin{aligned}
G^*_{fp,gu,hz_n}\bigl(\phi(t)\bigr)\geq \min\bigl\{&G^*_{Rp,Su,Tz_n}(t),G^*_{fp,Rp,Rp}(t),G^*_{gu,Su,Su}(t),G^*_{hz_n,Tz_n,Tz_n}(t),\\
&\bigl[G^*_{fp,Su,Tz_n}\oplus G^*_{Rp,gu,Tz_n}\oplus G^*_{Rp,Su,hz_n}\bigr](3t),\\
&\bigl[G^*_{fp,gu,Tz_n}\oplus G^*_{fp,Su,hz_n}\oplus G^*_{Rp,gu,hz_n}\bigr](3t)\bigr\}.
\end{aligned}$$

Letting $n\to\infty$ on the two sides of the above inequality, we get

$$\begin{aligned}
G^*_{a,gu,a}\bigl(\phi(t)+o\bigr)\geq{}& \min\Bigl\{1,1,G^*_{gu,a,a}(t),1,\lim_{n\to\infty}\bigl[G^*_{fp,Su,Tz_n}\oplus G^*_{Rp,gu,Tz_n}\oplus G^*_{Rp,Su,hz_n}\bigr](3t),\\
&\lim_{n\to\infty}\bigl[G^*_{fp,gu,Tz_n}\oplus G^*_{fp,Su,hz_n}\oplus G^*_{Rp,gu,hz_n}\bigr](3t)\Bigr\}\\
\geq{}& \min\bigl\{1,1,G^*_{gu,a,a}(t),1,G^*_{a,gu,a}(t),G^*_{a,gu,a}(t)\bigr\}=G^*_{a,gu,a}(t).
\end{aligned}$$

By Lemma 2.3, we can also obtain $gu = a$, and so u is the coincidence point of the pair (g, S).

Since $gX \subseteq TX$, there exists $v \in X$ such that $a = Tv$. We claim that $hv = a$. If not, from (3.1), we have

$$\begin{aligned}
G^*_{fp,gu,hv}\big(\phi(t) + o\big) &\geq G^*_{fp,gu,hv}\big(\phi(t)\big) \\
&\geq \min\big\{G^*_{Rp,Su,Tv}(t), G^*_{fp,Rp,Rp}(t), G^*_{gu,Su,Su}(t), G^*_{hv,Tv,Tv}(t), \\
&\quad \big[G^*_{fp,Su,Tv} \oplus G^*_{Rp,gu,Tv} \oplus G^*_{Rp,Su,hv}\big](3t), \\
&\quad \big[G^*_{fp,gu,Tv} \oplus G^*_{fp,Su,hv} \oplus G^*_{Rp,gu,hv}\big](3t)\big\} \\
&\geq \min\big\{1, 1, 1, G^*_{hv,a,a}(t), \big[G^*_{a,a,a} \oplus G^*_{a,a,a} \oplus G^*_{a,a,hv}\big](3t), \\
&\quad \big[G^*_{a,a,a} \oplus G^*_{a,a,hv} \oplus G^*_{a,a,hv}\big](3t)\big\} \\
&\geq \min\big\{1, 1, 1, G^*_{hv,a,a}(t), G^*_{a,a,hv}(t), G^*_{a,a,hv}(t)\big\} = G^*_{a,a,hv}(t).
\end{aligned}$$

By Lemma 2.3, we have $hv = a = Tv$, so v is the coincidence point of the pair (h, T).

Therefore, in all the above cases, we obtain $fp = Rp = a$, $gu = Su = hv = Tv = a$. Now, weak compatibility of the pairs (f, R), (g, S), and (h, T) give $fa = Ra$, $ga = Sa$, and $ha = Ta$.

Next, we show that $fa = a$. In fact, if $fa \neq a$, then from (3.1) we have

$$\begin{aligned}
G^*_{fa,a,a}\big(\phi(t) + o\big) &\geq \min\big\{G^*_{Ra,Su,Tv}(t), G^*_{fa,Ra,Ra}(t), G^*_{gu,Su,Su}(t), G^*_{hv,Tv,Tv}(t), \\
&\quad \big[G^*_{fa,Su,Tv} \oplus G^*_{Ra,gu,Tv} \oplus G^*_{Ra,Su,hv}\big](3t), \\
&\quad \big[G^*_{fa,gu,Tv} \oplus G^*_{fa,Su,hv} \oplus G^*_{Ra,gu,hv}\big](3t)\big\} \\
&\geq \min\big\{G^*_{Ra,a,a}(t), 1, 1, 1, \big[G^*_{fa,a,a} \oplus G^*_{Ra,a,a} \oplus G^*_{Ra,a,a}\big](3t), \\
&\quad \big[G^*_{fa,a,a} \oplus G^*_{fa,a,hv} \oplus G^*_{Ra,a,a}\big](3t)\big\} \\
&\geq \min\big\{1, 1, 1, G^*_{fa,a,a}(t), G^*_{fa,a,a}(t), G^*_{fa,a,a}(t)\big\} = G^*_{fa,a,a}(t).
\end{aligned}$$

From Lemma 2.3 we know $fa = a$ and so $fa = Ra = a$. Similarly, it can be show that $ga = Sa = a$ and $ha = Ta = a$, so we get $fa = ga = ha = Ra = Sa = Ta = a$, which means that a is a common fixed point of f, g, h, R, S, and T.

Next, we will show the uniqueness. Actually, suppose that $w \in X$, $w \neq a$ is another common fixed point of f, g, h, R, S, and T. Then by (3.1), we have

$$\begin{aligned}
G^*_{w,a,a}\big(\phi(t) + o\big) &\geq \min\big\{G^*_{Rw,Sa,Ta}(t), G^*_{fw,Rw,Rw}(t), G^*_{ga,Sa,Sa}(t), G^*_{ha,Ta,Ta}(t), \\
&\quad \big[G^*_{fw,Sa,Ta} \oplus G^*_{Rw,ga,Ta} \oplus G^*_{Rw,Sa,ha}\big](3t), \\
&\quad \big[G^*_{fw,ga,Ta} \oplus G^*_{fw,Sa,ha} \oplus G^*_{Rw,ga,ha}\big](3t)\big\} \\
&\geq \min\big\{G^*_{w,a,a}(t), 1, 1, 1, \big[G^*_{fa,a,a} \oplus G^*_{w,a,a} \oplus G^*_{w,a,a}\big](3t), \\
&\quad \big[G^*_{w,a,a} \oplus G^*_{w,a,hv} \oplus G^*_{w,a,a}\big](3t)\big\} \\
&\geq \min\big\{1, 1, 1, G^*_{w,a,a}(t), G^*_{w,a,a}(t), G^*_{w,a,a}(t)\big\} = G^*_{w,a,a}(t).
\end{aligned}$$

By Lemma 2.3 we have $a = w$, a contradiction, so, f, g, h, R, S, and T have a unique common fixed point.

Finally, if condition (ii) or (iii) holds, then the argument is similar to the above, so we omit it. This completes the proof of Theorem 3.1. □

Taking $\phi(t) = \lambda t$, $\lambda \in (0,1)$, then we can obtain the following results.

Corollary 3.1 *Let (X, G^*, Δ) be a Menger PGM-space. Suppose the self-mappings f, g, h, R, S, and $T : X \to X$ satisfy the following conditions*:

$$\begin{aligned} G^*_{fx,gy,hz}(\lambda t) \geq \min\bigl\{ & G^*_{Rx,Sy,Tz}(t), G^*_{fx,Rx,Rx}(t), G^*_{gy,Sy,Sy}(t), G^*_{hz,Tz,Tz}(t), \\ & \bigl[G^*_{fx,Sy,Tz} \oplus G^*_{Rx,gy,Tz} \oplus G^*_{Rx,Sy,hz}\bigr](3t), \\ & \bigl[G^*_{fx,gy,Tz} \oplus G^*_{fx,Sy,hz} \oplus G^*_{Rx,gy,hz}\bigr](3t)\bigr\} \end{aligned}$$

for all x, y, and $z \in X$, where $\lambda \in (0,1)$. If one of the following conditions is satisfied, then the pairs (f,R), (g,S), and (h,T) have a common fixed point of coincidence in X:

(i) *the subspace Rx is closed in X, $fx \subseteq Sx$, $gx \subseteq Tx$, and the two pairs of (f,R) and (g,S) satisfy the common $(E.A)$ property*;

(ii) *the subspace Sx is closed in X, $gx \subseteq Tx$, $hx \subseteq Rx$, and the two pairs of (g,S) and (h,T) satisfy the common $(E.A)$ property*;

(iii) *the subspace Tx is closed in X, $fx \subseteq Sx$, $hx \subseteq Rx$, and the two pairs of (f,R) and (h,T) satisfy the common $(E.A)$ property.*

Moreover, if the pairs (f,R), (g,S), and (h,T) are weakly compatible, the f, g, h, R, S, and T have a unique common fixed point in X.

Theorem 3.2 *Let (X, G^*, Δ) be a Menger PGM-space. Suppose self-mappings f, g, h, R, S, and $T : X \to X$ satisfying the following conditions*:

$$G^*_{fx,gy,hz}(t) \geq \psi\bigl\{m(x,y,z,t)\bigr\} \tag{3.4}$$

for all x, y, and $z \in X$, where

$$\begin{aligned} m(x,y,z,t) = \min\bigl\{ & G^*_{Rx,Sy,Tz}(t), G^*_{fx,Rx,Rx}(t), G^*_{gy,Sy,Sy}(t), G^*_{hz,Tz,Tz}(t), \\ & \bigl[G^*_{fx,Sy,Tz} \oplus G^*_{Rx,gy,Tz} \oplus G^*_{Rx,Sy,hz}\bigr](3t), \\ & \bigl[G^*_{fx,gy,Tz} \oplus G^*_{fx,Sy,hz} \oplus G^*_{Rx,gy,hz}\bigr](3t)\bigr\}, \end{aligned}$$

ψ is continuous and $\psi(t) > t$ for all $t > 0$. If one of the following conditions is satisfied, then the pairs (f,R), (g,S), and (h,T) have a common fixed point of coincidence in X:

(i) *the subspace Rx is closed in X, $fx \subseteq Sx$, $gx \subseteq Tx$, and the two pairs of (f,R) and (g,S) satisfy the common $(E.A)$ property*;

(ii) *the subspace Sx is closed in X, $gx \subseteq Tx$, $hx \subseteq Rx$, and the two pairs of (g,S) and (h,T) satisfy the common $(E.A)$ property*;

(iii) *the subspace Tx is closed in X, $fx \subseteq Sx$, $hx \subseteq Rx$, and the two pairs of (f,R) and (h,T) satisfy the common $(E.A)$ property.*

Moreover, if the pairs (f,R), (g,S), and (h,T) are weakly compatible, the f, g, h, R, S, and T have a unique common fixed point in X.

Proof First, we suppose that condition (i) is satisfied. Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Sy_n = t,$$

for some $t \in X$.

Since $gx \subseteq Tx$, there exists a sequence $\{z_n\}$ in X such that $gy_n = Tz_n$. Hence $\lim_{n\to\infty} Tz_n = a$. We claim that $\lim_{n\to\infty} hz_n = a$. In fact, if $\lim_{n\to\infty} hz_n = z \neq a$, it is not difficult to prove that there exists $t_0 > 0$ such that

$$\psi\big(G^*_{a,a,z}(t_0)\big) > G^*_{a,a,z}(t_0). \tag{3.5}$$

If not, we have $G^*_{a,a,z}(t) \geq \psi(G^*_{a,a,z}(t)) > G^*_{a,a,z}(t)$ for all $t > 0$, which is a contradiction. Then by (3.4), there exists $t_0 > 0$ such that

$$G^*_{fx_n,gy_n,hz_n}(t_0) \geq \psi\big\{m(x_n,y_n,z_n,t_0)\big\}, \tag{3.6}$$

where

$$\begin{aligned} &\psi\big\{m(x_n,y_n,z_n,t_0)\big\} \\ &\quad= \psi\big\{\min\big\{G^*_{Rx_n,Sy_n,Tz_n}(t_0), G^*_{fx_n,Rx_n,Rx_n}(t_0), G^*_{gy_n,Sy_n,Sy_n}(t_0), G^*_{hz_n,Tz_n,Tz_n}(t_0), \\ &\qquad\big[G^*_{fx_n,Sy_n,Tz_n} \oplus G^*_{Rx_n,gy_n,Tz_n} \oplus G^*_{Rx_n,Sy_n,hz_n}\big](3t_0), \\ &\qquad\big[G^*_{fx_n,gy_n,Tz_n} \oplus G^*_{fx_n,Sy_n,hz_n} \oplus G^*_{Rx_n,gy_n,hz_n}\big](3t_0)\big\}\big\}. \end{aligned} \tag{3.7}$$

Letting $n \to \infty$ in (3.7) and by the property of ψ, we can obtain

$$\begin{aligned} &\lim_{n\to\infty} \psi\big\{m(x_n,y_n,z_n,t_0)\big\} \\ &\quad= \psi\Big\{\lim_{n\to\infty} \min\big\{G^*_{Rx_n,Sy_n,Tz_n}(t_0), G^*_{fx_n,Rx_n,Rx_n}(t_0), G^*_{gy_n,Sy_n,Sy_n}(t_0), G^*_{hz_n,Tz_n,Tz_n}(t_0), \\ &\qquad\big[G^*_{fx_n,Sy_n,Tz_n} \oplus G^*_{Rx_n,gy_n,Tz_n} \oplus G^*_{Rx_n,Sy_n,hz_n}\big](3t_0), \\ &\qquad\big[G^*_{fx_n,gy_n,Tz_n} \oplus G^*_{fx_n,Sy_n,hz_n} \oplus G^*_{Rx_n,gy_n,hz_n}\big](3t_0)\big\}\Big\}. \end{aligned} \tag{3.8}$$

As the proof of Theorem 3.1, we know

$$\begin{aligned} &\lim_{n\to\infty}\big[G^*_{fx_n,Sy_n,Tz_n} \oplus G^*_{Rx_n,gy_n,Tz_n} \oplus G^*_{Rx_n,Sy_n,hz_n}\big](3t_0) \geq G^*_{a,a,z}(t_0), \\ &\lim_{n\to\infty}\big[G^*_{fx_n,gy_n,Tz_n} \oplus G^*_{fx_n,Sy_n,hz_n} \oplus G^*_{Rx_n,gy_n,hz_n}\big](3t_0) \geq G^*_{a,a,z}(t_0). \end{aligned}$$

Then (3.8) is

$$\begin{aligned} \lim_{n\to\infty} \psi\big\{m(x_n,y_n,z_n,t_0)\big\} &\geq \psi\big\{\min\big\{1,1,1,G^*_{z,a,a}(t_0), G^*_{z,a,a}(t_0), G^*_{z,a,a}(t_0)\big\}\big\} \\ &= \psi\big\{G^*_{z,a,a}(t_0)\big\}. \end{aligned} \tag{3.9}$$

Without loss of generality, we assume that t_0 in (3.5) is a continuous point of $G_{a,a,z}(\cdot)$. By the left-continuity of the distribution function and the continuity of ψ, there exists $\delta > 0$ such that

$$\psi\left(G^*_{a,a,z}(t)\right) > G^*_{a,a,z}(t),$$

for all $t \in (t_0 - \delta, t_0]$. Since $G_{a,a,z}(\cdot)$ is nondecreasing, the set of all discontinuous points of $G_{a,a,z}(\cdot)$ is a countable set at most. Thus, when t_0 is a discontinuous point of $G_{Ta,Ta,Sa}(\cdot)$, we can choose a continuous point t_1 of $G_{Ta,Ta,Sa}(\cdot)$ in $(t_0 - \delta, t_0]$ to replace t_0.

Let $n \to \infty$ in (3.6), then we have $G^*_{a,a,z}(t_0) \geq \psi\{G^*_{a,a,z}(t_0)\}$, which contradicts (3.5). Then $a = z$, $\lim_{n\to\infty} hz_n = a$.

Since Rx is a closed subset of X and $\lim_{n\to\infty} Rx_n = a$, there exists p in X such that $a = Rp$, we claim that $fp = a$. Suppose not, then by using (3.4), we obtain

$$\begin{aligned} G^*_{fp,gy_n,hz_n}(t) \geq \psi\big\{\min\big\{&G^*_{Rp,Sy_n,Tz_n}(t), G^*_{fp,Rp,Rp}(t), G^*_{gy_n,Sy_n,Sy_n}(t), G^*_{hz_n,Tz_n,Tz_n}(t),\\ &\big[G^*_{fp,Sy_n,Tz_n} \oplus G^*_{Rp,gy_n,Tz_n} \oplus G^*_{Rp,Sy_n,hz_n}\big](3t),\\ &\big[G^*_{fp,gy_n,Tz_n} \oplus G^*_{fp,Sy_n,hz_n} \oplus G^*_{Rp,gy_n,hz_n}\big](3t)\big\}\big\}. \end{aligned}$$

Similarly, we can get $fp = Rp = a$. Hence, p is the coincidence point of the pair (f,R).

By the condition $fx \subseteq Sx$ and $fp = a$, there exists $u \in X$ such that $a = Su$. Now we claim that $gu = a$. In fact, if $gu \neq a$, then from (3.4), we have

$$\begin{aligned} G^*_{fp,gu,hz_n}(t) \geq \psi\big\{\min\big\{&G^*_{Rp,Su,Tz_n}(t), G^*_{fp,Rp,Rp}(t), G^*_{gu,Su,Su}(t), G^*_{hz_n,Tz_n,Tz_n}(t),\\ &\big[G^*_{fp,Su,Tz_n} \oplus G^*_{Rp,gu,Tz_n} \oplus G^*_{Rp,Su,hz_n}\big](3t),\\ &\big[G^*_{fp,gu,Tz_n} \oplus G^*_{fp,Su,hz_n} \oplus G^*_{Rp,gu,hz_n}\big](3t)\big\}\big\}; \end{aligned}$$

in the same way, we can also obtain $gu = a$, and so u is the coincidence point of the pair (g,S).

Since $gX \subset TX$, there exists $v \in X$ such that $a = Tv$. We claim that $hv = a$. If not, from (3.4) and the property of ψ, we have

$$\begin{aligned} G^*_{fp,gu,hv}(t) &\geq \psi\big\{\min\big\{G^*_{Rp,Su,Tv}(t), G^*_{fp,Rp,Rp}(t), G^*_{gu,Su,Su}(t), G^*_{hv,Tv,Tv}(t),\\ &\quad \big[G^*_{fp,Su,Tv} \oplus G^*_{Rp,gu,Tv} \oplus G^*_{Rp,Su,hv}\big](3t),\\ &\quad \big[G^*_{fp,gu,Tv} \oplus G^*_{fp,Su,hv} \oplus G^*_{Rp,gu,hv}\big](3t)\big\}\big\}\\ &= \psi\big\{\min\big\{1,1,1,G^*_{hv,a,a}(t), \big[G^*_{a,a,a} \oplus G^*_{a,a,a} \oplus G^*_{a,a,hv}\big](3t),\\ &\quad \big[G^*_{a,a,a} \oplus G^*_{a,a,hv} \oplus G^*_{a,a,hv}\big](3t)\big\}\big\}\\ &\geq \psi\big\{\min\big\{1,1,1,G^*_{hv,a,a}(t), G^*_{a,a,hv}(t), G^*_{a,a,hv}(t)\big\}\big\}\\ &= \psi\big\{G^*_{a,a,hv}(t)\big\} > G^*_{a,a,hv}(t), \end{aligned}$$

a contradiction. Hence $hv = Tv = a$, and so v is the coincidence point of the pair (h,T).

Therefore, in all the above cases, we obtain $fp = Rp = a$, $gu = Su = a$, and $hv = Tv = a$. Now, the weak compatibility of the pairs (f,R), (g,S), and (h,T) give $fa = Ra$, $ga = Sa$, and $ha = Ta$.

Next, we show that $fa = a$. In fact, if $fa \neq a$, then from (3.4) we have

$$\begin{aligned} G^*_{fa,a,a}(t) &\geq \psi\bigl\{\min\bigl\{G^*_{Ra,Su,Tv}(t), G^*_{fa,Ra,Ra}(t), G^*_{gu,Su,Su}(t), G^*_{hv,Tv,Tv}(t), \\ &\qquad \bigl[G^*_{fa,Su,Tv} \oplus G^*_{Ra,gu,Tv} \oplus G^*_{Ra,Su,hv}\bigr](3t), \bigl[G^*_{fa,gu,Tv} \oplus G^*_{fa,Su,hv} \oplus G^*_{Ra,gu,hv}\bigr](3t)\bigr\}\bigr\} \\ &= \psi\bigl\{\min\bigl\{G^*_{Ra,a,a}(t), 1, 1, 1, \bigl[G^*_{fa,a,a} \oplus G^*_{Ra,a,a} \oplus G^*_{Ra,a,a}\bigr](3t), \\ &\qquad \bigl[G^*_{fa,a,a} \oplus G^*_{fa,a,hv} \oplus G^*_{Ra,a,a}\bigr](3t)\bigr\}\bigr\} \\ &\geq \psi\bigl\{\min\bigl\{1, 1, 1, G^*_{fa,a,a}(t), G^*_{fa,a,a}(t), G^*_{fa,a,a}(t)\bigr\}\bigr\} \\ &= \psi\bigl\{G^*_{fa,a,a}(t)\bigr\} > G^*_{fa,a,a}(t), \end{aligned}$$

which is a contradiction, hence $fa = a$ and so $fa = Ra = a$. Similarly, it can be shown that $ga = Sa = a$ and $ha = Ta = a$, so we get $fa = ga = ha = Ra = Sa = Ta = a$, which means that a is a common fixed point of f, g, h, R, S, and T.

Next, we will show the uniqueness. Actually, suppose that $w \in X$, $w \neq a$ is another common fixed point of f, g, h, R, S, and T. Then by (3.4), we have

$$\begin{aligned} G^*_{w,a,a}(t) &\geq \psi\bigl\{\min\bigl\{G^*_{Rw,Sa,Ta}(t), G^*_{fw,Rw,Rw}(t), G^*_{ga,Sa,Sa}(t), G^*_{ha,Ta,Ta}(t), \\ &\qquad \bigl[G^*_{fw,Sa,Ta} \oplus G^*_{Rw,ga,Ta} \oplus G^*_{Rw,Sa,ha}\bigr](3t), \\ &\qquad \bigl[G^*_{fw,ga,Ta} \oplus G^*_{fw,Sa,ha} \oplus G^*_{Rw,ga,ha}\bigr](3t)\bigr\}\bigr\} \\ &\geq \psi\bigl\{\min\bigl\{G^*_{w,a,a}(t), 1, 1, 1, \bigl[G^*_{fa,a,a} \oplus G^*_{w,a,a} \oplus G^*_{w,a,a}\bigr](3t), \\ &\qquad \bigl[G^*_{w,a,a} \oplus G^*_{w,a,hv} \oplus G^*_{w,a,a}\bigr](3t)\bigr\}\bigr\} \\ &\geq \psi\bigl\{\min\bigl\{1, 1, 1, G^*_{w,a,a}(t), G^*_{w,a,a}(t), G^*_{w,a,a}(t)\bigr\}\bigr\} \\ &= \psi\bigl\{G^*_{w,a,a}(t)\bigr\} > G^*_{w,a,a}(t), \end{aligned}$$

which is a contradiction, so f, g, h, R, S, and T have a unique common fixed point.

Finally, if condition (ii) or (iii) holds, then the argument is similar to the above, so we omit it. This completes the proof of Theorem 3.2. □

Taking $\psi(t) = \rho t$, $\rho \in (1, +\infty)$, then we can obtain the following results.

Corollary 3.2 *Let* (X, G^*, Δ) *be a Menger PGM-space. Suppose the self-mappings* f, g, h, R, S, *and* $T : X \to X$ *satisfy the following conditions*:

$$\begin{aligned} G^*_{fx,gy,hz}(t) \geq \rho \min\bigl\{&G^*_{Rx,Sy,Tz}(t), G^*_{fx,Rx,Rx}(t), G^*_{gy,Sy,Sy}(t), G^*_{hz,Tz,Tz}(t), \\ &\bigl[G^*_{fx,Sy,Tz} \oplus G^*_{Rx,gy,Tz} \oplus G^*_{Rx,Sy,hz}\bigr](3t), \\ &\bigl[G^*_{fx,gy,Tz} \oplus G^*_{fx,Sy,hz} \oplus G^*_{Rx,gy,hz}\bigr](3t)\bigr\}, \end{aligned}$$

for all x, y, *and* $z \in X$, *where* $\rho \in (1, +\infty)$. *If one of the following conditions is satisfied, then the pairs* (f, R), (g, S), *and* (h, T) *have a common fixed point of coincidence in* X:

(i) *the subspace* Rx *is closed in* X, $fx \subseteq Sx$, $gx \subseteq Tx$, *and the two pairs of* (f, R) *and* (g, S) *satisfy the common* (*E.A*) *property*;

(ii) *the subspace Sx is closed in X, $gx \subseteq Tx$, $hx \subseteq Rx$, and the two pairs of (g,S) and (h,T) satisfy the common $(E.A)$ property*;

(iii) *the subspace Tx is closed in X, $fx \subseteq Sx$, $hx \subseteq Rx$, and the two pairs of (f,R) and (h,T) satisfy the common $(E.A)$ property.*

Moreover, if the pairs (f,R), (g,S), and (h,T) are weakly compatible, the f, g, h, R, S, and T have a unique common fixed point in X.

4 An application

In this section, we will provide an example to exemplify the validity of the main result.

Example 4.1 Let $X = [0,1]$, $G^*_{x,y,z}(t) = \frac{t}{t+|x-y|+|y-z|+|z-x|}$, from Example 2.1, we know (X, G^*, Δ) is a PGM-space. We define the mappings f, g, h, R, S, and T by

$$fx = \begin{cases} 0, & x \in [0,\frac{1}{2}], \\ \frac{1}{7}, & x \in (\frac{1}{2},1], \end{cases} \qquad gx = \begin{cases} 0, & x \in [0,\frac{1}{2}], \\ \frac{1}{8}, & x \in (\frac{1}{2},1], \end{cases}$$

$$hx = \begin{cases} 0, & x \in [0,\frac{1}{2}], \\ \frac{1}{6}, & x \in (\frac{1}{2},1], \end{cases} \qquad Rx = \begin{cases} 0, & x \in [0,\frac{1}{2}], \\ \frac{2}{3}, & x \in (\frac{1}{2},1], \end{cases}$$

$$Sx = \begin{cases} 0, & x \in [0,\frac{1}{2}), \\ \frac{1}{7}, & x = \frac{1}{2}, \\ \frac{3}{4}, & x \in (\frac{1}{2},1], \end{cases} \qquad Tx = \begin{cases} 0, & x \in [0,\frac{1}{2}], \\ \frac{1}{8}, & x = \frac{1}{2}, \\ \frac{2}{5}, & x \in (\frac{1}{2},1]. \end{cases}$$

Noting that f, g, h, R, S, and T are discontinuous mappings, RX is closed in X. From the definition of f, g, h, R, S, and T, we have $fx \subseteq Sx$, $gx \subseteq Tx$; let $x_n = \frac{1}{n} + \frac{1}{3}$, $y_n = \frac{1}{n} + \frac{1}{4}$, then the pairs (f,R) and (g,S) satisfy the common $(E.A)$ property. Thus, the condition (i) in Theorem 3.1 is satisfied. It is not difficult to find that (f,R), (g,S), and (h,T) are weakly compatible. Let $\phi(t) = \frac{5}{12}(t)$. Next we will show that (3.1) is also satisfied.

To prove (3.1), we just need to show $G^*_{fx,gy,hz}(\phi(t)) \geq G^*_{Rx,Sy,Tz}(t)$; we discuss the following cases.

Case (1). For $x,y,z \in [0,\frac{1}{2}]$, we have $G^*_{fx,gy,hz}(\phi(t)) = 1$, then (3.1) is obviously satisfied.

Case (2). For $x,y,z \in (\frac{1}{2},1]$, we have

$$G^*_{fx,gy,hz}(\phi(t)) = \frac{t}{t+\frac{1}{5}} \geq \frac{t}{t+\frac{7}{10}} = G^*_{Rx,Sy,Tz}(t).$$

Case (3). For $x,y \in [0,\frac{1}{2}]$, $z \in (\frac{1}{2},1]$, it is not difficult to find that $G^*_{Rx,Sy,Tz}(t) = \frac{t}{t+\frac{4}{5}}$, neither $y \in [0,\frac{1}{2})$ nor $y = \frac{1}{2}$. On the other hand, $G^*_{fx,gy,hz}(\phi(t)) = \frac{t}{t+\frac{4}{5}}$, we have

$$G^*_{fx,gy,hz}(\phi(t)) \geq G^*_{Rx,Sy,Tz}(t).$$

Case (4). For $x,z \in [0,\frac{1}{2}]$, $y \in (\frac{1}{2},1]$, similar to Case (3), we have

$$G^*_{fx,gy,hz}(\phi(t)) = \frac{t}{t+\frac{3}{5}} \geq \frac{t}{t+\frac{3}{2}} = G^*_{Rx,Sy,Tz}(t).$$

Case (5). For $y,z \in [0,\frac{1}{2}]$, $x \in (\frac{1}{2},1]$, we have $G^*_{fx,gy,hz}(\phi(t)) = \frac{t}{t+\frac{4}{5}}$. Next we divide the study into two subcases.

(a) If $y = z = \frac{1}{2}$, $x \in (\frac{1}{2}, 1]$, we have $G^*_{Rx,Sy,Tz}(t) = \frac{t}{t+\frac{22}{21}}$, then

$$G^*_{fx,gy,hz}\bigl(\phi(t)\bigr) \geq G^*_{Rx,Sy,Tz}(t).$$

(b) If $y \neq \frac{1}{2}$ or $z \neq \frac{1}{2}$, $x \in (\frac{1}{2}, 1]$, we have $G^*_{Rx,Sy,Tz}(t) = \frac{t}{t+\frac{4}{3}}$, then

$$G^*_{fx,gy,hz}\bigl(\phi(t)\bigr) \geq G^*_{Rx,Sy,Tz}(t)$$

is also satisfied.

Case (6). For $x \in [0, \frac{1}{2}]$, $y, z \in (\frac{1}{2}, 1]$, we have

$$G^*_{fx,gy,hz}\bigl(\phi(t)\bigr) = \frac{t}{t+\frac{4}{5}} \geq \frac{t}{t+\frac{3}{2}} = G^*_{Rx,Sy,Tz}(t).$$

Case (7). For $y \in [0, \frac{1}{2}]$, $x, z \in (\frac{1}{2}, 1]$, we have $G^*_{fx,gy,hz}(\phi(t)) = \frac{t}{t+\frac{4}{5}}$. Next we divide the study into two subcases.

(a) If $y \in [0, \frac{1}{2})$, $x, z \in (\frac{1}{2}, 1]$, we have $G^*_{Rx,Sy,Tz}(t) = \frac{t}{t+\frac{4}{3}}$, then

$$G^*_{fx,gy,hz}\bigl(\phi(t)\bigr) \geq G^*_{Rx,Sy,Tz}(t).$$

(b) If $y = \frac{1}{2}$, $x, z \in (\frac{1}{2}, 1]$, we have $G^*_{Rx,Sy,Tz}(t) = \frac{t}{t+\frac{22}{21}}$, then

$$G^*_{fx,gy,hz}\bigl(\phi(t)\bigr) \geq G^*_{Rx,Sy,Tz}(t).$$

Case (8). For $z \in [0, \frac{1}{2}]$, $x, y \in (\frac{1}{2}, 1]$, we have $G^*_{fx,gy,hz}(\phi(t)) = \frac{t}{t+\frac{24}{35}}$. Next we divide the study into two subcases.

(a) If $z \in [0, \frac{1}{2})$, $x, y \in (\frac{1}{2}, 1]$, we have $G^*_{Rx,Sy,Tz}(t) = \frac{t}{t+\frac{3}{2}}$, then

$$G^*_{fx,gy,hz}\bigl(\phi(t)\bigr) \geq G^*_{Rx,Sy,Tz}(t).$$

(b) If $z = \frac{1}{2}$, $x, y \in (\frac{1}{2}, 1]$, we have $G^*_{Rx,Sy,Tz}(t) = \frac{t}{t+\frac{5}{4}}$, then

$$G^*_{fx,gy,hz}\bigl(\phi(t)\bigr) \geq G^*_{Rx,Sy,Tz}(t).$$

Then in all the above cases, f, g, h, R, S, and T satisfy the conditions (3.1) and (i) of Theorem 3.1. So, f, g, h, R, S, and T have a unique common fixed point in $[0,1]$. In fact, 0 is the unique common fixed point of f, g, h, R, S, and T.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally. All authors read and approved the final manuscript.

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