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Fixed points of multivalued mappings in modular function spaces with a graph

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Abstract

Let $\rho \in \Re$ (the class of all nonzero regular function modulars defined on a nonempty set Ω) and G be a directed graph defined on a subset C of L_ρ. In this paper, we discuss the existence of fixed points of monotone G-contraction and G-nonexpansive mappings in modular function spaces. These results are the modular version of Jachymski fixed point results for mappings defined in a metric space endowed with a graph.

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1 Introduction

Fixed point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering have been widely investigated. These theorems are hybrids of the two most fundamental and useful theorems in fixed point theory: Banach's contraction principle ([1], Theorem 2.1) and Tarski's fixed point theorem [2, 3]. Generalizing the Banach contraction principle for multivalued mappings to metric spaces, Nadler [4] obtained the following result.

Theorem 1.1 [4] *Let (X,d) be a complete metric space. Denote by $\mathrm{CB}(X)$ the set of all nonempty closed bounded subsets of X. Let $F : X \to \mathrm{CB}(X)$ be a multivalued mapping. If there exists $k \in [0,1)$ such that*

$$H\big(F(x),F(y)\big) \leq kd(x,y)$$

for all $x,y \in X$, where H is the Hausdorff metric on $\mathrm{CB}(X)$, then F has a fixed point in X.

A number of extensions and generalizations of Nadler's theorem were obtained by different authors; see, for instance, [5, 6] and the references cited therein. Tarski's theorem was extended to multivalued mappings by different authors; see [7–9]. Investigation of the existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings in [10] who proved the following result.

Theorem 1.2 [10] *Let $(X,\preceq)$ be a partially ordered set such that every pair $x,y \in X$ has an upper and lower bound. Let d be a metric on X such that (X,d) is a complete metric*

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space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:

(1) *There exists $k \in (0,1)$ with*

$$d\big(f(x),f(y)\big) \leq kd(x,y) \quad \text{for all } x \succeq y.$$

(2) *There exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$.*

Then f is a Picard operator (PO), that is, f has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.

After this, different authors considered the problem of existence of a fixed point for contraction mappings in partially ordered sets; see [11–14] and the references cited therein. Nieto *et al.* in [14] proved the following theorem.

Theorem 1.3 [14] *Let (X,d) be a complete metric space endowed with a partial ordering $\preceq$. Let $f : X \to X$ be an order preserving mapping such that there exists $k \in [0,1)$ with*

$$d\big(f(x),f(y)\big) \leq kd(x,y) \quad \text{for all } x \succeq y.$$

Assume that one of the following conditions holds:

(1) *f is continuous and there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $x_0 \succeq f(x_0)$;*
(2) *$(X,d,\preceq)$ is such that for any nondecreasing $(x_n)_{n\in N}$, if $x_n \to x$, then $x_n \preceq x$ for $n \in N$, and there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$;*
(3) *$(X,d,\preceq)$ is such that for any nonincreasing $(x_n)_{n\in N}$, if $x_n \to x$, then $x_n \succeq x$ for $n \in N$, and there exists $x_0 \in X$ with $x_0 \succeq f(x_0)$.*

Then f has a fixed point. Moreover, if $(X,\preceq)$ is such that every pair of elements of X has an upper or a lower bound, then f is a PO.

Recently, two results have appeared, giving sufficient conditions for f to be a PO, if (X,d) is endowed with a graph. The first result in this direction was given by Jachymski and Lukawska [15, 16], which generalized the results of [12, 14, 17, 18] to a single-valued mapping in metric spaces with a graph instead of partial ordering. Subsequently, Beg *et al.* [19] tried to extend the results of [15] to multivalued mappings, but their extension was not carried correctly (see [20]). The aim of this paper is to give the correct extension by studying the existence of fixed points for multivalued mappings in modular function spaces endowed with a graph G. Recall that the fixed point theory in modular function spaces was initiated by Khamsi *et al.* [21]. The reader interested in fixed point theory in modular function spaces is referred to [22–25].

2 Preliminaries

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Σ such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we denote the space of all extended measurable functions, *i.e.*, all functions $f : \Omega \to [-\infty,\infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.1 Let $\rho : \mathcal{M}_\infty \to [0,\infty]$ be a nontrivial even function. We say that ρ is a regular function pseudomodular if:

(i) $\rho(0) = 0$;

(ii) ρ is monotone, *i.e.*, $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;

(iii) ρ is orthogonally subadditive, *i.e.*, $\rho(f1_{A\cup B}) \leq \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}$;

(iv) ρ has the Fatou property, *i.e.*, $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;

(v) ρ is order continuous in $\mathcal{E}$, *i.e.*, $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind, we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty \ \rho\text{-a.e. for every } \omega \in \Omega\}, \tag{1}$$

where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists, we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.2 Let ρ be a regular function pseudomodular.

(1) We say that ρ is a regular function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ-a.e.;

(2) We say that ρ is a regular function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

The class of all nonzero regular function modulars defined on Ω will be denoted by $\Re$.

Let us denote $\rho(f, E) = \rho(f1_E)$ for $f \in \mathcal{M}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudomodular in the sense of Definition 2.1.1 in [22] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function spaces as per the framework defined by Kozlowski in [22, 26, 27].

Definition 2.3 [22, 26, 27] Let ρ be a function modular.

(a) A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

(b) The following formula defines a norm in L_ρ (frequently called Luxemburg norm):

$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \leq 1\}.$$

In the following theorem we recall some of the properties of modular spaces.

Theorem 2.1 [22, 26, 27] *Let* $\rho \in \Re$.

(1) $(L_\rho, \|\cdot\|_\rho)$ *is complete and the norm* $\|\cdot\|_\rho$ *is monotone w.r.t. the natural order in* $\mathcal{M}$.

(2) $\|f_n\|_\rho \to 0$ *if and only if* $\rho(\alpha f_n) \to 0$ *for every* $\alpha > 0$.
(3) *If* $\rho(\alpha f_n) \to 0$ *for* $\alpha > 0$, *then there exists a subsequence* $\{g_n\}$ *of* $\{f_n\}$ *such that* $g_n \to 0$ ρ-*a.e.*
(4) $\rho(f) \leq \liminf_{n\to\infty} \rho(f_n)$, *whenever* $f_n \to f$ ρ-*a.e. Note this property will be referred to as the Fatou property.*

The following definition plays an important role in the theory of modular function spaces.

Definition 2.4 Let $\rho \in \Re$. We say that ρ has the Δ_2-property if $\sup_n \rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $D_k \downarrow \emptyset$ and $\sup_n \rho(f_n, D_k) \to 0$, where $\{f_n\}_{n\geq 1} \subset M$ and $D_k \in \Sigma$.

We have the following interesting result.

Theorem 2.2 [22] *Let* $\rho \in \Re$. *The following conditions are equivalent*:
(a) ρ *has the* Δ_2-*property*,
(b) *if* $\rho(f_n) \to 0$, *then* $\rho(2f_n) \to 0$,
(c) *if* $\rho(\alpha f_n) \to 0$ *for* $\alpha > 0$, *then* $\|f_n\|_\rho \to 0$, *i.e., the modular convergence is equivalent to the norm convergence.*

Moreover, if ρ *has the* Δ_2-*type condition, i.e., there exists* $k \in [0, +\infty)$ *such that*

$$\rho(2f) \leq k\rho(f) \quad \text{for any } f \in L_\rho,$$

then ρ *has the* Δ_2-*property. In general, the converse is not true* (*see Example* 3.2.6 *of* [27]).

Remark 2.1 It is easy to check that ρ has the Δ_2-type condition if and only if for any $\lambda > 1$, there exists $k \in [0, +\infty)$ such that

$$\rho(\lambda f) \leq k\rho(f)$$

for any $f \in L_\rho$.

We will also use another type of convergence which is situated between norm and modular convergence. It is defined, among other important terms, in the following definition.

Definition 2.5 Let $\rho \in \Re$.
(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f(\rho)$ if and only if $\rho(f_n - f) \to 0$.
(b) A sequence $\{f_n\}$, where $f_n \in L_\rho$, is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.
(c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f(\rho)$ implies that f belongs to B.
(d) A set $B \subset L_\rho$ is called ρ-bounded if $\sup\{\rho(f - g); f, g \in B\} < \infty$.
(e) A set $C \subset L_\rho$ is called ρ-a.e. closed if for any $\{f_n\}$ in C which ρ-a.e. converges to some f, then we must have $f \in C$.
(f) A set $C \subset L_\rho$ is called ρ-a.e. compact if for any $\{f_n\}$ in C, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e. converges to some $f \in C$.

Let us note that ρ-convergence does not necessarily imply ρ-Cauchy condition. Also, $f_n \to f$ does not imply in general $\lambda f_n \to \lambda f$, $\lambda > 1$. Using Theorem 2.1 it is not difficult to prove the following.

Proposition 2.1 *Let $\rho \in \Re$.*

(i) *L_ρ is ρ-complete.*

(ii) *L_ρ is a lattice, i.e., for any $f, g \in L_\rho$, we have $\max\{f, g\} \in L_\rho$ and $\min\{f, g\} \in L_\rho$.*

(iii) *ρ-balls $B_\rho(x, r) = \{y \in L_\rho; \rho(x - y) \le r\}$ are ρ-closed and ρ-a.e. closed.*

Using the property (3) of Theorem 2.1, we get the following result.

Theorem 2.3 *Let $\rho \in \Re$ and $\{f_n\}$ be a ρ-Cauchy sequence in L_ρ. Assume that $\{f_n\}$ is monotone increasing, i.e., $f_n \le f_{n+1}$ ρ-a.e. (resp. decreasing, i.e., $f_{n+1} \le f_n$ ρ-a.e.) for any $n \ge 1$. Then there exists $f \in L_\rho$ such that $\rho(f_n - f) \to 0$ and $f_n \le f$ ρ-a.e. (resp. $f \le f_n$ ρ-a.e.) for any $n \ge 1$.*

It seems that the terminology of graph theory instead of partial ordering sets can give clearer pictures and yield to generalize the contraction theorems. Let us finish this section with such terminology for the modular space mapping which will be studied throughout.

Let $C \subseteq L_\rho$ with $\rho \in \Re$. Let G be a directed graph (digraph) with a set of vertices of G being the elements of C and a set of edges $E(G)$ containing all the loops, *i.e.*, $(x, x) \in E(G)$ for any $x \in V(G)$. We also assume that G has no parallel edges (arcs) and so we can identify G with the pair $(V(G), E(G))$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [28] and [29]. Moreover, we may treat G as a weighted graph (see [29], p.309]) by assigning to each edge the distance between its vertices. By G^{-1} we denote the conversion of a graph G, *i.e.*, the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(y, x) \mid (x, y) \in E(G)\}.$$

A digraph G is called an oriented graph if whenever $(u, v) \in E(G)$, then $(v, u) \notin E(G)$. The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

We call (V', E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$ and for any edge $(x, y) \in E'$, $x, y \in V'$.

If x and y are vertices in a graph G, then a (directed) path in G from x to y of length N is a sequence $\{x_i\}_{i=1}^{i=N}$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$, and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \dots, N$. A graph G is connected if there is a directed path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:

yRz if there is a (directed) path in G from y to z.

Clearly, G_x is connected.

In the sequel we assume that $\rho \in \Re$ is a convex, σ-finite modular function, and C is a nonempty subset of the modular function space L_ρ. We denote by $\mathcal{C}(C)$ the collection of all nonempty ρ-closed subsets of C, and by $\mathcal{K}(C)$ the collection of all nonempty ρ-compact subsets of C.

Definition 2.6 We say that a mapping $T : C \to C$ is G-edge preserving if

$$\forall f, g \in C, \quad (f,g) \in E(G) \quad \Rightarrow \quad \bigl(T(f), T(g)\bigr) \in E(G).$$

Definition 2.7 Let $\rho \in \Re$ and $C \subset L_\rho$ be nonempty ρ-closed and ρ-bounded. A multivalued mapping $T : C \to 2^C$ is said to be a monotone increasing G-contraction if there exists $\alpha \in [0,1)$ such that for any $f, h \in C$ with $(f,h) \in E(G)$ and any $F \in T(f)$ there exists $H \in T(h)$ such that

$$(F,H) \in E(G) \quad \text{and} \quad \rho(F-H) \le \alpha\rho(f-h).$$

Similarly we will say that the multivalued mapping $T : C \to 2^C$ is monotone increasing G-nonexpansive if for any $f, h \in C$ with $(f,h) \in E(G)$ and any $F \in T(f)$ there exists $H \in T(h)$ such that

$$(F,H) \in E(G) \quad \text{and} \quad \rho(F-H) \le \rho(f-h).$$

$f \in C$ is called a fixed point of T if and only if $f \in T(f)$. The set of all fixed points of a mapping T is denoted by $\operatorname{Fix} T$.

Our definition of monotone multivalued mappings is slightly different from the one used in [19]. Indeed in Definition 2.6 in [19], one may let α go to 0, which is very restrictive. Because of this, the proof of the main result is incorrect. In this work, we will show how our definition will give the correct proof.

Definition 2.8 [30] Let $\rho \in \Re$ be convex and satisfy the Δ_2-type condition. Define the growth function ω by

$$\omega(\alpha) = \sup\left\{\frac{\rho(\alpha f)}{\rho(f)}; f \in L_\rho, f \neq 0\right\}$$

for any $\alpha \ge 0$.

The following properties were proved in [30].

Lemma 2.1 [30] *Let $\rho \in \Re$ be convex and satisfy the Δ_2-type condition. Then the growth function ω satisfies the following properties*:

(1) $\omega(\alpha) < \infty$ *for any* $\alpha > 0$,
(2) ω *is a strictly increasing function, and* $\omega(1) = 1$,
(3) $\omega(\alpha\beta) \le \omega(\alpha)\omega(\beta)$ *for any* $\alpha, \beta \in (0,\infty)$,
(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \le \omega^{-1}(\alpha\beta)$, *where* ω^{-1} *is the function inverse of* ω,

(5) *for any* $f \in L_\rho, f \neq 0$, *we have*

$$\|f\|_\rho \leq \frac{1}{\omega^{-1}(1/\rho(f))}.$$

The following technical lemma will be useful later on in this work.

Lemma 2.2 [30] *Let* $\rho \in \Re$ *be convex and satisfy the* Δ_2*-type condition. Let* $\{f_n\}$ *be a sequence in* L_ρ *such that*

$$\rho(f_{n+1} - f_n) \leq K\alpha^n, \quad n = 1, \ldots,$$

where K *is an arbitrary nonzero constant and* $\alpha \in (0,1)$. *Then* $\{f_n\}$ *is Cauchy for* $\|\cdot\|_\rho$ *and* ρ*-Cauchy.*

Note that this lemma is crucial since the main assumption on $\{f_n\}$ will not be enough to imply that $\{f_n\}$ is ρ-Cauchy since ρ fails the triangle inequality.

Property 2.1 For any sequence $\{f_n\}_{n\in\mathbb{N}}$ in C, if f_n ρ-converges to f and $(f_n, f_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(f_n, f) \in E(G)$.

3 Main results

We begin with the following theorem that gives the existence of a fixed point for monotone multivalued mappings in modular spaces endowed with a graph. The key feature in this theorem is that the Lipschitzian condition on the nonlinear map is only assumed to hold on elements that are comparable in the natural partial order of L_ρ.

Theorem 3.1 *Let* $\rho \in \Re$ *be convex. Let* $C \subset L_\rho$ *be a nonempty and* ρ*-closed subset. Assume that* ρ *satisfies the* Δ_2*-type condition. Let* $T : C \to \mathcal{C}(C)$ *be a monotone increasing* ρ*-contraction mapping and* $C_T := \{f \in C; f \leq g \ \rho\text{-a.e. for some } g \in T(f)\}$. *If* $C_T \neq \emptyset$, *then* T *has a fixed point in* C.

Proof Recall that T is a monotone increasing ρ-contraction mapping if and only if there exists $\alpha \in [0,1)$ such that for all $f, g \in C$ with $f \leq g$ ρ-a.e., then for any $F \in T(f)$, there exists $G \in T(g)$ such that $F \leq G$ ρ-a.e. and

$$\rho(F - G) \leq \alpha\rho(f - g).$$

Fix $f_0 \in C_T$. Then there exists $f_1 \in T(f_0)$ such that $f_0 \leq f_1$ ρ-a.e. Since T is a monotone increasing ρ-contraction, there exists $f_2 \in T(f_1), f_1 \leq f_2$ ρ-a.e., such that

$$\rho(f_2 - f_1) \leq \alpha\rho(f_1 - f_0),$$

where $\alpha < 1$ is associated to the definition of T being a monotone increasing ρ-contraction. Without loss of generality, we may assume $\alpha > 0$. By induction, we construct a sequence $\{f_n\}$ such that $f_{n+1} \in T(f_n), f_n \leq f_{n+1}$ ρ-a.e. and

$$\rho(f_{n+1} - f_n) \leq \alpha\rho(f_n - f_{n-1}) \leq \alpha^n\rho(f_1 - f_0)$$

for any $n \geq 1$. The technical Lemma 2.2 implies that $\{f_n\}$ is ρ-Cauchy and converges to some $f \in C$ since L_ρ is ρ-complete. We claim that $f \in T(f)$, *i.e.*, f is a fixed point of T. Indeed Theorem 2.3 implies that $f_n \leq f$ ρ-a.e. for any $n \geq 0$. Since T is a monotone increasing ρ-contraction, there exists $g_n \in T(f)$ such that $f_{n+1} \leq g_n$ and

$$\rho(f_{n+1} - g_n) \leq \alpha\rho(f_n - f).$$

Hence $\{f_{n+1} - g_n\}$ ρ-converges to 0. Since ρ satisfies the Δ_2-type condition, we conclude that $\{\|f_{n+1} - g_n\|_\rho\}$ converges to 0. Hence $\{g_n\}$ also ρ-converges to f. Since $T(f)$ is ρ-closed, we conclude that $f \in T(f)$. □

Note that the fixed point may not be unique. Indeed if we take A, any nonempty ρ-closed subset of C, then the multivalued map $T : C \to \mathcal{C}(C)$ defined by $T(f) = A$, for any $f \in C$, is a monotone increasing ρ-contraction mapping. The set of fixed points of T is exactly the set A.

An easy consequence of Theorem 3.1 is the following result.

Proposition 3.1 *Let $\rho \in \Re$ be convex. Let $C \subset L_\rho$ be a nonempty and ρ-closed convex subset. Assume that ρ satisfies the Δ_2-type condition and C is ρ-bounded. Let $T : C \to \mathcal{C}(C)$ be a monotone increasing ρ-nonexpansive mapping and $C_T := \{f \in C; f \leq g$ ρ-a.e. for some $g \in T(f)\}$. If $C_T \neq \emptyset$, then T has an approximate fixed point sequence $\{f_n\} \in C$, that is, for any $n \geq 1$, there exists $g_n \in T(f_n)$ such that*

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

In particular, we have $\lim_{n\to\infty} \operatorname{dist}_\rho(f_n, T(f_n)) = 0$, *where*

$$\operatorname{dist}_\rho\bigl(f_n, T(f_n)\bigr) = \inf\bigl\{\rho(f_n - g); g \in T(f_n)\bigr\}.$$

Proof Recall that T is a monotone increasing ρ-nonexpansive mapping if and only if for all $f, g \in C$ with $f \leq g$ ρ-a.e. and for any $F \in T(f)$, there exists $G \in T(g)$ such that $F \leq G$ ρ-a.e. and

$$\rho(F - G) \leq \rho(f - g).$$

Fix $\lambda \in (0,1)$ and $h_0 \in C$. Define the multivalued map T_λ on C by

$$T_\lambda(f) = \lambda h_0 + (1 - \lambda)T(f) = \bigl\{\lambda h_0 + (1 - \lambda)g; g \in T(f)\bigr\}.$$

Note that $T_\lambda(f)$ is nonempty and ρ-closed subset of C. Let us show that T_λ is a monotone increasing ρ-contraction. Let $f, g \in C$ such that $f \leq g$ ρ-a.e. Since T is a monotone increasing ρ-nonexpansive mapping, for any $F \in T(f)$, there exists $G \in T(g)$ such that $F \leq G$ ρ-a.e. and $\rho(F - G) \leq \rho(f - g)$. Since

$$\rho\bigl(\bigl(\lambda h_0 + (1 - \lambda)F\bigr) - \bigl(\lambda h_0 + (1 - \lambda)G\bigr)\bigr) = \rho\bigl((1 - \lambda)(F - G)\bigr)$$

and ρ is convex, we get

$$\rho\big((\lambda h_0 + (1-\lambda)F) - (\lambda h_0 + (1-\lambda)G)\big) \le (1-\lambda)\rho(F-G).$$

Using the basic properties of the partial order of L_ρ, we get $\lambda h_0 + (1-\lambda)F \le \lambda h_0 + (1-\lambda)G$ ρ-a.e. This clearly shows that T_λ is a monotone increasing ρ-contraction as claimed. Note that if C_T is not empty, then C_{T_λ} is also nonempty. Using Theorem 3.1 we conclude that T_λ has a fixed point $f_\lambda \in C$. Thus there exists $F_\lambda \in T(f_\lambda)$ such that

$$f_\lambda = \lambda h_0 + (1-\lambda)F_\lambda.$$

In particular we have

$$\rho(f_\lambda - F_\lambda) = \rho\big(\lambda(h_0 - F_\lambda)\big) \le \lambda\delta_\rho(C),$$

which implies $\mathrm{dist}_\rho(f_\lambda, T(f_\lambda)) \le \lambda\delta_\rho(C)$. If we choose $\lambda = \frac{1}{n}$ for $n \ge 1$, there exist $f_n \in C$ and $F_n \in T(f_n)$ such that $\rho(f_n - F_n) \le \frac{1}{n}\delta_\rho(C)$, which implies

$$\mathrm{dist}_\rho\big(f_n, T(f_n)\big) \le \frac{1}{n}\delta_\rho(C).$$

The proof of Proposition 3.1 is therefore complete. □

Remark 3.1 We can modify slightly the above proof to show that the approximate fixed point sequence $\{f_n\}$ and its associated sequence $\{F_n\}$ satisfy $f_n \le f_{n+1} \le F_{n+1}$ ρ-a.e. Indeed, set $\{\lambda_n\} = \{1/(n+1)\}_{n\ge1}$. Let $f_0 \in C_T$. Then from the above proof, there exists a fixed point $f_1 = \lambda_1 f_0 + (1-\lambda_1)F_1$ with $f_0 \le f_1$ ρ-a.e. It is easy to check that $f_0 \le f_1 \le F_1$ ρ-a.e. Clearly $f_1 \in C_T$. By induction we build the sequences $\{f_n\}$ and $\{F_n\}$ with $F_n \in T(f_n)$, $f_{n+1} = \lambda_{n+1}f_n + (1-\lambda_{n+1})F_{n+1}$, and $f_n \le f_{n+1} \le F_{n+1}$ for every $n \ge 1$. Since $\lambda_n \to 0$ as n go to ∞, we conclude that $\{f_n\}$ is an approximate fixed point sequence of T.

Using the above results, we are now ready to prove the main fixed point theorem for ρ-nonexpansive monotone multivalued mappings. This theorem may be seen as the monotone version of Theorem 2.4 of [31]. Note that the authors of [31] must assume that ρ is additive to be able to have their conclusion. To avoid the additivity of ρ, we need the following property.

Definition 3.1 Let $\rho \in \Re$. We will say that L_ρ satisfies the ρ-a.e.-Opial property if for every (f_n) in L_ρ ρ-a.e. convergent to 0, such that there exists $\beta > 1$ for which $\sup_n \rho(\beta f_n) < +\infty$, then we have

$$\liminf_{n\to+\infty} \rho(f_n) < \liminf_{n\to+\infty} \rho(f_n + f)$$

for every $f \in L_\rho$ not equal to 0.

Theorem 3.2 *Let $\rho \in \Re$ be convex. Let $C \subset L_\rho$ be a nonempty, ρ-closed, and ρ-bounded convex subset. Assume that ρ satisfies the Δ_2-type condition, L_ρ satisfies the ρ-a.e.-Opial property, and C is ρ-a.e. compact. Then each monotone increasing ρ-nonexpansive map $T : C \to \mathcal{K}(C)$ has a fixed point.*

Proof Proposition 3.1 and Remark 3.1 ensure us with the existence of a sequence $\{f_n\}$ in C and a sequence $\{F_n\}$ such that $F_n \in T(f_n)$, $f_n \preceq f_{n+1} \preceq F_{n+1}$ ρ-a.e., for every $n \geq 1$, and $\lim_{n\to\infty} \rho(f_n - F_n) = 0$. Without loss of generality, we may assume that f_n ρ-a.e. converges to $f \in C$, and F_n ρ-a.e. converges to $F \in C$. The Fatou property implies

$$\rho(f - F) \leq \lim_{n\to\infty} \inf \rho(f_n - F_n) = 0.$$

Hence $f = F$. Clearly we have $f_n \preceq f$ ρ-a.e. for any $n \geq 1$. Since T is a monotone increasing ρ-nonexpansive map, then there exists a sequence $\{H_n\}$ in $T(f)$ such that $F_n \preceq H_n$ ρ-a.e. and

$$\rho(F_n - H_n) \leq \rho(f_n - f)$$

for all $n \geq 1$. Since $T(f)$ is ρ-compact, we may assume that $\{H_n\}$ is ρ-convergent to some $h \in T(f)$. Since ρ satisfies the Δ_2-condition, Lemma 4.2 in [32] implies

$$\liminf_{n\to\infty} \rho(f_n - h) = \liminf_{n\to\infty} \rho(f_n - F_n + F_n - H_n + H_n - h) = \liminf_{n\to\infty} \rho(F_n - H_n).$$

Since $\rho(F_n - H_n) \leq \rho(f_n - f)$, we get

$$\liminf_{n\to\infty} \rho(f_n - h) \leq \liminf_{n\to\infty} \rho(f_n - f).$$

Since C is ρ-bounded and ρ satisfies the Δ_2-condition, the ρ-a.e.-Opial property implies that $f = h$, *i.e.*, $f \in T(f)$. Hence f is a fixed point of T. □

Next we give the graph versions of our results found above.

Theorem 3.3 *Let $\rho \in \Re$ be convex. Let $C \subset L_\rho$ be a nonempty, ρ-closed subset that has Property* 2.1. *Assume that ρ satisfies the Δ_2-type condition and C is ρ-bounded. Let $T : C = V(G) \to \mathcal{C}(C)$ be a monotone increasing G-contraction mapping and $C_T := \{f \in C : (f,g) \in E(G)$ for some $g \in T(f)\}$. If $C_T \neq \emptyset$, then the following statements hold*:

1. *For any $f \in C_T$, $T|_{[f]_{\widetilde{G}}}$ has a fixed point.*
2. *If $f \in C$ with $(\bar{f},f) \in E(G)$, where $\bar{f}$ is a fixed point of T, then there exists a sequence $\{f_n\}$ such that $f_{n+1} \in T(f_n)$ for every $n \geq 0$, and $\{f_n\}$ ρ-converges to $\bar{f}$.*
3. *If G is weakly connected, then T has a fixed point in G.*
4. *If $C' := \bigcup\{[f]_{\widetilde{G}_\rho} : f \in C_T\}$, then $T|_{C'}$ has a fixed point in C.*

Proof 1. Let $f_0 \in C_T$, then there exists $f_1 \in T(f_0)$ with $(f_0,f_1) \in E(G)$. Since T is a monotone increasing G-contraction, there exists $f_2 \in T(f_1)$ such that $(f_1,f_2) \in E(G)$ and

$$\rho(f_1,f_2) \leq \alpha\rho(f_0,f_1),$$

where $\alpha < 1$ is the constant associated to the contraction definition of T. Similarly, there exists $f_3 \in T(f_2)$ such that $(f_2,f_3) \in E(G)$ and

$$\rho(f_2,f_3) \leq \alpha\rho(f_1,f_2).$$

By induction we build $\{f_n\}$ in C with $f_{n+1} \in T(f_n)$ and $(f_n,f_{n+1}) \in E(G)$ such that

$$\rho(f_{n+1},f_n) \leq \alpha\rho(f_n,f_{n-1})$$

for every $n \geq 1$. Hence

$$\rho(f_{n+1},f_n) \leq \alpha^n \rho(f_1,f_0)$$

for every $n \geq 0$. The technical Lemma 2.2 implies that $\{f_n\}$ is ρ-Cauchy. Since L_ρ is ρ-complete and C is ρ-closed, therefore $\{f_n\}$ ρ-converges to some point $g \in C$. Since $(f_n,f_{n+1}) \in E(G)$, for every $n \geq 1$, then $(f_n,g) \in E(G)$ by Property 2.1. Since T is a monotone increasing G-contraction, there exists $g_n \in T(g)$ such that

$$\rho(f_{n+1},g_n) \leq \alpha\rho(f_n,g)$$

for every $n \geq 1$. Hence

$$\begin{aligned}\rho\left(\frac{g_n - g}{2}\right) &\leq \rho(g_n - f_{n+1}) + \rho(f_{n+1} - g)\\ &\leq \alpha\rho(f_n - g) + \rho(f_{n+1} - g)\end{aligned}$$

for every $n \geq 1$. Since $\{f_n\}$ ρ-converges to g, we conclude that $\lim_{n\to\infty} \rho((g_n - g)/2) = 0$. The Δ_2-type condition satisfied by ρ implies that $\lim_{n\to\infty} \rho(g_n - g) = 0$, *i.e.*, $\{g_n\}$ ρ-converges to g. Since $T(g)$ is ρ-closed, we conclude that $g \in T(g)$, *i.e.*, g is a fixed point of T.

2. Let $f \in C$ such that $(\bar{f},f) \in E(G)$. Since T is a monotone increasing G-contraction, then there exists $f_1 \in T(f)$ such that $(\bar{f},f_1) \in E(G)$ and

$$\rho(\bar{f} - f_1) \leq \alpha\rho(\bar{f} - f).$$

By induction, we construct a sequence $\{f_n\}$ such that $f_{n+1} \in T(f_n)$, $(\bar{f},f_n) \in E(G)$, and

$$\rho(\bar{f} - f_{n+1}) \leq \alpha\rho(\bar{f} - f_n)$$

for every $n \geq 0$. Hence we have

$$\rho(\bar{f} - f_n) \leq \alpha^n\rho(\bar{f} - f)$$

for every $n \geq 0$. Since $\alpha < 1$, we conclude that $\{f_n\}$ ρ-converges to $\bar{f}$.

3. Since $C_T \neq \emptyset$, there exists $f_0 \in C_T$, and since G is weakly connected, then $[f_0]_{\widetilde{G}_\rho} = C$, and by 1 the mapping T has a fixed point.

4. It follows easily from 1 and 3. □

Remark 3.2 If we assume that G is such that $E(G) := C \times C$, then clearly G is connected and our Theorem 3.3 gives Nadler's theorem. Moreover, if T is single-valued, then we get the Banach contraction theorem and if T is multivalued, then we get the corrected version of the analogue of the main result of Beg *et al.* [19] in modular function spaces.

The following is a direct consequence of Theorem 3.3.

Corollary 3.1 *Let $\rho \in \Re$ be convex. Let $C \subset L_\rho$ be a nonempty and ρ-closed subset that has Property* 2.1. *Assume that ρ satisfies the Δ_2-type condition and C is ρ-bounded. Assume that G is weakly connected. Let $T : C = V(G) \to \mathcal{C}(C)$ be a monotone increasing G-contraction multivalued mapping such that there exist f_0 and $f_1 \in T(f_0)$ with $(f_0, f_1) \in E(G)$. Then T has a fixed point.*

Let us give an example which will illustrate the role of the above defined notions.

Example 3.1 Consider $L^\infty[0,1]$ (the space of all bounded measurable functions on $[0,1]$ or rather all measurable functions which are bounded except possibly on a subset of measure zero). Notice that we identify functions which are equivalent, *i.e.*, we should say that the elements of $L^\infty[0,1]$ are not functions both, rather equivalent classes of functions. Then $L^\infty[0,1]$ is a normed linear space with

$$\|f\| = \|f\|_\infty = \inf\{M : |f(t)| \leq M \text{ a.e.}\} = \inf\{M : m\{t : f(t) > M\} = 0\}.$$

Let $C = \{f_0, f_1, f_2\}$, where:

(1) f_0 is the step function on $[0,1]$ with partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ such that $f_0 = c_i = \frac{1}{i+1}$ on (x_{i-1}, x_i) and $f_0(x_i) = d_i$ arbitrary real number. Note that $\|f_0\|_\infty = \max|c_i| = \frac{1}{2}$.

(2) $f_1 = e^x$ on $[0,1]$. So $\|f_1\|_\infty = e$.

(3) $f_2 = X^2$ on $[0,1]$. So $\|f_2\|_\infty = 1$.

Let $\rho := \|\cdot\|_\infty$ and $T : C \to \mathcal{C}(C)$ be defined as follows:

$$T(f) = \begin{cases} f_0 & \text{if } f = f_0, \\ f_2 & \text{if } f = f_1, f_2. \end{cases}$$

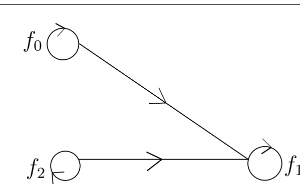
Therefore, $E(G) = \{(f_0, f_0), (f_1, f_1), (f_2, f_2), (f_0, f_1), (f_2, f_1)\}$. The digraph of G is shown in Figure 1.

Now, for all $(f, g) \in E(G)$, T is a G-contraction. Also all other assumptions of Theorem 3.3 are satisfied and T has a fixed point.

As an application of Theorem 3.3, we have the following result whose proof is similar to Proposition 3.1.

Proposition 3.2 *Let $\rho \in \Re$ be convex. Let $C \subset L_\rho$ be a nonempty and ρ-closed convex subset. Assume that ρ satisfies the Δ_2-type condition and C is ρ-bounded that has Property* 2.1.

Figure 1 The digraph of *G*.



Let $T : C \to \mathcal{C}(C)$ be a monotone increasing G-nonexpansive map. Then there exists an approximate fixed points sequence $\{f_n\}$ in C, i.e., for any $n \geq 1$, there exists $F_n \in T(f_n)$ such that

$$\lim_{n\to\infty} \rho(f_n - F_n) = 0.$$

Note that a similar conclusion to Theorem 3.2 in terms of graph may be found under strong properties satisfied by the graph G.

Competing interests
The author declares that they have no competing interests.

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