Martínez-Moreno et al. *Fixed Point Theory and Applications* (2015) 2015:53
DOI 10.1186/s13663-015-0297-2

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Multi-dimensional coincidence point theorems for weakly compatible mappings with the CLR_g-property in (fuzzy) metric spaces

J Martínez-Moreno[1], A Roldán[1], C Roldán[2] and Yeol Je Cho[3,4*]

*Correspondence: yjcho@gnu.ac.kr
[3]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Korea
[4]Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we extend the concepts of the $E.A$-property and the CLR_g-property in fuzzy metric spaces to the setting of multi-dimensional fuzzy metric spaces and show the existence of multi-dimensional coincidence point and fixed point theorems for weakly compatible mappings with the $E.A$-property and the CLR_g-property.

Keywords: fuzzy metric space; coincidence point; fixed point; contraction; the $E.A$ property; the CLR_g-property

1 Introduction

Since Banach's fixed point theorem in 1922, many authors have improved, extended and generalized this theorem in many different ways. One of the newest branches of this theorem is devoted to the existence of *coupled fixed point*, which was introduced by Guo and Lakshmikantham [1] in 1987. Later, Berinde and Borcut [2] introduced the concept of *tripled fixed point* and proved some tripled fixed point theorems using mixed monotone mappings (see also [3, 4]). Recently, Roldán *et al.* [5] proposed the notion of *coincidence point* for nonlinear mappings with any number of variables and showed the existence and uniqueness theorems that extended the mentioned previous results for nonlinear mappings, not necessarily permuted or ordered, in the framework of partially ordered complete metric spaces by using some weaker contractive condition, which also generalized other works by Berzig and Samet [6].

Especially in [7], the existence results of coincidence points for the nonlinear mappings in any number of variables in fuzzy metric spaces were presented.

In 2002, Aamri and Moutawakil [8] defined the notion of the $E.A$-property for nonlinear self-mappings which contained the class of non-compatible mappings in metric spaces. It was pointed out that the $E.A$-property allows replacing the completeness requirement of the space with a more natural condition of closedness of the range as well as relaxes the completeness of the whole space, continuity of one or more mappings and containment of the range of one mapping into the range of another, which is utilized to construct the sequence of some joint iterates. Since Aamri and Moutawakil, many authors have also proved common fixed point theorems in fuzzy metric spaces for different contractive conditions.

Recently, Sintunavarat and Kumam [9] defined the notion of the CLR_g-property in fuzzy metric spaces and improved the results of Mihet [10] without any requirement of the closedness of the space.

In this paper, we extend the notions of the $E.A$-property and the CLR_g-property for nonlinear mappings with any number of variables and use these notions to present the existence results of coincidence points for weakly compatible mappings in fuzzy metric spaces. Our results improve, extend and generalize many fixed point theorems in metric spaces and fuzzy metric spaces given by some authors.

2 Preliminaries

Let n be a positive integer and let $\Lambda_n = \{1, 2, \dots, n\}$. Henceforth, X denotes a nonempty set and X^n denotes the product space $X \times X \times \overset{n}{\cdots} \times X$. We represent the identity mapping on X as I_X.

Throughout this manuscript, m and p denote non-negative integers, t is a positive real number and $i, j, s \in \{1, 2, \dots, n\}$. Unless otherwise stated, 'for all m' will mean 'for all $m \geq 0$', 'for all t' will mean 'for all $t > 0$' and 'for all i' will mean 'for all $i \in \{1, 2, \dots, n\}$'. Let us denote $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{I} = [0, 1]$.

In the sequel, let $F : X^n \to X$ and $g : X \to X$ be two mappings. For brevity, $g(x)$ is denoted by gx.

Let $F : X^n \to X$ and $g : X \to X$ be two mappings. Henceforth, let $\sigma_1, \sigma_2, \dots, \sigma_n : \Lambda_n \to \Lambda_n$ be n mappings from Λ_n into itself, and let Φ be the n-tuple $(\sigma_1, \sigma_2, \dots, \sigma_n)$.

Definition 1 ([5]) A point $(x_1, x_2, \dots, x_n) \in X^n$ is called a Φ-*coincidence point* of the mappings F and g if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) = gx_i \tag{1}$$

for all i. If g is the identity mapping on X, then $(x_1, x_2, \dots, x_n) \in X^n$ is called a Φ-*fixed point* of the mapping F.

If $\sigma : \Lambda_n \to \Lambda_n$ is a mapping, then, from its ordered image, *i.e.*, $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$, we have the following:

(1) Gnana-Bhaskar and Lakshmikantham's election [11] in $n = 2$ is $\sigma_1 = \tau = (1, 2)$ and $\sigma_2 = (2, 1)$;

(2) Berinde and Borcut's election [2] in $n = 3$ is $\sigma_1 = \tau = (1, 2, 3)$, $\sigma_2 = (2, 1, 2)$ and $\sigma_2 = (3, 2, 1)$;

(3) Karapınar's election in $n = 4$ is $\sigma_1 = \tau = (1, 2, 3, 4)$, $\sigma_2 = (2, 3, 4, 1)$, $\sigma_3 = (3, 4, 1, 2)$ and $\sigma_4 = (4, 1, 2, 3)$.

There exist different notions of *fuzzy metric space* (see [12]). For our purposes, we use the following one.

Definition 2 (George and Veeramani [13]) A triple $(X, M, *)$ is called a *fuzzy metric space* (briefly, an *FMS*) if X is an arbitrary nonempty set, $*$ is a continuous t-norm and $M : X \times X \times \mathbb{R}^+ \to \mathbb{I}$ is a fuzzy set satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$:

(FM1) $M(x,y,t) > 0$;
(FM2) $M(x,y,t) = 1$ if and only if $x = y$;
(FM3) $M(x,y,t) = M(y,x,t)$;
(FM4) $M(x,y,\cdot) : \mathbb{R}^+ \to \mathbb{I}$ is continuous;
(FM5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$.
In this case, we also say that (X,M) is an *FMS* under $*$.
In the sequel, we only consider an *FMS* satisfying the following:
(FM6) $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in X$.

Lemma 3 (Grabiec [14]) *If (X,M) is an FMS under some t-norm and $x,y \in X$, then $M(x,y,\cdot)$ is a non-decreasing function on $(0,\infty)$.*

Let (X,M) be an *FMS* under some t-norm. For all $t,r > 0$, the *open ball* with center $x \in X$ is $B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$. A subset $A \subseteq X$ is *open* if, for all $x \in X$, there exists an open ball $B(x,r,t)$ such that $B(x,r,t) \subseteq A$.

George and Veeramani [13] proved that the family of all open sets of X is a Hausdorff topology τ_M on X. In this topology, we may consider the following notions.

Definition 4 (1) A sequence $\{x_m\}_{m\geq 0} \subset X$ is a *Cauchy sequence* if, for any $\varepsilon > 0$ and $t > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$M(x_m, x_{m+p}, t) > 1 - \varepsilon$$

for all $m \geq m_0$ and $p \geq 1$;

(2) A sequence $\{x_m\}_{m\geq 0} \subset X$ is *convergent* to a point $x \in X$ denoted by $\lim_{m\to\infty} x_m = x$ if, for any $\varepsilon > 0$ and $t > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$M(x_m, x, t) > 1 - \varepsilon$$

for all $m \geq m_0$;

(3) An *FMS* in which every Cauchy sequence is convergent is said to be *complete*.

Lemma 5 (Rodríguez-López and Romaguera [15]) *If (X,M) is an FMS under some t-norm, then M is a continuous mapping on $X^2 \times (0,\infty)$.*

For any t-norm $*$, it is easy to prove that $* \leq \min$. Therefore, if (X,M) is an *FMS* under min, then (X,M) is an *FMS* under any (continuous or not) t-norm. This is the case in the following examples.

Example 6 From a metric space (X,d), we can consider an FMS in different ways. For all $t > 0$ and $x,y \in X$ with $x \neq y$, define

$$M^d(x,y,t) = \frac{t}{t + d(x,y)}, \qquad M^{\mathrm{e}}(x,y,t) = \mathrm{e}^{-\frac{d(x,y)}{t}}.$$

It is well known that (X,M^d) is an *FMS* under the product $* = \cdot$ called the *standard FMS* on (X,d) since it is the standard way of viewing the metric space (X,d) as an *FMS*. However, it is also true that (X,M^d) and (X,M^{e}) are *FMS* under min.

Furthermore, (X,d) is a complete metric space if and only if (X,M^d) (or (X,M^e)) is a complete *FMS*. For instance, this is the case for any nonempty closed subset of $\mathbb{R}$ provided with its Euclidean metric.

The concept of the *E.A*-property in a metric space has been recently introduced by Aamri and Moutawakil [8] and the concept of the CLR_g-property by Sintunavarat and Kumam [9] is as follows.

Definition 7 ([8, 9]) (1) Two self-mappings f and g of a metric space (X,d) are said to satisfy the *E.A-property* if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$;

(2) If $t \in g(X)$, then f and g are said to satisfy the *CLR_g-property*.

Similarly, we say that two self-mappings f, g of a fuzzy metric space $(X,M,*)$ satisfy the *E.A-property* if there exist a sequence $\{x_n\}$ in X and z in X such that fx_n and gx_n converge to z in the sense of Definition 4. Similarly, we say that if $t \in g(X)$, then f and g are said to satisfy the *CLR_g-property*.

3 Main results

Henceforth, fix a partition $\{A,B\}$ of $\Lambda_n = \{1,2,\ldots,n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$. We denote the following:

$$\Omega_{A,B} := \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A, \sigma(B) \subseteq B\}$$

and

$$\Omega'_{A,B} := \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}.$$

If $\preccurlyeq$ is a partial order on X (*i.e.*, $(X,\preccurlyeq)$ is a partially ordered set), then we use the following notation:

$$x \preccurlyeq_i y \quad \Longleftrightarrow \quad \begin{cases} x \preccurlyeq y & \text{if } i \in A, \\ x \succcurlyeq y & \text{if } i \in B. \end{cases}$$

Consider on the product space X^n the following partial order:

for all $(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n) \in X^n$,

$$(x_1,x_2,\ldots,x_n) \le (y_1,y_2,\ldots,y_n) \quad \Longleftrightarrow \quad x_i \preccurlyeq_i y_i$$

for all i.

Recently, Roldán *et al.* [7] proved the following result.

Theorem 8 ([7]) *Let $\{A,B\}$ be a partition of $\Lambda_n = \{1,2,\ldots,n\}$ and*

$$\Omega_{A,B} := \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A, \sigma(B) \subseteq B\}$$

and

$$\Omega'_{A,B} := \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}.$$

Let $(X,M,*,\preccurlyeq)$ be a complete partially ordered FMS such that $*$ is a t-norm of H-type. Let $\Phi = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ be an n-tuple of mappings from $\{1,2,\ldots,n\}$ into itself with $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed $(g,\preccurlyeq)$-monotone property on X, $F(X^n) \subseteq g(X)$ and g is continuous and Φ-compatible with F. Assume that there exists $k \in (0,1)$ such that

$$M\big(F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n),kt\big) \ge \gamma\Big(\mathop{*}_{i=1}^{n} M(gx_i,gy_i,t)\Big) \tag{2}$$

for all $t > 0$ and all $x_1,\ldots,x_n,y_1,\ldots,y_n \in X$ with $gx_i \leq_i gy_i$ for all i, where $\gamma : [0,1] \to [0,1]$ is a continuous mapping such that $*^n\gamma(a) \ge a$ for each $a \in [0,1]$. Suppose that

$$\gamma\Big(\mathop{*}_{i=1}^{n} M(gx_{\sigma_j(i)},gy_{\sigma_j(i)},t)\Big) \ge \gamma\Big(\mathop{*}_{i=1}^{n} M(gx_i,gy_i,t)\Big) \tag{3}$$

for all $j \in \{1,2,\ldots,n\}$ and $x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n \in X$ such that $gx_i \preccurlyeq_i gy_i$ for all i. Suppose that either F is continuous or $(X,\tau_M,\preccurlyeq)$ has the sequential g-monotone property. If there exist $x_0^1,x_0^2,\ldots,x_0^n \in X$ such that

$$gx_0^i \preccurlyeq_i F\big(x_0^{\sigma_i(1)},x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)}\big)$$

for all i, then F and g have at least one Φ-coincidence point in X.

We are going to give a version of the above result using a pair of mappings satisfying the CLR_g-property. The following definitions extend previous considerations from other authors.

Definition 9 Let $\Phi = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ be an n-tuple of mappings from $\{1,2,\ldots,n\}$ into itself. The mappings $F : X^n \to X$ and $g : X \to X$ are said to be Φ-*weakly compatible* if

$$gF(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)}) = F(gx_{\sigma_i(1)},gx_{\sigma_i(2)},\ldots,gx_{\sigma_i(n)})$$

whenever $gx_i = F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)})$ for all i and some $(x_1,x_2,\ldots,x_n) \in X^n$.

Definition 10 Let $(X,M,*)$ be an FMS and $\Phi = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ be an n-tuple of mappings from $\{1,2,\ldots,n\}$ into itself. Two mappings $F : X^n \to X$ and $g : X \to X$ are said to satisfy the Φ-*common limit* in the range of the g-property (the CLR_g-*property* for short) if there exist n sequences $\{x_m^1\}_{m\ge0},\{x_m^2\}_{m\ge0},\ldots,\{x_m^n\}_{m\ge0}$ such that

$$gx_i = \lim_{m\to\infty} gx_m^i = \lim_{m\to\infty} F\big(x_m^{\sigma_i(1)},x_m^{\sigma_i(2)},\ldots,x_m^{\sigma_i(n)}\big)$$

for all i and some $x_1,\ldots,x_n \in X$.

If $\sigma : \Lambda_n \to \Lambda_n$ is a mapping, then, from its ordered image, *i.e.*, $\sigma = (\sigma(1),\sigma(2),\ldots,\sigma(n))$, we have the following:

(1) Sintunavarat and Kumam's election [9] in $n = 1$;
(2) Jain, Tas, Kumar and Gupta's election [16] and Khan and Sumitra's election [17] in $n = 2$ is $\sigma_1 = \tau = (1,2)$ and $\sigma_2 = (2,1)$;
(3) Wairojjana, Sintunavarat and Kumam's election [18] in $n = 3$ is $\sigma_1 = \tau = (1,2,3)$, $\sigma_2 = (2,3,1)$ and $\sigma_3 = (3,2,1)$ in abstract metric spaces.

Definition 11 Let $(X,M,*)$ be an *FMS* and $\Phi = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ be an n-tuple of mappings from $\{1,2,\ldots,n\}$ into itself. Two mappings $F : X^n \to X$ and $g : X \to X$ are said to satisfy the Φ-*E.A-property* (shortly, the *E.A-property*) if there exist n sequences $\{x_m^1\}_{m\geq 0}, \{x_m^2\}_{m\geq 0}, \ldots, \{x_m^n\}_{m\geq 0}$ such that

$$x_i = \lim_{m\to\infty} gx_m^i = \lim_{m\to\infty} F\bigl(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\bigr)$$

for all i and some $x_1,\ldots,x_n \in X$.

Example 12 Let $(\mathbb{R},M,*)$ be a fuzzy metric space and $*$ be a continuous t-norm. Define $M(x,y,t) = \frac{t}{t+|x-y|}$ for all $x,y \in \mathbb{R}$ and $t > 0$. Define the mappings $F : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by

$$F(x_1,\ldots,x_n) = \sum_{i=1}^{n} (-1)^{i+1} x_i, \qquad g(x) = nx$$

for all $x_1,\ldots,x_n,x \in \mathbb{R}$ and consider the sequences

$$x_m^i = \frac{(-1)^{i+1}}{m}$$

for all $i = 1,\ldots,n$. Then we have

$$\begin{aligned} &F\bigl(x_m^1, x_m^2, \ldots, x_m^{n-1}, x_m^n\bigr) = \frac{n}{m} = g\bigl(x_m^1\bigr), \\ &F\bigl(x_m^2, x_m^3, \ldots, x_m^n, x_m^1\bigr) = \frac{-n}{m} = g\bigl(x_m^2\bigr), \\ &\ldots \\ &F\bigl(x_m^n, x_m^1, \ldots, x_m^{n-2}, x_m^{n-1}\bigr) = g\bigl(x_m^n\bigr). \end{aligned}$$

Therefore, we have

$$\lim_{m\to\infty} M\bigl(F\bigl(x_m^1, x_m^2, \ldots, x_m^{n-1}, x_m^n\bigr)\bigr) = \lim_{m\to\infty} g\bigl(x_m^1\bigr) = 0 = g(0)$$

and so on. Therefore, F and g satisfy the CLR_g-property and the $E.A$-property with $\Phi = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ given by

$$\sigma_1 = (1,2,\ldots,n), \quad \sigma(2) = (2,3,\ldots,n,1), \quad \ldots, \quad \sigma_n = (n,1,\ldots,n-2,n-1).$$

Note that the $E.A$-property does not imply the CLR_g-property and, in [17], there is an example showing that the mappings satisfying the CLR_g-property need not be continuous.

The following result does not require the conditions on the completeness (or the closedness) of the underlying space together with the conditions on continuity and Hadžić's condition of t.

Theorem 13 *Let $(X, M, *)$ be an FMS and $\Phi = (\sigma_1, \sigma_2, \dots, \sigma_n)$ be an n-tuple of mappings from $\{1, 2, \dots, n\}$ into itself. Let $F : X^n \to X$ and $g : X \to X$ be two mappings satisfying the CLR_g-property. Assume that there exists $k \in (0,1)$ such that*

$$M\bigl(F(x_1, x_2, \dots, x_n), F(y_1, y_2, \dots, y_n), kt\bigr) \geq \gamma\Bigl(\mathop{*}_{i=1}^{n} M(gx_i, gy_i, t)\Bigr) \tag{4}$$

for all $t > 0$, $x_1, \dots, x_n, y_1, \dots, y_n \in X$, where $\gamma : [0,1] \to [0,1]$ is a continuous mapping such that $*^n \gamma(a) \geq a$ for each $a \in [0,1]$. Then F and g have at least one Φ-coincidence point.

Proof Since F and g satisfy the CLR_g-property, there exist n sequences $\{x_m^1\}_{m\geq 0}, \{x_m^2\}_{m\geq 0}, \dots, \{x_m^n\}_{m\geq 0}$ such that

$$gx_i = \lim_{m\to\infty} gx_m^i = \lim_{m\to\infty} F\bigl(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \dots, x_m^{\sigma_i(n)}\bigr)$$

for all i and some $x_1, \dots, x_n \in X$. Then we have

$$\begin{aligned} &M\bigl(F\bigl(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \dots, x_m^{\sigma_i(n)}\bigr), F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}), kt\bigr) \\ &\quad \geq \gamma\Bigl(\mathop{*}_{j=1}^{n} M\bigl(gx_m^{\sigma_i(j)}, gx_{\sigma_i(j)}, t\bigr)\Bigr). \end{aligned} \tag{5}$$

By letting $m \to \infty$, we have

$$M\bigl(gx_i, F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}), kt\bigr) \geq \gamma\Bigl(\mathop{*}_{j=1}^{n} M(gx_{\sigma_i(j)}, gx_{\sigma_i(j)}, t)\Bigr) = \gamma(1). \tag{6}$$

Since γ verifies $1 \leq *^n\gamma(1) \leq \min(\gamma(1), \dots, \gamma(1)) = \gamma(1)$, then we have $\gamma(1) = 1$. Therefore, it follows that

$$gx_i = F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)})$$

and so $(x_1, x_2, \dots, x_n)$ is a Φ-coincidence point of F and g. This completes the proof. □

Corollary 14 *Under the hypothesis of Theorem* 13, *assume also that F and g are weakly compatible. If $(x_1, x_2, \dots, x_n) \in X^n$ is a Φ-coincidence point of F and g, then $(gx_1, gx_2, \dots, gx_n)$ is also a Φ-coincidence point of F and g.*

Proof Now, we let $z_i = gx_i = F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)})$. Since F and g are weakly compatible mappings, we have

$$\begin{aligned} gz_i &= gF(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) \\ &= F(gx_{\sigma_i(1)}, gx_{\sigma_i(2)}, \dots, gx_{\sigma_i(n)}) \\ &= F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \dots, z_{\sigma_i(n)}). \end{aligned}$$

Thus, from Theorem 13, we have the conclusion. □

Corollary 15 *Let $(X, M, *)$ be an FMS and $\Phi = (\sigma_1, \sigma_2, \dots, \sigma_n)$ be an n-tuple of mappings from $\{1, 2, \dots, n\}$ into itself. Let $F : X^n \to X$ and $g : X \to X$ be two mappings satisfying the E.A-property. Assume that there exists $k \in (0, 1)$ such that inequality* (4) *is satisfied, where $\gamma : [0, 1] \to [0, 1]$ is a continuous mapping such that $*^n \gamma(a) \geq a$ for each $a \in [0, 1]$. If $g(X)$ is a closed subspace of X, then F and g have at least one Φ-coincidence point.*

Proof Since F and g satisfy the *E.A*-property, there exist n sequences $\{x^1_m\}_{m\geq 0}, \{x^2_m\}_{m\geq 0}, \dots, \{x^n_m\}_{m\geq 0}$ such that

$$x_i = \lim_{m\to\infty} gx^i_m = \lim_{m\to\infty} F\bigl(x^{\sigma_i(1)}_m, x^{\sigma_i(2)}_m, \dots, x^{\sigma_i(n)}_m\bigr)$$

for all i and some $x_1, \dots, x_n \in X$. Since $g(X)$ is a closed subspace of X, it follows that $x_i = g(w_i)$ for some $w_i \in X$ and for all i, and so F and g satisfy the CLR_g-property. Therefore, by Theorem 13, F and g have at least one Φ-coincidence point. □

Remark 16 Corollary 15 is a version of Theorem 13 for the *E.A*-property. Similarly, we can write a new version by replacing the CLR_g-property for the *E.A*-property and the closedness of $g(X)$ for all the result in this paper.

4 The uniqueness of Φ-coincidence points

Theorem 17 *Under the hypothesis of Corollary* 14, *suppose additionally that*

$$\gamma\Bigl(\mathop{*}_{i=1}^{n} M(gx_{\sigma_j(i)}, gy_{\sigma_j(i)}, t)\Bigr) \geq \gamma\Bigl(\mathop{*}_{i=1}^{n} M(gx_i, gy_i, t)\Bigr) \tag{7}$$

for all $j \in \{1, 2, \dots, n\}$ and $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in X$. Then F and g have a unique Φ-coincidence point $(z_1, z_2, \dots, z_n) \in X^n$ such that $gz_i = z_i$ for all i.

Proof From Theorem 13, the set of Φ-coincidence points of F and g is nonempty. The proof is divided in two steps.

Step 1. We claim that if $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$ are two Φ-coincidence points of F and g, then we have

$$gx_i = gy_i \tag{8}$$

for all i. Let $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in X^n$ be two Φ-coincidence points of F and g. For all t and m, define

$$\delta_m(t) = \mathop{*}_{j=1}^{m} M\bigl(gy^j_m, gx_j, t\bigr).$$

Now, we claim that $\delta_{m+1}(kt) \geq \delta_m(t)$ for all m and $t > 0$. By inequalities (4) and (7), it follows that, for all m, t and j,

$$\begin{aligned} M\bigl(gy^j_{m+1}, gx_j, kt\bigr) &= M\bigl(F\bigl(y^{\sigma_j(1)}_m, y^{\sigma_j(2)}_m, \dots, y^{\sigma_j(n)}_m\bigr), F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \dots, x_{\sigma_j(n)}), kt\bigr) \\ &\geq \gamma\Bigl(\mathop{*}_{i=1}^{m} M\bigl(gy^{\sigma_j(i)}_m, gx_{\sigma_j(i)}, t\bigr)\Bigr) \\ &\geq \gamma\Bigl(\mathop{*}_{i=1}^{m} M\bigl(gy^i_m, gx_i, t\bigr)\Bigr) \\ &= \gamma\bigl(\delta_m(t)\bigr). \end{aligned}$$

Since $*^n\gamma(a) \geq a$ for each $a \in [0,1]$, it follows that

$$\delta_{m+1}(kt) = \mathop{*}_{j=1}^{m} M\big(gy^j_{m+1}, gx_j, t\big) \geq \mathop{*}_{j=1}^{n} \gamma\left(\delta_m(t)\right) = *^n\gamma\left(\delta_m(t)\right) \geq \delta_m(t),$$

i.e., $\delta_{m+1}(kt) \geq \delta_m(t) \geq \cdots \geq \delta_0(t/k^m)$ and, as a consequence, $\lim_{m\to\infty} \delta_m(t) = 1$ for all $t > 0$. It follows that, for all i and t,

$$M\big(gy^i_m, gx_i, t\big) = 1 * \cdots * 1 * M\big(gy^i_m, gx_i, t\big) * 1 * \cdots * 1 \geq \mathop{*}_{j=1}^{n} M\big(gy^j_m, gx_j, t\big) = \delta_m(t),$$

which means that $\lim_{m\to\infty} M(gy^i_m, gx_i, t) = 1$ for all t, *i.e.*,

$$\lim_{m\to\infty} gy^i_m = gx_i \tag{9}$$

for all i. Let $(x_1, x_2, \ldots, x_n) \in X^n$ be a Φ-coincidence point of F and g and define $z_i = gx_i$ for all i. Since $(z_1, z_2, \ldots, z_n) = (gx_1, gx_2, \ldots, gx_n)$, from Corollary 14 it follows that $(z_1, z_2, \ldots, z_n)$ is also a Φ-coincidence point of F and g.

Step 2. We claim that $(z_1, z_2, \ldots, z_n)$ is the unique Φ-coincidence point of F and g such that $gz_i = z_i$ for all i. Indeed, by Step 1, we observe that $gz_i = gx_i = z_i$ for all i. Suppose that $(z'_1, z'_2, \ldots, z'_n) \in X^n$ is another Φ-coincidence point of F and g such that $gz'_i = z'_i$ for all i. Since $(z_1, z_2, \ldots, z_n)$ and $(z'_1, z'_2, \ldots, z'_n)$ are two Φ-coincidence points of F and g, by Step 1 it follows that $gz_i = gz'_i$ for all i and so $z_i = gz_i = gz'_i = z'_i$ for all i. Therefore, $(z_1, z_2, \ldots, z_n)$ is the unique Φ-coincidence point of F and g such that $gz_i = z_i$ for all i. This completes the proof. □

Corollary 18 *In addition to the hypotheses of Theorem* 17, *suppose that* $* = \min$, $\gamma(a) = a$ *for all* $a \in \mathbb{I}$ *and* $(z_1, z_2, \ldots, z_n) \in X^n$ *is the unique* Φ*-coincidence point of* F *and* g. *Then* $z_1 = z_2 = \cdots = z_n$. *In particular, there exists a unique* $z \in X$ *such that* $F(z, z, \ldots, z) = z$, *which verifies* $gz = z$.

Proof For all $t > 0$, define

$$\lambda(t) = \mathop{*}_{j,s=1}^{n} M(z_j, z_s, t) = \min_{1\leq j,s\leq n} M(z_j, z_s, t).$$

Clearly, λ is continuous and non-decreasing on $\mathbb{R}^+$. Take any $j, s \in \{1, 2, \ldots, n\}$. Since $* = \min$ and $\gamma(a) = a$ for all $a \in \mathbb{I}$, applying (4) we have

$$\begin{aligned} M(z_j, z_s, t) &= M(gz_j, gz_s, t) \\ &= M\big(F(z_{\sigma_j(1)}, z_{\sigma_j(2)}, \ldots, z_{\sigma_j(n)}), F(z_{\sigma_s(1)}, z_{\sigma_s(2)}, \ldots, z_{\sigma_s(n)}), t\big) \\ &\geq \min_{1\leq i\leq n} M(gz_{\sigma_j(i)}, gz_{\sigma_s(i)}, t/k) \\ &= \min_{1\leq i\leq n} M(z_{\sigma_j(i)}, z_{\sigma_s(i)}, t/k) \end{aligned}$$

for all $t > 0$. Therefore, for all $t > 0$, we have

$$\begin{aligned} \lambda(t) &= \min_{1\leq j,s\leq n} M(z_j, z_s, t) \\ &\geq \min_{1\leq j,s\leq n} \Big(\min_{1\leq i\leq n} M(z_{\sigma_j(i)}, z_{\sigma_s(i)}, t/k)\Big) \end{aligned}$$

$$\begin{aligned}&\geq \min_{1\leq j,s\leq n} M(z_j, z_s, t/k)\\&= \lambda(t/k).\end{aligned}$$

Repeating this process, $\lambda(t) \geq \lambda(t/k) \geq \cdots \geq \lambda(t/k^m)$ for all m. Taking the limit $m \to \infty$, we deduce that $\lambda(t) = 1$ for all $t > 0$. For any $j, s \in \{1, 2, \ldots, n\}$, we note that $M(z_j, z_s, t) \geq \lambda(t) = 1$ for all t, *i.e.*, $z_j = z_s$. This completes the proof. □

5 Some results in metric spaces

It seems natural to introduce the analogous definitions and results in the setting of metric spaces by using the results in fuzzy metric spaces.

Definition 19 Let (X, d) be a metric space and $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be an n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself. Two mappings $F : X^n \to X$ and $g : X \to X$ are said to satisfy the Φ-*common limit* in the range of the g-property (shortly, *CLR_g-property*) if there exist n sequences $\{x_m^1\}_{m\geq 0}, \{x_m^2\}_{m\geq 0}, \ldots, \{x_m^n\}_{m\geq 0}$ such that

$$gx_i = \lim_{m\to\infty} gx_m^i = \lim_{m\to\infty} F\bigl(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}\bigr)$$

for all i and some $x_1, \ldots, x_n \in X$.

Remark 20 Let (X, d) be a metric space, $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be an n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself. If we consider the FMS $(X, M^{\mathrm{e}}, \min)$ induced by d, see Example 6, then the *CLR_g-property* on (X, d) coincides with the *CLR_g-property* on $(X, M^{\mathrm{e}}, \min)$.

From Theorem 13, we have the following in metric spaces.

Corollary 21 *Let (X, d) be a metric space and $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be an n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself. Let $F : X^n \to X$ and $g : X \to X$ be two mappings satisfying the CLR_g-property. Assume that there exists $k \in [0, 1[$ such that*

$$d\bigl(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)\bigr) \leq k \max_{1\leq i\leq n} d(gx_i, gy_i) \tag{10}$$

for all i. Then F and g have at least one Φ-coincidence point in X.

Proof It is easy to show that the mapping $\delta : X^n \times X^n \to [0, \infty)$ defined by

$$\delta\bigl((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\bigr) = \max_{1\leq i\leq n} d(gx_i, gy_i)$$

is a metric on X^n. The proof follows from Theorem 13 for M^{e} induced by δ, γ the identity and $* = \min$. □

If $n = 1$, then we have the following.

Corollary 22 *Let (X, d) be a metric space. Let $f : X \to X$ and $g : X \to X$ be two mappings satisfying the CLR_g-property. Assume that there exists $k \in [0, 1[$ such that*

$$d(fx, fy) \leq kd(gx, gy) \tag{11}$$

for all $x, y \in X$. Then f and g have at least one coincidence point.

It is well known that in the setting of metric spaces multi-dimensional results only depend on their first argument. Therefore, multi-dimensional results reduce to the one dimensional case.

Theorem 23 *Corollaries* 21 *and* 22 *are equivalent*.

Proof Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, $F : X^n \to X$ and $g : X \to X$ be two mappings with the conditions of Corollary 21. It is easy to show that the mapping $\delta : X^n \times X^n \to [0, \infty)$ defined by

$$\delta\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\big) = \max_{1 \leq i \leq n} d(gx_i, gy_i)$$

is a metric on X^n.

Now, define the mappings $T : X^n \to X^n$ and $G : X^n \to X^n$ by

$$T(x_1, x_2, \ldots, x_n) = \big(F\big(x_1^{\sigma_1(1)}, x_1^{\sigma_1(2)}, \ldots, x_1^{\sigma_1(n)}\big), \ldots, F\big(x_n^{\sigma_n(1)}, x_n^{\sigma_n(2)}, \ldots, x_n^{\sigma_n(n)}\big)\big),$$
$$G(x_1, x_2, \ldots, x_n) = (gx_1, gx_2, \ldots, gx_n),$$

respectively. It is easy to show that

(a) F and g satisfy the CLR_g-property if and only if T and G satisfy the CLR_g-property;

(b) $(x_1, x_2, \ldots, x_n) \in X^n$ is a Φ-coincidence point of the mappings F and g if and only if $(x_1, x_2, \ldots, x_n) \in X^n$ is a coincidence point of the mappings T and G.

Thus, from (10) it follows that

$$\delta\big(T(x_1, x_2, \ldots, x_n), T(y_1, y_2, \ldots, y_n)\big) \leq k\delta\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\big).$$

Thus the mappings T and G satisfy the conditions of Corollary 21. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, University of Jaén, Jaén, Spain. [2]Department of Statistics and Operations Research, University of Granada, Granada, Spain. [3]Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Korea. [4]Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia.

Acknowledgements
The fourth author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and future Planning (2014R1A2A2A01002100).

Received: 24 December 2014 Accepted: 18 March 2015 Published online: 11 April 2015

References
1. Guo, D, Lakshmikantham, V: Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. **11**, 623-632 (1987)
2. Berinde, V, Borcut, M: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. **74**, 4889-4897 (2011)
3. Berinde, V: Approximating common fixed points of noncommuting almost contractions in metric spaces. Fixed Point Theory **11**, 179-188 (2010)
4. Borcut, M, Berinde, V: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218**, 5929-5936 (2012)

5. Roldán, A, Martínez-Moreno, J, Roldán, C: Multidimensional fixed point theorems in partially ordered complete metric spaces. J. Math. Anal. Appl. **396**, 536-545 (2012)
6. Berzig, M, Samet, B: An extension of coupled fixed point's concept in higher dimension and applications. Comput. Math. Appl. **63**, 1319-1334 (2012)
7. Roldń, A, Martínez-Moreno, J, Roldán, C, Cho, YJ: Multidimensional coincidence point results for compatible mappings in partially ordered fuzzy metric spaces. Fuzzy Sets Syst. **251**, 71-82 (2014)
8. Aamri, M, El Moutawakil, D: Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl. **270**, 181-188 (2002)
9. Sintunavarat, W, Kumam, P: Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces. J. Appl. Math. **2011**, Article ID 637958 (2011)
10. Mihet, D: Fixed point theorems in fuzzy metric space using *E.A*. Nonlinear Anal. **73**, 2184-2188 (2010)
11. Gnana-Bhaskar, T, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379-1393 (2006)
12. Roldán, A, Martínez-Moreno, J, Roldán, C: On interrelationships between fuzzy metric structures. Iran. J. Fuzzy Syst. **10**, 133-150 (2013)
13. George, A, Veeramani, PV: On some results in fuzzy metric spaces. Fuzzy Sets Syst. **64**, 395-399 (1994)
14. Grabiec, M: Fixed points in fuzzy metric spaces. Fuzzy Sets Syst. **27**, 385-389 (1988)
15. Rodríguez-López, J, Romaguera, S: The Hausdorff fuzzy metric on compact sets. Fuzzy Sets Syst. **147**, 273-283 (2004)
16. Jain, M, Tas, K, Kumar, S, Gupta, N: Coupled fixed point theorems for a pair of weakly compatible maps along with CLR_g-property in fuzzy metric spaces. J. Appl. Math. **2012**, Article ID 961210 (2012)
17. Khan, MA, Sumitra: *CLRg* Property for coupled fixed point theorems in fuzzy metric spaces. Int. J. Appl. Phys. Math. **2**, 355-358 (2012)
18. Wairojjana, N, Sintunavarat, W, Kumam, P: Common tripled fixed point theorems for *w*-compatible mappings along with the CLR_g property in abstract metric spaces. J. Inequal. Appl. **2014**, Article ID 133 (2014)