Ni and Yao *Fixed Point Theory and Applications* (2015) 2015:35
DOI 10.1186/s13663-015-0283-8

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

The modified Ishikawa iterative algorithm with errors for a countable family of Bregman totally quasi-D-asymptotically nonexpansive mappings in reflexive Banach spaces

Ren-Xing Ni[1] and Jen-Chih Yao[2,3*]

*Correspondence: yaojc@mail.cmu.edu.tw
[2]Center for General Education, China Medical University, Taichung, 40402, Taiwan
[3]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this paper, a new modified Ishikawa iterative algorithm with errors by a shrinking projection method for generalized mixed equilibrium problems and a countable family of uniformly Bregman totally quasi-D-asymptotically nonexpansive mappings is introduced and investigated in the framework of a real Banach space. Strong convergence of the sequence generated by the proposed algorithm is derived under some suitable assumptions. These results are new and develop some recent results in this field.

MSC: 26B25; 46T99; 47H04; 47H05; 47H09; 47H10; 47J05; 47J20; 47J25; 52A41; 54C20

Keywords: asymptotically Bregman totally quasi-D-asymptotically nonexpansive mapping; generalized mixed equilibrium problem; Bregman distance; Bregman projection; fixed point; shrinking projection method; Banach space

1 Introduction and preliminaries

In this paper, without other specifications, let N^* and R be the sets of positive integers and real numbers, respectively, C be a nonempty, closed, and convex subset of a real Banach space E with the dual space E^*. The norm and the dual pair between E^* and E are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $g: E \to R\cup\{+\infty\}$ be a proper convex and lower semicontinuous function. Denote the domain of g by $\operatorname{dom} g$, *i.e.*, $\operatorname{dom} g = \{x \in E: g(x) < +\infty\}$. The Fenchel conjugate of g is the function $g^*: E^* \to (-\infty,+\infty]$ defined by $g^*(\zeta) = \sup_{x\in E}\{\langle\zeta,x\rangle - g(x)\}$. Let $T: E \to C$ be a nonlinear mapping. For all $x \in E$ and $x^* \in E^*$, denote by $F(T) = \{x \in C: Tx = x\}$ the set of fixed points of T and by $\langle x,x^*\rangle$ the value of x^* at x. A mapping T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x,y \in E$.

Let $\{x_n\}$ be a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$. For any $x \in \operatorname{int}(\operatorname{dom} g)$, the right-hand derivative of g at x in the direction $y \in E$ is defined by $g'(x,y) := \lim_{t\to 0}\frac{g(x+ty)-g(x)}{t}$. The mapping g is called Gâteaux differentiable at x if, for all $y \in E$, $\lim_{t\to 0}\frac{g(x+ty)-g(x)}{t}$ exists. In this case, $g'(x,y)$ coincides with $\nabla g(x)$ and the value of the gradient of g at x. The mapping g is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} g)$. g is called Fréchet differentiable at x if this limit is attained uniformly for $\|y\| = 1$. We say that g is uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The Legendre function $g : E \to (-\infty, +\infty]$ is defined in [1]. From [1], if E is a reflexive Banach space, then g is the Legendre function if and only if it satisfies the conditions (L1) and (L2):

(L1) The interior of the domain of g, $\operatorname{int}(\operatorname{dom} g)$, is nonempty, g is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} g)$ and $\operatorname{dom}(g) = \operatorname{int}(\operatorname{dom} g)$.

(L2) The interior of the domain of g^*, $\operatorname{int}(\operatorname{dom} g^*)$, is nonempty, g^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} g^*)$ and $\operatorname{dom} g^* = \operatorname{int}(\operatorname{dom} g^*)$, where the function $g^* : E^* \to (-\infty, +\infty]$ is the Fenchel conjugate of g.

Examples of Legendre functions are given in [2, 3]. One important and interesting Legendre function is $\frac{1}{s}\|\cdot\|^s$ $(1 < s < +\infty)$, in the Banach space E which is smooth and strictly convex and, in particular, a Hilbert space.

By Bauschke *et al.* [1], Theorem 5.4, the conditions (L1) and (L2) also show that the functions g and g^* are strictly convex on the interior of their respective domains. From now on, we assume that the convex function $g : E \to (-\infty, +\infty]$ is Legendre.

Definition 1.1 [4, 5] Let $g : E \to R$ be a Gâteaux differentiable and convex function. The function $D(\cdot,\cdot) : \operatorname{dom} g \times \operatorname{int}(\operatorname{dom} g) \to [0, +\infty)$ defined by $D(y,x) = g(y) - g(x) - \langle y - x, \nabla g(x)\rangle$ is called the Bregman distance with respect to g.

It follows from the strict convexity of g that $D(x,y) \geq 0$ for all x, y in E. However, $D(\cdot,\cdot)$ might not be symmetric and $D(\cdot,\cdot)$ might not satisfy the triangular inequality.

Remark 1.1 [4] The Bregman distance has the following properties:

(1) the three point identity, for any $x \in \operatorname{dom} g$ and $y, z \in \operatorname{int}(\operatorname{dom} g)$,
$D(x,z) = D(x,y) + D(y,z) + \langle \nabla g(y) - \nabla g(z), x - y\rangle$;

(2) the four point identity, for any $y, w \in \operatorname{dom} g$ and $x, z \in \operatorname{int}(\operatorname{dom} g)$,
$D(y,x) - D(y,z) - D(w,x) + D(w,z) = \langle \nabla g(z) - \nabla g(x), y - w\rangle$.

Definition 1.2 [4] Let $g : E \to R$ be a Gâteaux differentiable and convex function. The Bregman projection of $x \in \operatorname{int}(\operatorname{dom} g)$ onto the nonempty, closed and convex set $C \subset \operatorname{dom} g$ is the necessarily unique vector $\operatorname{Proj}_C^g(x) \in C$ satisfying the following:

$$D\bigl(\operatorname{Proj}_C^g(x), x\bigr) = \inf\bigl\{D(y,x) : y \in C\bigr\}.$$

Definition 1.3 [6] Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by $J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$, $\phi : E \times E \to R^+$ be the Lyapunov functional defined by $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, $\forall x, y \in E$. The generalized projection $\Pi_C(x)$ defined by

$$\phi\bigl(\Pi_C(x), x\bigr) = \inf\bigl\{\phi(y,x) : y \in C\bigr\}.$$

Remark 1.2 (1) If E is a smooth Banach space and $g(x) = \|x\|^2$ for all $x \in E$, then we have $\nabla g(x) = 2Jx$ for all x in E. Hence, $D(\cdot,\cdot)$ reduces to the usual map $\phi(\cdot,\cdot)$ as $D(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 = \phi(x,y)$, $\forall x, y \in E$. The Bregman projection $\operatorname{Proj}_C^g(x)$ reduces to the generalized projection $\Pi_C(x)$ [6]. It is obvious from the definition of ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2$.

(2) If E is a Hilbert space and $g(x) = \|x\|^2$ for all $x \in E$, then $D(x,y) = \|x - y\|^2$ and the Bregman projection $\mathrm{Proj}_C^g(x)$ is reduced to the metric projection $P_C(x)$ of x onto C. For more details we refer the readers to [5].

Let C be a nonempty, closed, and convex subset of E and T be a mapping from E to C. A point $p \in C$ is said to be an asymptotic fixed point of T [7] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. A point $p \in C$ is said to be a strong asymptotic fixed point of T [7] if C contains a sequence which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the sets of asymptotic fixed points and strong asymptotic fixed points of T by $\widehat{F}(T)$ and $\widetilde{F}(T)$, respectively.

Definition 1.4 (1) A mapping T from E to C is said to be Bregman relatively nonexpansive [5, 8], if $\widehat{F}(T) = F(T) \neq \emptyset$ and $D(p, Tx) \le D(p,x)$ for all $x \in E$ and $p \in F(T)$.

(2) T is said to be Bregman weak relatively nonexpansive [4, 5, 8], if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $D(p, Tx) \le D(p,x)$ for all $x \in E$ and $p \in F(T)$.

(3) T is said to be Bregman quasi-D-nonexpansive [7, 9], if $F(T) \neq \emptyset$ and $D(p, Tx) \le D(p,x)$ for all $x \in E$ and $p \in F(T)$.

(4) T is said to be Bregman firmly nonexpansive [4], if $\langle \nabla g(Tx) - \nabla g(Ty), Tx - Ty\rangle \le \langle \nabla g(x) - \nabla g(y), Tx - Ty\rangle$, $\forall x, y \in E$, or, equivalently, $D(Tx, Ty) + D(Ty, Tx) + D(Tx, x) + D(Ty, y) \le D(Tx, y) + D(Ty, x)$, $\forall x, y \in E$.

(5) T is said to be Bregman strongly nonexpansive [10], if $\widehat{F}(T) \neq \emptyset$ and $D(p, Tx) \le D(p,x)$ for all $x \in E$ and $p \in \widehat{F}(T)$ and if whenever $\{x_n\} \subset E$ is bounded, $p \in \widehat{F}(T)$ and $\lim_{n\to+\infty}[D(p,x_n) - D(p, Tx_n)] = 0$, it follows that $\lim_{n\to+\infty} D(Tx_n, x_n) = 0$.

(6) T is said to be relatively quasi-nonexpansive [4], if $\widehat{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p,x)$ for all $x \in E$ and $p \in F(T)$.

(7) T is said to be weak relatively nonexpansive [11–14], if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p,x)$ for all $x \in E$ and $p \in F(T)$.

(8) T is said to be quasi-ϕ-nonexpansive [11–14], if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p,x)$ for all $x \in E$ and $p \in F(T)$.

Definition 1.5 (1) A mapping $T : E \to C$ is said to be Bregman totally quasi-D-asymptotically nonexpansive [15], if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{u_n\}$ with $\nu_n, u_n \to 0$ (as $n \to +\infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that

$$D\bigl(p, T^n x\bigr) \le D(p,x) + \nu_n \cdot \zeta\bigl[D(p,x)\bigr] + u_n, \quad \forall n \ge 1, x \in E, p \in F(T). \tag{1.1}$$

(2) A mapping $T : E \to C$ is said to be Bregman quasi-D-asymptotically nonexpansive [15], if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to+\infty} k_n = 1$ such that

$$D\bigl(p, T^n x\bigr) \le k_n D(p,x) \quad \text{for all } x \in E, p \in F(T) \text{ and } n \ge 1. \tag{1.2}$$

(3) A mapping $T : E \to C$ is said to be Bregman quasi-D-asymptotically nonexpansive in the intermediate sense with sequence $\{\nu_n\}$, if $F(T) \neq \emptyset$ and there exists a sequence $\{\nu_n\}$ in $[0, +\infty)$ with $\lim_{n\to+\infty} \nu_n = 0$ such that

$$\limsup_{n\to+\infty} \sup_{x\in E, p\in F(T)} \bigl[D\bigl(p, T^n x\bigr) - (1 + \nu_n) D(p,x)\bigr] \le 0. \tag{1.3}$$

(4) A mapping $T : E \to C$ is said to be totally quasi-ϕ-asymptotically nonexpansive [16], if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{u_n\}$ with $\nu_n, u_n \to 0$ (as $n \to +\infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \nu_n \cdot \zeta[\phi(p, x)] + u_n, \quad \forall n \geq 1, x \in E, p \in F(T). \tag{1.4}$$

(5) A mapping $T : E \to C$ is said to be quasi-ϕ-asymptotically nonexpansive [16], if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to+\infty} k_n = 1$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $x \in E$, $p \in F(T)$ and $n \geq 1$.

(6) A mapping $T : E \to C$ is said to be quasi-ϕ-asymptotically nonexpansive in the intermediate sense with sequence $\{\nu_n\}$, if $F(T) \neq \emptyset$ and there exists a sequence $\{\nu_n\}$ in $[0, +\infty)$ with $\lim_{n\to+\infty} \nu_n = 0$ such that

$$\limsup_{n\to+\infty} \sup_{x\in E, p\in F(T)} [\phi(p, T^n x) - (1 + \nu_n)\phi(p, x)] \leq 0. \tag{1.5}$$

Remark 1.3 (1) If $\zeta(t) = t$, $t \geq 0$, then (1.1) reduces to

$$D(p, T^n x) \leq (1 + \nu_n) \cdot D(p, x) + u_n, \quad \forall n \geq 1, x \in E, p \in F(T). \tag{1.6}$$

In addition, if $u_n \equiv 0$ for all $n \geq 1$, then Bregman totally quasi-D-asymptotically nonexpansive mappings coincide with Bregman quasi-D-asymptotically nonexpansive mappings. If $u_n \equiv 0$ and $\nu_n \equiv 0$ for all $n \geq 1$, we obtain from (1.6) the class of mappings that includes the class of Bregman quasi-nonexpansive mappings. If $\nu_n \equiv 0$ and $u_n = \sigma_n = \max\{0, \sup_{x\in E, p\in F(T)}(D(p, T^n x) - D(p, x))\}$, for all $n \geq 1$, then (1.6) reduces to (1.3) which has been studied as mappings Bregman quasi-D-asymptotically nonexpansive in the intermediate sense.

(2) From the definitions, it is obvious that if $\widehat{F}(T) = F(T) \neq \emptyset$, then a Bregman strongly nonexpansive mapping is a Bregman relatively nonexpansive mapping; a Bregman relatively nonexpansive mapping is a Bregman quasi-D-nonexpansive mapping. A Bregman quasi-D-nonexpansive mapping is a Bregman quasi-D-asymptotically nonexpansive mapping, but the converse is not true.

If taking $\zeta(t) = t$, $t \geq 0$, $\nu_n = k_n - 1$, $u_n = 0$, $\lim_{n\to+\infty} k_n = 1$, then (1.1) can be rewritten as (1.2). This implies that each Bregman quasi-D-asymptotically nonexpansive mapping must be a Bregman total quasi-D-asymptotically nonexpansive mapping, but the converse is not true. In [10], Chang *et al.* gave an example of Bregman total quasi-D-asymptotically nonexpansive mapping. A Bregman relatively nonexpansive mapping is a Bregman weak relatively nonexpansive mapping, but the converse in not true in general. Indeed, for any mapping $T : E \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. If T is Bregman relatively nonexpansive, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$. In [7], Naraghirad and Yao have given two examples of a Bregman weak relatively nonexpansive mapping which is not a Bregman relatively nonexpansive mapping, and a Bregman quasi-nonexpansive mapping which is neither a Bregman relatively nonexpansive mapping nor a Bregman weak relatively nonexpansive mapping.

(3) The class of quasi-ϕ-(asymptotically) nonexpansive mappings is more general than that of relatively nonexpansive mappings which requires the restriction $\widehat{F}(T) = F(T)$.

A quasi-ϕ-nonexpansive mapping with a nonempty fixed point set $F(T)$ is a quasi-ϕ-asymptotically nonexpansive mapping, but the converse may not be true. In the framework of Hilbert spaces, quasi-ϕ-(asymptotically) nonexpansive mappings is reduced to quasi-(asymptotically) nonexpansive mappings.

The idea of the definition of a total asymptotically nonexpansive mappings is to unify various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and to prove a general convergence theorems applicable to all these classes of nonlinear mappings.

Definition 1.6 [7] Let E be a Banach space. The function $g : E \to R$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D(x,y) \leq r\}$ is bounded for all $x \in E$ and $r > 0$.

The theory of fixed points with respect to Bregman distances have been studied in the last ten years and much intensively in the last six years. In [17], Bauschke and Combettes introduced an iterative method to construct the Bregman projection of a point onto a countable intersection of closed and convex sets in reflexive Banach spaces. They proved strong convergence theorem of the sequence produced by their method; for more details, see [17], Theorem 4.6. To find a point of the intersection of m closed and convex subsets in a Banach space, in 2007, Alber [18] first studied the iterative method with Bregman projections. In [6], Alber investigated the generalized projections in a Banach space. For some recent articles on the existence of fixed points for Bregman nonexpansive type mappings, we refer the reader to [1–10, 19–34].

It is well known that the following conclusions hold:

Lemma 1.1 [5, 10] *Let E be a Banach space and $g : E \to R$ a Gâteaux differentiable function which is locally uniformly convex on E. Let $\{y_n\}$ and $\{z_n\}$ be sequences in E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. Then* $\lim_{n\to+\infty} D(y_n,z_n) = 0 \Leftrightarrow \lim_{n\to+\infty} \|y_n - z_n\| = 0$.

Lemma 1.2 *Let C be a nonempty closed convex subset of Banach space E and $g : E \to (-\infty,+\infty]$ be a Legendre function which is total convex on bounded subsets of E. Let $T : E \to C$ be a closed and Bregman totally quasi-D-asymptotically nonexpansive mapping with nonnegative real sequences $\{v_n\}$, $\{u_n\}$ and a strictly increasing and continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$. If $v_n, u_n \to 0$ (as $n \to +\infty$). Then $F(T)$ is a closed convex subset of C.*

Proof Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x^*$ (as $n \to +\infty$). We have $Tx_n = x_n \to x^*$ (as $n \to +\infty$) and by the closeness of T, we have $Tx^* = x^*$. This implies that $F(T)$ is closed.

Let $p,q \in F(T)$ and $t \in (0,1)$, and put $w = tp + (1-t)q$. Next we prove that $w \in F(T)$. Indeed, in view of the definition of D, we have

$$\begin{aligned} D(w,T^n w) &= g(w) - g(T^n w) - \langle \nabla g(T^n w), w - T^n w\rangle \\ &= g(w) - g(T^n w) - \langle \nabla g(T^n w), tp + (1-t)q - T^n w\rangle \\ &= g(w) + tD(p,T^n w) + (1-t)D(q,T^n w) - tg(p) - (1-t)g(q). \end{aligned} \tag{1.7}$$

Since

$$
\begin{aligned}
&tD\big(p,T^n w\big)+(1-t)D\big(q,T^n w\big)\\
&\quad\le t\big\{D(p,w)+\nu_n\zeta\big[D(p,w)\big]+u_n\big\}\\
&\qquad+(1-t)\big\{D(q,w)+\nu_n\zeta\big[D(q,w)\big]+u_n\big\}\\
&\quad=t\big\{g(p)-g(w)-\big\langle\nabla g(w),p-w\big\rangle+\nu_n\zeta\big[D(p,w)\big]+u_n\big\}\\
&\qquad+(1-t)\big\{g(q)-g(w)-\big\langle\nabla g(w),q-w\big\rangle+\nu_n\zeta\big[D(q,w)\big]+u_n\big\}\\
&\quad=tg(p)+(1-t)g(q)-g(w)+(1-t)\nu_n\zeta\big[D(q,w)\big]\\
&\qquad+u_n+t\nu_n\zeta\big[D(p,w)\big].
\end{aligned}
\tag{1.8}
$$

Substituting (1.7) into (1.8) and simplifying it, we have

$$0\le D\big(w,T^n w\big)\le t\nu_n\zeta\big[D(p,w)\big]+(1-t)\nu_n\zeta\big[D(q,w)\big]+u_n \quad (\text{as } n\to+\infty).$$

Hence, we have $T^n w\to w$. This implies that $T(T^n w)=T^{n+1}w\to w$. Since T is closed, we have $w\in Tw$, *i.e.*, $w\in F(T)$. This completes the proof of Lemma 1.2. □

Definition 1.7 [35] Let $g:E\to(-\infty,+\infty]$ be a convex and Gâteaux differentiable function. g is called

(1) totally convex at $x\in\operatorname{int}(\operatorname{dom}g)$ if its modulus of total convexity at x, that is, the function $\nu_g:\operatorname{int}(\operatorname{dom}g)\times[0,+\infty)\to[0,+\infty)$, defined by $\nu_g(x,t):=\inf\{D(y,x):y\in\operatorname{dom}g,\|y-x\|=t\}$, is positive whenever $t>0$;
(2) totally convex if it is totally convex at every point $x\in\operatorname{int}(\operatorname{dom}g)$;
(3) totally convex on bounded sets if $\nu_g(B,t)$ is positive for any nonempty bounded subset B of E and $t>0$, where the modulus of total convexity of the function g on the set B is the function $\nu_g:\operatorname{int}(\operatorname{dom}g)\times[0,+\infty)\to[0,+\infty)$ defined by $\nu_g(B,t):=\inf\{\nu_g(x,t):x\in B\cap\operatorname{dom}g\}$.

Definition 1.8 [9, 35] Let B be the closed unit ball of a Banach space E. A function $g:E\to R$ is said to be

(1) cofinite if $\operatorname{dom}g^*=E^*$;
(2) coercive if $\lim_{\|x\|\to\infty}(g(x)/\|x\|)=+\infty$;
(3) sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded, $\lim_{n\to+\infty}D(y_n,x_n)=0\Rightarrow\lim_{n\to+\infty}\|y_n-x_n\|=0$;
(4) locally bounded if $g(rB)$ is bounded for all $r>0$;
(5) locally uniformly smooth on E if the function $\sigma_r:[0,+\infty)\to[0,+\infty)$, defined by

$$
\begin{aligned}
\sigma_r(t)=&\sup_{x\in rB,y\in E,\|y\|=1,\alpha\in(0,1)}\big[\alpha g\big(x+(1-\alpha)ty\big)+(1-\alpha)g(x-\alpha ty)-g(x)\big]\\
&\times\big[\alpha(1-\alpha)\big]^{-1/2},
\end{aligned}
$$

satisfies $\lim_{t\to0}\frac{\sigma_r(t)}{t}=0$, $\forall r>0$;

(6) locally uniformly convex on E (or uniformly convex on bounded subsets of E) if the gauge $\rho_r : [0, +\infty) \to [0, +\infty)$ of uniform convexity of g, defined by

$$\rho_r(t) = \inf_{x,y \in rB, \|x-y\|=t, \alpha\in(0,1)} \bigl[\alpha g(x) + (1-\alpha)g(y) - g\bigl(\alpha x + (1-\alpha)y\bigr)\bigr] \times \bigl[\alpha(1-\alpha)\bigr]^{-1/2},$$

satisfies $\rho_r(t) > 0$, $\forall r, t > 0$.

Lemma 1.3 [9] *If $g : E \to (-\infty, +\infty]$ is Fréchet differentiable and totally convex, then g is cofinite.*

Lemma 1.4 [32] *Let $g : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points. Then the following statements hold*:

(1) *g is sequentially consistent if and only if it is totally convex on bounded sets.*
(2) *If g is lower semicontinuous, then g is sequentially consistent if and only if it is uniformly convex on bounded sets.*
(3) *If g is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when g is lower semicontinuous, Fréchet differentiable on its domain and the Fréchet derivative ∇g is uniformly continuous on bounded sets.*

Lemma 1.5 [36] *Let $g : E \to R$ be uniformly Fréchet differentiable and bounded on bounded subsets of E. Then ∇g is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 1.6 ([9], Lemma 3.1) *Let $g : E \to R$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 1.7 [7] *Let E be a Banach space, $r > 0$ be a positive number and $g : E \to R$ be a continuous and convex function which is uniformly convex on bounded subsets of E. Then*

$$g\left(\sum_{n=1}^{m} \lambda_n x_n\right) \le \sum_{n=1}^{m} \lambda_n g(x_n) - \lambda_i \lambda_j \rho_r\bigl(\|x_i - x_j\|\bigr)$$

for any given infinite subset $\{x_n\} \subset B_r(0) = \{x \in E : \|x\| \le r\}$ and for any given sequence $\{\lambda_n\}$ of positive numbers with $\sum_{n=1}^{m} \lambda_n = 1$, for any $i, j \in \{1, 2, \dots, m\}$ with $i < j$, where ρ_r is the gauge of uniformly convexity of g.

Lemma 1.8 [37] *Let $g : E \to (-\infty, +\infty]$ be Gâteaux differentiable and totally convex on* $\operatorname{int}(\operatorname{dom} g)$. *Let $x \in \operatorname{int}(\operatorname{dom} g)$ and $C \subset \operatorname{int}(\operatorname{dom} g)$ be a nonempty, closed, and convex set. If $\hat{x} \in C$, then the following statements are equivalent*:

(1) *the vector $\hat{x}$ is the Bregman projection of x onto C with respect to g*;
(2) *the vector $\hat{x}$ is the unique solution of the variational inequality*:
$\langle \nabla g(x) - \nabla g(z), z - y\rangle \ge 0$, $\forall y \in C$;
(3) *the vector $\hat{x}$ is the unique solution of the inequality*: $D(y, z) + D(z, x) \le D(y, x)$, $\forall y \in C$.

Lemma 1.9 ([7], Theorem 2.1) *Let E be a reflexive Banach space and let $g : E \to R$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent*:

(1) *g is strongly coercive and uniformly convex on bounded subsets of E*;
(2) $\operatorname{dom} g^* = E^*$, *$g^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets of E^**;
(3) $\operatorname{dom} g^* = E^*$, *$g^*$ is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*.*

Lemma 1.10 ([7], Theorem 2.2) *Let E be a reflexive Banach space and let $g : E \to R$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent*:

(1) *g is bounded on bounded subsets and uniformly smooth on bounded subsets of E*;
(2) *g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^**;
(3) $\operatorname{dom} g^* = E^*$, *$g^*$ is strongly coercive and uniformly convex on bounded subsets of E^*.*

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to R$ satisfies the following conditions:

(C1) $f(x,x) = 0$, $\forall x \in C$;
(C2) f is monotone, *i.e.*, $f(x,y) + f(y,x) \leq 0$, $\forall x,y \in C$;
(C3) for each $y \in C$, the function $x \mapsto f(x,y)$ is upper semicontinuous;
(C4) $\forall x \in C$, $y \mapsto f(x,y)$ is convex and lower semicontinuous.

Lemma 1.11 [7] *Let E be a reflexive Banach space and $g : E \to R$ a convex, continuous and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subset of E. Let C be a nonempty, closed and convex subset of E and $f : C \times C \to R$ a bifunction satisfying conditions* (C1)-(C4) *and* $\mathrm{EP}(G) \neq \emptyset$, *$\varphi : C \to R$ be a lower semicontinuous and convex functional, $A : C \to E^*$ be a continuous and monotone mapping. For $r > 0$ and $x \in E$, define a mapping $T_r^G : E \to C$ as follows*:

$$T_r^G x = \left\{ z \in C : G(z,y) + \frac{1}{r}\langle y - z, \nabla g(z) - \nabla g(x)\rangle \geq 0, \forall y \in C \right\}, \tag{1.9}$$

where $G(x,y) = f(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle$, $\forall x,y \in E$. Then the following statements hold:

(1) $\operatorname{dom}(T_r^G) = E$;
(2) *T_r^G is single-valued*;
(3) *T_r^G is a Bregman firmly nonexpansive mapping*;
(4) $F(T_r^G) = \mathrm{GMEP}(f,\varphi)$;
(5) $\mathrm{GMEP}(f,\varphi)$ *is closed and convex of C*;
(6) $D(q, T_r^G x) + D(T_r^G x, x) \leq D(q,x)$, $\forall q \in F(T_r^G)$.

In 2010, Saewan *et al.* [38] studied the following generalized mixed equilibrium problem: find $z \in C$ such that

$$f(z,y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.10}$$

where f is a bifunction from $C \times C$ to R, $\varphi : C \to R$ is a real-valued function and $A : C \to E^*$ is a nonlinear mapping. Denote the set of solutions of the problem (1.10) by $\mathrm{GMEP}(f, \varphi)$, *i.e.*,

$$\mathrm{GMEP}(f,\varphi) = \left\{z \in C | f(z,y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \geq 0, \forall y \in C\right\}.$$

Special cases: (I) If $A = 0$, then the problem (1.10) is equivalent to find $z \in C$ such that

$$f(z,y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.11}$$

which is called the mixed equilibrium problem. Denote the set of solutions of (1.11) by $\mathrm{MEP}(f, \varphi)$.

(II) If $f = 0$, then the problem (1.10) is equivalent to find $z \in C$ such that

$$\langle Az, y - z\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.12}$$

which is called the mixed variational inequality of Browder-type. Denote the set of solutions of (1.12) by $\mathrm{VI}(C, A, \varphi)$. In particular, we denote $\mathrm{VI}(C, A, 0)$ by $\mathrm{VI}(C, A)$.

(III) If $\varphi = 0$, then the problem (1.10) is equivalent to finding $z \in C$ such that

$$f(z,y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C, \tag{1.13}$$

which is called the generalized equilibrium problem. Denote the set of solutions of (1.13) by $\mathrm{GEP}(f)$.

(IV) If $A = 0$, $\varphi = 0$, then the problem (1.10) is equivalent to finding $z \in C$ such that

$$f(z,y) \geq 0, \quad \forall y \in C, \tag{1.14}$$

which is called the equilibrium problem. Denote the set of solutions of (1.14) by $\mathrm{EP}(f)$.

It is well known that mixed equilibrium problems and their generalizations have been important tools for solving problems arising in the fields of linear or nonlinear programming, variational inequalities, complementary problems, optimization problems, and fixed point problems, and they have been widely applied to physics, structural analysis, management science, economics, *etc.* One of the most important and interesting topics in the theory of equilibria is to develop efficient and implementable algorithms for solving equilibrium problems and their generalizations (see, *e.g.*, [39–44] and the references therein). Since the generalized mixed equilibrium problems have very close connections with both the fixed point problems and the variational inequalities problems, finding the common elements of these problems has drawn many researchers' attention and has become one of the hot topics in the related fields in the past few years (see, *e.g.*, [11–14, 45–48] and the references therein). Some methods have been proposed to solve the generalized mixed equilibrium problem (see, for example, [11–14, 38, 40–48]). Numerous problems in physics, optimization and economics help to find a solution of problem (1.10).

It is well known that, in an infinite-dimensional Hilbert space, only weak convergence theorems for the segmenting Mann iteration were established even for nonexpansive mappings. Attempts to modify the segmenting Mann iteration for nonexpansive mappings and asymptotically nonexpansive mappings by hybrid projection algorithms have recently been made so that strong convergence theorems are obtained; see, for example, [11–14, 38, 40–48] and the references therein.

In [44], Martinez-Yanes and Xu introduced the following iterative scheme for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|z-y_n\|^2 \le \|z-x_n\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.15}$$

where P_C denotes the metric projection of H onto a closed and convex subset C of H. They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

In [11], Qin and Su extended the results of Martinez-Yanes and Xu [44] from Hilbert spaces to Banach spaces and proved the following result: Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E and let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, n \ge 0. \end{cases} \tag{1.16}$$

If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

In 2009, Wangkeeree and Wangkeeree [14] introduced the following hybrid projection algorithm for approximation of common fixed point of two families of relatively quasi-nonexpansive mappings, which is also a solution to a variational inequality problem in a Banach space E:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_{1,i} = C, \quad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \\ x_i = \Pi_{C_1} x_0, \\ w_{n,i} = \Pi_C J^{-1}(Jx_n - \lambda_{n,i}Bx_n), \\ z_{n,i} = J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i}), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 + (1-\alpha_{n,i})Jz_{n,i}), \\ C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \le \phi(z, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle Jx_n - Jx_0, z\rangle)\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{1.17}$$

They proved under appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (1.17) converges strongly to a common element of the set of common fixed points of the two families $\{T_i\}$ and $\{S_i\}$ and the set of solutions to a variational inequality problem.

In 1967, Bregman [34] discovered an elegant and effective technique for using the so-called Bregman distance function $D(\cdot,\cdot)$ in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique has been applied in various ways in order to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, for computing fixed points of nonlinear mappings, and so on (see, *e.g.*, [7, 9, 19–31] and the references therein). In 2006, Butnariu and Resmerita [32] presented Bregman-type iterative algorithms and studied the convergence of the Bregman-type iterative method of solving some nonlinear operator equations.

In 2010, by using the Bregman projection, Reich and Sabach [9] presented the following proximal algorithms for finding common zeroes of maximal monotone operators $A_i : E \to 2^{E^*}$ $(i = 1, 2, \dots, m)$ in a reflexive Banach space E. More precisely, they proved the following strong convergence theorem.

Theorem RS *Let E be a reflexive Banach space and let $A_i : E \to 2^{E^*}$ $(i = 1, 2, \dots, m)$ be m maximal monotone operators such that $Z := \bigcap_{i=1}^m A_i^{-1}(0^*) \neq \emptyset$. Let $g : E \to R$ be a Legendre function that is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{x_n\}$ be a sequence defined by the following iterative algorithm*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ y_n^i = \operatorname{Res}^g_{\lambda_n^i}(x_n + e_n^i), \\ C_n^i = \{z \in E : D(z, y_n^i) \leq D(z, x_n + e_n^i)\}, \\ C_n = \bigcap_{i=1}^m C_n^i, \\ Q_n = \{z \in E : \langle \nabla g(x_0) - \nabla g(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = \operatorname{Proj}^g_{C_n \cap Q_n}(x_0), \quad \forall n \geq 0. \end{cases} \tag{1.18}$$

If, for each $i = 1, 2, \dots, m$, $\liminf_{n\to+\infty} \lambda_n^i > 0$ and the sequences of errors $\{e_n^i\} \subset E$ satisfy $\lim_{n\to+\infty} e_n^i = 0$, then each such sequence $\{x_n\}$ converges strongly to $\operatorname{Proj}^g_Z(x_0)$ as $n \to +\infty$. Further, under some suitable conditions, they obtained two strong convergence theorems of maximal monotone operators in a reflexive Banach space. Reich and Sabach [28] *also studied the convergence of two iterative algorithms for finitely many Bregman strongly nonexpansive operators in a Banach space.*

In [26], Reich and Sabach proposed the following algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators $T_i : E \to E$ $(i = 1, 2, \dots, m)$ in a reflexive Banach space E, if $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$:

$$\begin{cases} x_0 \in E, \\ Q_0^i = E, \quad i = 1, 2, \dots, m, \\ y_n^i = T_i(x_n + e_n^i), \\ Q_{n+1}^i = \{z \in Q_n^i : \langle \nabla g(x_n + e_n^i) - \nabla g(y_n^i), z - y_n^i \rangle \leq 0\}, \\ Q_n = \bigcap_{i=1}^m Q_n^i, \\ x_{n+1} = \operatorname{Proj}^g_{Q_{n+1}}(x_0), \quad \forall n \geq 0. \end{cases} \tag{1.19}$$

Under some suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.19) converges strongly to $\operatorname{Proj}^g_F(x_0)$ and applied the result to the solution of convex feasibility and equilibrium problems, where $g : E \to R$ and $\{e_n^i\} \subset E$ satisfying $\lim_{n\to+\infty} e_n^i = 0$ for each $n \geq 1$ and $i = 1, 2, \dots, m$.

Very recently, Chen *et al.* [33] introduced the concept of weak Bregman relatively nonexpansive mappings in a reflexive Banach space and gave an example to illustrate the existence of a weak Bregman relatively nonexpansive mapping and the difference between a weak Bregman relatively nonexpansive mapping and a Bregman relatively nonexpansive mapping. They also proved the strong convergence of the sequences generated by the constructed algorithms with errors for finding a fixed point of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings under some suitable conditions.

Motivated by the above mentioned results and the on-going research, in this paper, using Bregman function and the shrinking projection method, we introduce new modified Ishikawa iterative algorithms with errors for finding a common element of solutions to the generalized mixed equilibrium problems (1.10) and fixed points to a countable family of Bregman totally quasi-D-asymptotically nonexpansive mappings in Banach spaces. We prove strong convergence theorems for the sequences generated by the proposed algorithm. Furthermore, these algorithms take into account possible computational errors. No assumption $\widehat{F}(T) = F(T)$ is imposed on the mapping T in reflexive Banach space setting. Our results improve and develop many known results in the current literature; see, for example, [9, 11, 43, 44].

2 Main results

We now state and prove the main result of this paper.

Theorem 2.1 *Let E be a reflexive Banach space and $g : E \to R$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E. For each $k = 1, 2, \ldots, m$, let $A_k : C \to E^*$ be a continuous and monotone mapping, $\varphi_k : C \to R$ be a lower semicontinuous and convex functional, let $f_k : C \times C \to R$ be a bifunction satisfying* (C1)-(C4) *and $T_i : E \to \operatorname{int}(\operatorname{dom} g)$, $\forall i \in N$ be an infinite family of closed and uniformly Bregman totally quasi-D-asymptotically nonexpansive mappings with nonnegative real sequences $\{v_n^{(i)}\}$, $\{u_n^{(i)}\}$ and a strictly increasing and continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$. If $\lim_{n\to+\infty} \sup_{i\in N^*}\{v_n^{(i)}\} = 0$ and $\lim_{n\to+\infty} \sup_{i\in N^*}\{u_n^{(i)}\} = 0$. Assume that T_i is uniformly asymptotically regular on E for all $i \ge 1$, i.e., $\lim_{n\to+\infty} \sup_{x\in K} \|T_i^{n+1}x - T_i^n x\| = 0$ holds for any bounded subset K of E and $F = [\bigcap_{i=1}^{+\infty} F(T_i)] \cap [\bigcap_{k=1}^{m} \mathrm{GMEP}(f_k, \varphi_k)] \neq \emptyset$. For all $z, y \in C$, $G_k(z, y) = f_k(z, y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z\rangle$, $T_{r_{k,n}}^{G_k}(x) = \{z \in C : G_k(z, y) + \frac{1}{r_{k,n}}\langle y - z, \nabla g(z) - \nabla g(x)\rangle \ge 0, \forall y \in C\}$. For an initial point $x_1 \in E$, let $C_1^i = C$ for each $i \ge 1$ and $C_1 = \bigcap_{i=1}^{\infty} C_1^i$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n^i = \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(z_n^i)], \\ z_n^i = \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(T_i^n(x_n + e_n^i))], \\ u_n^i = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} y_n^i, \\ C_{n+1}^i = \{z \in C_n : D(z, u_n^i) \le \alpha_n D(z, x_n) + (1-\alpha_n) D(z, z_n^i) \le D(z, x_n) + \zeta_n^i\}, \\ C_{n+1} = \bigcap_{i=1}^{+\infty} C_{n+1}^i, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}}^g (x_1), \end{cases} \tag{2.1}$$

where the sequences $\{\zeta_n^i\}$, $\{e_n^i\}$, $\{r_{k,n}\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(1) $\zeta_n^i = D(x_n, x_n + e_n^i) + \sup_{p\in C}\langle x_n - p, \nabla g(x_n + e_n^i) - \nabla g(x_n)\rangle + v_n^{(i)} \cdot \sup_{p\in C} \zeta[D(p, x_n + e_n^i)] + u_n^{(i)}$, $e_n^i \in E$ *satisfying* $\lim_{n\to+\infty} \sup_{i\in N^*}\{\|e_n^i\|\} = 0$ *for each* $n \ge 1$ *and* $i \ge 1$;

(2) *for each* $k = 1, 2, \dots, m$, $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0, +\infty)$ *satisfy* $\liminf_{n\to+\infty} r_{k,n} > 0$;

(3) $\{\alpha_n\}$, $\{\beta_n\}$ *are real sequences in* $[0,1]$ *such that* $\liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n)\beta_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $\mathrm{Proj}_F^g(x_1)$.

Proof We define a bifunction $G_k : C \times C \to R$ by

$$G_k(x, y) = f_k(x, y) + \varphi_k(y) - \varphi_k(x) + \langle A_k x, y - x\rangle, \quad \forall x, y \in C.$$

Then we prove from Lemma 1.11 that the bifunction G_k satisfies conditions (C1)-(C4) for each $k = 1, 2, \dots, m$. Therefore, the generalized mixed equilibrium problem (1.10) is equivalent to the following equilibrium problem: find $x \in C$ such that $G_k(x, y) \geq 0$, $\forall y \in C$. Hence, $\mathrm{GMEP}(f_k, \varphi_k) = \mathrm{EP}(G_k)$. By taking $\theta_n^k = T_{r_{k,n}}^{G_k} T_{r_{k-1,n}}^{G_{k-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1}$, $k = 1, 2, \dots, m$, and $\theta_n^0 = I$ for all $n \geq 1$, we obtain $u_n = \theta_n^m y_n$.

In view of Lemma 1.2 and Lemma 1.11, we find that F is closed and convex, so that $\mathrm{Proj}_F^g(x_1)$ is well defined for any $x_1 \in E$.

We divide the proof of Theorem 2.1 into six steps:

Step 1. We first show that C_n is closed and convex for each $n \geq 1$.

In fact, from the definition, $C_1 = \bigcap_{i=1}^{\infty} C_1^i = C$ for all $i \geq 1$ is closed and convex. Suppose that C_{n+1}^i is closed and convex for some $n \geq 1$. For any $z \in C_{n+1}^i$, we know that

$$D\big(z, u_n^i\big) \leq \alpha_n D(z, x_n) + (1-\alpha_n) D\big(z, z_n^i\big) \leq D(z, x_n) + \zeta_n^i$$

is equivalent to the following:

$$\big\langle z - u_n^i, \alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g\big(z_n^i\big) - \nabla g\big(u_n^i\big)\big\rangle \leq \alpha_n D\big(u_n^i, x_n\big) + (1-\alpha_n) D\big(u_n^i, z_n^i\big) - g\big(u_n^i\big)$$

and

$$(1-\alpha_n)\big\langle z - x_n, \nabla g(x_n) - \nabla g\big(z_n^i\big)\big\rangle \leq -(1-\alpha_n) D\big(x_n, z_n^i\big) + \zeta_n^i, \quad \forall i \geq 1.$$

Since the left-hand sides of the last two inequalities are affine with respect to z as functions of z, C_{n+1}^i is closed and convex. Hence $C_{n+1} = \bigcap_{i=1}^{+\infty} C_{n+1}^i$ is closed and convex for all $n \geq 1$.

Step 2. Assume that $F \subset C_n$ for all $n \geq 1$. Then the sequence $\{x_n\}$ is bounded.

In fact, by $x_{n+1} = \mathrm{Proj}_{C_{n+1}}^g(x_1)$, it then follows from Lemma 1.8 that

$$D(x_{n+1}, x_0) = D\big(\mathrm{Proj}_{C_{n+1}}^g(x_1), x_0\big) \leq D(p, x_0) - D(p, x_{n+1}) \leq D(p, x_0)$$

for each $p \in F \subset C_n$, $\forall n \geq 1$. Hence, the sequence $\{D(x_{n+1}, x_0)\}$ is bounded, by Lemma 1.6, $\{x_n\}$ is bounded and so are $\{T_i x_n\}$, $\{y_n^i\}$, $\{z_n^i\}$, and $\{u_n^i\}$.

Step 3. Next, we show, by induction, that $F \subset C_n$ for all $n \geq 1$.

In fact, it is obvious that $F \subset C_1 = C$. Suppose that $F \subset C_n$ for some $n \geq 1$. Let $p \in F$, since $T_i : E \to C$ ($\forall i \in N$) is an infinite family of closed and uniformly Bregman totally quasi-D-asymptotically nonexpansive mappings, by the definition of $D(\cdot,\cdot)$ and Remark 1.1, for each $i \geq 1$, we have

$$\begin{aligned} D\big(p, z_n^i\big) &= D\big(p, \nabla g^*\big[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g\big(T_i^n\big(x_n + e_n^i\big)\big)\big]\big) \\ &\leq \beta_n D(p, x_n) + (1-\beta_n) D\big(p, T_i^n\big(x_n + e_n^i\big)\big) \end{aligned}$$

$$
\begin{aligned}
&\le \beta_n D(p,x_n) + (1-\beta_n)\bigl\{D\bigl(p,x_n+e_n^i\bigr) + \nu_n^{(i)}\cdot\zeta\bigl[D\bigl(p,x_n+e_n^i\bigr)\bigr] + u_n^{(i)}\bigr\}\\
&= \beta_n D(p,x_n) + (1-\beta_n)\bigl\{D(p,x_n) + D\bigl(x_n,x_n+e_n^i\bigr)\\
&\quad + \bigl\langle x_n - p, \nabla g\bigl(x_n+e_n^i\bigr) - \nabla g(x_n)\bigr\rangle + \nu_n^{(i)}\cdot\zeta\bigl[D\bigl(p,x_n+e_n^i\bigr)\bigr] + u_n^{(i)}\bigr\}\\
&= D(p,x_n) + (1-\beta_n)\bigl\{D\bigl(x_n,x_n+e_n^i\bigr) + \bigl\langle x_n - p, \nabla g\bigl(x_n+e_n^i\bigr) - \nabla g(x_n)\bigr\rangle\\
&\quad + \nu_n^{(i)}\cdot\zeta\bigl[D\bigl(p,x_n+e_n^i\bigr)\bigr] + u_n^{(i)}\bigr\}\\
&\le D(p,x_n) + \zeta_n^i.
\end{aligned}
\tag{2.2}
$$

Observe that $p \in F$ implies $p \in C$. Thus, by (2.2), Lemma 1.11, and the fact that $T_{r_{k,n}}^{G_k}$ ($k = 1,2,\ldots,m$) is a Bregman quasi-D-nonexpansive mapping, for each $p \in F$, we have

$$
\begin{aligned}
D\bigl(p,u_n^i\bigr) &= D\bigl(p,\theta_n^m y_n^i\bigr)\\
&\le D\bigl(p,y_n^i\bigr)\\
&= D\bigl(p,\nabla g^*\bigl[\alpha_n\nabla g(x_n) + (1-\alpha_n)\nabla g\bigl(z_n^i\bigr)\bigr]\bigr)\\
&\le \alpha_n D(p,x_n) + (1-\alpha_n)D\bigl(p,z_n^i\bigr)\\
&\le \alpha_n D(p,x_n) + (1-\alpha_n)\bigl[D(p,x_n) + \zeta_n^i\bigr]\\
&\le D(p,x_n) + \zeta_n^i.
\end{aligned}
\tag{2.3}
$$

This shows that $p \in C_{n+1}$, which implies that $F \subset C_{n+1}$. Hence $F \subset C_n$ for all $n \ge 1$.

Step 4. Now, we show that $\{x_n\}$ is Cauchy sequence.

In fact, combining $x_{n+1} = \mathrm{Proj}_{C_{n+1}}^g(x_1) \in C_{n+1}$ and Lemma 1.8, we obtain $0 \le D(x_n,x_{n+1}) \le D(x_n,x_1) - D(x_{n+1},x_1)$ for all $n \ge 1$. Thus, the sequence $\{D(x_n,x_1)\}$ is nondecreasing. It follows from the boundedness of $\{D(x_n,x_1)\}$ that the limit of $\{D(x_n,x_1)\}$ exists.

For any positive integer m, it then follows from Lemma 1.8 that

$$
\begin{aligned}
D(x_{n+m},x_{n+1}) &= D\bigl(x_{n+m},\mathrm{Proj}_{C_{n+1}}^g(x_1)\bigr) \le D(x_{n+m},x_1) - D\bigl(\mathrm{Proj}_{C_{n+1}}^g(x_1),x_1\bigr)\\
&= D(x_{n+m},x_1) - D(x_{n+1},x_1),
\end{aligned}
\tag{2.4}
$$

from which it follows from (2.4) that $D(x_{n+m},x_{n+1}) \to 0$ as $m,n \to \infty$. We have from Lemma 1.1 and the boundedness of $\{x_n\}$,

$$
x_{n+m} - x_{n+1} \to 0, \quad m,n \to \infty.
$$

Hence, the sequence $\{x_n\}$ is Cauchy in C. Since E is a Banach space and C is closed convex, there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$. Now, since $D(x_{n+m},x_{n+1}) \to 0$ as $m,n \to \infty$, we have in particular that

$$
\lim_{n\to\infty} D(x_{n+2},x_{n+1}) = 0 \tag{2.5}
$$

and this further implies that

$$
\lim_{n\to\infty} \|x_{n+1} - x_{n+2}\| = 0 \tag{2.6}
$$

from Lemma 1.1.

From $\|x_n - (x_n + e_n^i)\| = \|e_n^i\| \to 0$ (as $n \to +\infty$, $\forall i \geq 1$), Lemma 1.4, and the boundedness of $\{\|\nabla g(x_n + e_n^i)\|\}$, we obtain

$$\begin{aligned} 0 &\leq D(x_n, x_n + e_n^i) = g(x_n) - g(x_n + e_n^i) + \langle e_n^i, \nabla g(x_n + e_n^i)\rangle \\ &\leq |g(x_n) - g(x_n + e_n^i)| + \|e_n^i\| \cdot \|\nabla g(x_n + e_n^i)\| \to 0 \quad \text{as } n \to +\infty, \forall i \geq 1. \end{aligned} \tag{2.7}$$

Since g is uniformly smooth on bounded subsets of E, by Lemma 1.10, we find that $\nabla g(\cdot)$ is uniformly norm-to-norm continuous on any bounded sets and $\|x_n - (x_n + e_n^i)\| = \|e_n^i\| \to 0$ (as $n \to +\infty$, $\forall i \geq 1$), and we obtain

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(x_n + e_n^i)\| = 0, \quad \forall i \geq 1. \tag{2.8}$$

Thus, it follows from (2.7), (2.8), $\lim_{n\to+\infty} \sup_{i\in N^*}\{v_n^{(i)}\} = 0$, and $\lim_{n\to+\infty} \sup_{i\in N^*}\{u_n^{(i)}\} = 0$ that

$$\begin{aligned} 0 &\leq |\zeta_n^i| \\ &\leq D(x_n, x_n + e_n^i) + \left|\sup_{p\in C}\langle x_n - p, \nabla g(x_n + e_n^i) - \nabla g(x_n)\rangle\right| \\ &\quad + \left|v_n^{(i)} \cdot \sup_{p\in C} \zeta\left[D(p, x_n + e_n^i)\right]\right| + |u_n^{(i)}| \\ &\leq D(x_n, x_n + e_n^i) + \left[\sup_{p\in C}\|x_n - p\|\right] \cdot \|\nabla g(x_n + e_n^i) - \nabla g(x_n)\| \\ &\quad + |v_n^{(i)}| \cdot \sup_{p\in C} \zeta\left[D(p, x_n + e_n^i)\right] + |u_n^{(i)}| \to 0 \quad \text{as } n \to +\infty, \forall i \geq 1. \end{aligned} \tag{2.9}$$

By $x_{n+2} = \mathrm{Proj}_{C_{n+2}}^g(x_1) \in C_{n+2} \subset C_{n+1}$ ($\subset C$) and by the definition of C_{n+2}, it follows from (2.5) and (2.9) that

$$0 \leq D(x_{n+2}, u_{n+1}^i) \leq D(x_{n+2}, x_{n+1}) + \zeta_{n+1}^i \to 0, \quad n \to \infty, \forall i \geq 1.$$

From Lemma 1.1, we obtain $\lim_{n\to\infty} \|x_{n+2} - u_{n+1}^i\| = 0$. Therefore

$$\|x_{n+1} - u_{n+1}^i\| \leq \|x_{n+1} - x_{n+2}\| + \|x_{n+2} - u_{n+1}^i\| \to 0. \tag{2.10}$$

It follows from $\lim_{n\to+\infty} \|x_n - p\| = 0$ and (2.10) that

$$u_n^i \to p, \quad n \to \infty, \forall i \geq 1. \tag{2.11}$$

Step 5. Now we prove that $p \in [\bigcap_{i=0}^{+\infty} F(T_i)] \cap [\bigcap_{k=1}^{m} \mathrm{GMEP}(f_k, \varphi_k)]$.

(a) First we prove that $p \in \bigcap_{i=0}^{+\infty} F(T_i)$.

Since g is uniformly smooth on bounded subsets of E, by Lemma 1.10, we find that $\nabla g(\cdot)$ is uniformly norm-to-norm continuous on any bounded sets and from (2.10), we obtain

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(u_n^i)\| = 0, \quad \forall i \geq 1. \tag{2.12}$$

It follows from the boundedness of the sequences $\{x_n\}$ and $D(p, T_i^n(x_n + e_n^i)) \leq D(p, x_n + e_n^i) + \nu_n^{(i)} \cdot \zeta[D(p, x_n + e_n^i)] + u_n^{(i)}$ for each $p \in F$ and $i \geq 1$ that the sequences $\{\nabla g(x_n)\}$ and $\{\nabla g(T_i^n(x_n + e_n^i))\}$ are bounded. Thus there exists $r > 0$ such that $\{\nabla g(x_n)\} \subset B_r(0)$ and $\{\nabla g(T_i^n(x_n + e_n^i))\} \subset B_r(0)$. For each $p \in F$, we have from Lemma 1.7 and Lemma 1.11

$$\begin{aligned}
D(p, u_n^i) &= D(p, \theta_n^m y_n^i) \leq D(p, y_n^i) = D(p, \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(z_n^i)]) \\
&\leq \alpha_n D(p, x_n) + (1-\alpha_n) D(p, z_n^i) \\
&\leq \alpha_n D(p, x_n) + (1-\alpha_n) \cdot [\beta_n D(p, x_n) + (1-\beta_n) D(p, T_i^n(x_n + e_n^i)) \\
&\quad - \beta_n(1-\beta_n)\rho_r^*(\|\nabla g(x_n) - \nabla g(T_i^n(x_n + e_n^i))\|)] \\
&\leq \alpha_n D(p, x_n) + (1-\alpha_n) \cdot \{\beta_n D(p, x_n) + (1-\beta_n)[D(p, x_n + e_n^i) + \nu_n^{(i)} \\
&\quad \cdot \zeta(D(p, x_n + e_n^i)) + u_n^{(i)}] - \beta_n(1-\beta_n)\rho_r^*(\|\nabla g(x_n) - \nabla g(T_i^n(x_n + e_n^i))\|)\} \\
&= \alpha_n D(p, x_n) + (1-\alpha_n) \cdot \{\beta_n D(p, x_n) + (1-\beta_n)\{\nu_n^{(i)} \cdot \zeta[D(p, x_n + e_n^i)]\} \\
&\quad + u_n^{(i)} + D(p, x_n) + D(x_n, x_n + e_n^i) + \langle x_n - p, \nabla g(x_n + e_n^i) - \nabla g(x_n)\rangle\} \\
&\quad - (1-\alpha_n)\beta_n(1-\beta_n)\rho_r^*(\|\nabla g(x_n) - \nabla g(T_i^n(x_n + e_n^i))\|) \\
&\leq \alpha_n D(p, x_n) + (1-\alpha_n) \cdot (D(p, x_n) + \zeta_n^i - \beta_n(1-\beta_n)\rho_r^*(\|\nabla g(x_n) \\
&\quad - \nabla g(T_i^n(x_n + e_n^i))\|)) \\
&\leq \alpha_n D(p, x_n) + (1-\alpha_n) D(p, x_n) + \zeta_n^i - (1-\alpha_n)\beta_n(1-\beta_n)\rho_r^*(\|\nabla g(x_n) \\
&\quad - \nabla g(T_i^n(x_n + e_n^i))\|) \\
&= D(p, x_n) + \zeta_n^i - (1-\alpha_n)\beta_n(1-\beta_n)\rho_r^*(\|\nabla g(x_n) - \nabla g(T_i^n(x_n + e_n^i))\|).
\end{aligned}$$

This implies that

$$\begin{aligned}
0 &\leq (1-\alpha_n)\beta_n(1-\beta_n)\rho_r^*(\|\nabla g(x_n) - \nabla g(T_i^n(x_n + e_n^i))\|) \\
&\leq D(p, x_n) - D(p, u_n^i) + \zeta_n^i.
\end{aligned} \tag{2.13}$$

On the other hand, we have

$$\begin{aligned}
|D(p, x_n) - D(p, u_n^i)| &= |-D(x_n, u_n^i) + \langle x_n - p, \nabla g(u_n^i) - \nabla g(x_n)\rangle| \\
&\leq D(x_n, u_n^i) + \|x_n - p\| \cdot \|\nabla g(u_n^i) - \nabla g(x_n)\|.
\end{aligned}$$

In view of (2.10) and (2.12), we obtain

$$D(p, x_n) - D(p, u_n^i) \to 0, \quad n \to \infty. \tag{2.14}$$

Combining (2.13) and (2.14), $\lim_{n\to+\infty} \zeta_n^i = 0$, and the assumption $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$, we have

$$\rho_r^*(\|\nabla g(x_n) - \nabla g(T_i^n(x_n + e_n^i))\|) \to 0, \quad n \to \infty.$$

It follows from the property of $\rho_r^*(\cdot)$ that

$$\lim_{n\to+\infty}\big\|\nabla g(x_n)-\nabla g\big(T_i^n\big(x_n+e_n^i\big)\big)\big\|=0. \tag{2.15}$$

Since $x_n\to p$ as $n\to\infty$ and $\nabla g(\cdot)$ is uniformly norm-to-norm continuous on any bounded sets, we obtain

$$\big\|\nabla g(x_n)-\nabla g(p)\big\|\to 0 \quad \text{as } n\to 0. \tag{2.16}$$

Note that

$$\big\|\nabla g\big(T_i^n\big(x_n+e_n^i\big)\big)-\nabla g(p)\big\|\le\big\|\nabla g(p)-\nabla g(x_n)\big\|+\big\|\nabla g(x_n)-\nabla g\big(T_i^n\big(x_n+e_n^i\big)\big)\big\|.$$

From (2.15) and (2.16), we see that

$$\lim_{n\to+\infty}\big\|\nabla g\big(T_i^n\big(x_n+e_n^i\big)\big)-\nabla g(p)\big\|=0. \tag{2.17}$$

By Lemma 1.10, note that $\nabla g^*(\cdot)$ is also uniformly norm-to-norm continuous on any bounded sets. It follows from (2.17) that

$$\lim_{n\to+\infty}\big\|T_i^n\big(x_n+e_n^i\big)-p\big\|=0. \tag{2.18}$$

Noting that $\|T_i^{n+1}(x_n+e_n^i)-p\|\le\|T_i^{n+1}(x_n+e_n^i)-T_i^n(x_n+e_n^i)\|+\|T_i^n(x_n+e_n^i)-p\|$, the uniformly asymptotic regularity of T and (2.18), we have $\lim_{n\to+\infty}\|T_i^{n+1}(x_n+e_n^i)-p\|=0$. That is, $T_i(T_i^n(x_n+e_n^i))\to p$ as $n\to\infty$, and it follows from the closeness of T_i that $T_ip=p$, $\forall i\ge 1$, *i.e.* $p\in\bigcap_{i=1}^{+\infty}F(T_i)$.

(b) Now we prove that $p\in\bigcap_{k=1}^m\mathrm{GMEP}(f_k,\varphi_k)=\bigcap_{k=1}^m\mathrm{EP}(G_k)$.

In fact, in view of $u_n^i=\theta_n^m y_n^i$, (2.3), and Lemma 1.11, for each $q\in F(\theta_n^k)$, we have

$$0\le D\big(u_n^i,y_n^i\big)=D\big(\theta_n^m y_n^i,y_n^i\big)\le D\big(p,y_n^i\big)-D\big(p,\theta_n^m y_n^i\big)\le D(p,x_n)-D\big(p,u_n^i\big)+\zeta_n^i.$$

It follows from (2.14) and $\lim_{n\to+\infty}\zeta_n^i=0$ that $D(u_n^i,y_n^i)\to 0$ as $n\to\infty$. Using Lemma 1.1, we see that $\|u_n^i-y_n^i\|\to 0$ as $n\to\infty$. Furthermore, $\|x_n-y_n^i\|\le\|x_n-u_n^i\|+\|u_n^i-y_n^i\|\to 0$ as $n\to\infty$. Since $x_n\to p$, $n\to\infty$ and $\|x_n-y_n^i\|\to 0$, $n\to\infty$, then $y_n^i\to p$, $n\to\infty$. By the fact that θ_n^k, $k=1,2,\ldots,m$ is Bregman relatively nonexpansive and using Lemma 1.11 again, we have

$$0\le D\big(\theta_n^k y_n^i,y_n^i\big)\le D\big(p,y_n^i\big)-D\big(p,\theta_n^k y_n^i\big)\le D(p,x_n)-D\big(p,\theta_n^k y_n^i\big)+\zeta_n^i. \tag{2.19}$$

Observe that

$$\begin{aligned} D\big(p,u_n^i\big)&=D\big(p,\theta_n^m y_n^i\big)=D\big(p,T_{r_{m,n}}^{G_m}T_{r_{m-1,n}}^{G_{m-1}}\cdots T_{r_{2,n}}^{G_2}T_{r_{1,n}}^{G_1}y_n^i\big)\\ &=D\big(p,T_{r_{m,n}}^{G_m}T_{r_{m-1,n}}^{G_{m-1}}\cdots\theta_n^k y_n^i\big)\le D\big(p,\theta_n^k y_n^i\big). \end{aligned} \tag{2.20}$$

Using (2.20) and (2.19), we obtain $0\le D(\theta_n^k y_n^i,y_n^i)\le D(p,x_n)-D(p,u_n^i)+\zeta_n^i\to 0$, $n\to\infty$. Then Lemma 1.1 implies that $\lim_{n\to\infty}\|\theta_n^k y_n^i-y_n^i\|=0$, $k=1,2,\ldots,m$. Now $\|\theta_n^k y_n^i-p\|\le$

$\|\theta_n^k y_n^i - y_n^i\| + \|y_n^i - p\| \to 0$, $n \to \infty$, $k = 1, 2, \ldots, m$. Similarly, $\lim_{n\to+\infty} \|\theta_n^{k-1} y_n^i - p\| = 0$, $k = 1, 2, \ldots, m$. This further implies that

$$\lim_{n\to+\infty} \left\|\theta_n^{k-1} y_n^i - \theta_n^k y_n^i\right\| = 0. \tag{2.21}$$

Also, since $\nabla g(\cdot)$ is uniformly norm-to-norm continuous on any bounded sets and using (2.21), we obtain $\lim_{n\to+\infty} \|\nabla g(\theta_n^k y_n^i) - \nabla g(\theta_n^{k-1} y_n^i)\| = 0$. From the assumption $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0, +\infty)$ satisfying $\liminf_{n\to+\infty} r_{k,n} > 0$ for each $k = 1, 2, \ldots, m$, we see that

$$\lim_{n\to\infty} \frac{\|\nabla g(\theta_n^k y_n^i) - \nabla g(\theta_n^{k-1} y_n^i)\|}{r_{k,n}} = 0. \tag{2.22}$$

By Lemma 1.11, we have, for each $k = 1, 2, \ldots, m$, $G_k(\theta_n^k y_n^i, y) + \frac{1}{r_{k,n}}\langle y - \theta_n^k y_n^i, \nabla g(\theta_n^k y_n^i) - \nabla g(\theta_n^{k-1} y_n^i)\rangle \geq 0$, $\forall y \in C$. Furthermore, replacing n by n_j in the last inequality and using condition (C2), we obtain

$$\begin{aligned}
&\left\|y - \theta_{n_j}^k y_{n_j}^i\right\| \cdot \frac{\|\nabla g(\theta_{n_j}^k y_{n_j}^i) - \nabla g(\theta_{n_j}^{k-1} y_{n_j}^i)\|}{r_{k,n_j}} \\
&\quad\geq \frac{1}{r_{k,n_j}}\left\langle y - \theta_{n_j}^k y_{n_j}^i, \nabla g\left(\theta_{n_j}^k y_{n_j}^i\right) - \nabla g\left(\theta_{n_j}^{k-1} y_{n_j}^i\right)\right\rangle \\
&\quad\geq -G_k\left(\theta_{n_j}^k y_{n_j}^i, y\right) \geq G_k\left(y, \theta_{n_j}^k y_{n_j}^i\right), \quad \forall y \in C.
\end{aligned}$$

By taking the limit as $j \to +\infty$ in the above inequality, for each $k = 1, 2, \ldots, m$, we have from the condition (C4), (2.22), and $\theta_{n_j}^k y_{n_j}^i \to p$ that $G_k(y, p) \leq 0$, $\forall y \in C$.

For $0 < t \leq 1$ and $y \in C$, define $y_t = ty + (1-t)p$. It follows from $y, p \in C$ that $y_t \in C$, which yields $G_k(y_t, p) \leq 0$. It follows from the conditions (C1) and (C4) that

$$0 = G_k(y_t, y_t) \leq tG_k(y_t, y) + (1-t)G_k(y_t, p) \leq tG_k(y_t, y),$$

that is,

$$G_k(y_t, y) \geq 0.$$

Letting $t \to 0^+$, from the condition (C3), we obtain $G_k(p, y) \geq 0$, $\forall y \in C$. This implies that $p \in \mathrm{EP}(G_k)$, $k = 1, 2, \ldots, m$, *i.e.*, $p \in \bigcap_{k=1}^m \mathrm{GMEP}(f_k, \varphi_k) = \bigcap_{k=1}^m \mathrm{EP}(G_k)$. Thus we have $p \in F$.

Step 6. Finally, we prove that $p = \mathrm{Proj}_F^g(x_1)$.

From Lemma 1.8 and the definition of $x_{n+1} = \mathrm{Proj}_{C_{n+1}}^g(x_1)$, we see that $\langle x_{n+1} - z, \nabla g(x_1) - \nabla g(x_{n+1})\rangle \geq 0$, $\forall z \in C_{n+1}$. Since $F \subset C_n$ for each $n \geq 1$, we have

$$\left\langle x_{n+1} - w, \nabla g(x_1) - \nabla g(x_{n+1})\right\rangle \geq 0, \quad \forall w \in F.$$

Let $n \to +\infty$ in the last inequality, we see that $\langle p - w, \nabla g(x_1) - \nabla g(p)\rangle \geq 0$, $\forall w \in F$. In view of Lemma 1.8, we can obtain $p = \mathrm{Proj}_F^g(x_1)$. This completes the proof of Theorem 2.1. □

Remark 2.1 (1) If we suppose that T_i is uniformly L_i-Lipschitz continuous on E for each $i \in N^+$, then the assumption that T_i is closed and uniformly asymptotically regular on E can be removed in Theorem 2.1.

(2) If we set $\alpha_n = \beta_n = 0$, $u_n^i = y_n^i$, and $T_i = \mathrm{Res}^g_{\lambda_n^i}$ $(i = 1, 2, \ldots, m)$ in (2.1), then (2.1) can be rewritten as (1.18), hence, Theorem 2.1 improves and generalizes Theorem RS [9].

(3) For the mappings, Theorem 2.1 extends the mapping in Theorem RS [9] from a finite family of relatively nonexpansive mapping to a countable family of Bregman totally quasi-D-asymptotically nonexpansive mappings. Theorem 2.1 also removes the assumption $\widehat{F}(T) = F(T)$ on the mapping T.

Setting $e_n^i \equiv 0$ for each $i \geq 1$ and $\forall n \geq 1$ in Theorem 2.1, we have the following.

Corollary 2.1 *Let E be a reflexive Banach space and $g : E \to R$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E. For each $k = 1, 2, \ldots, m$, let $A_k : C \to E^*$ be a continuous and monotone mapping, $\varphi_k : C \to R$ be a lower semicontinuous and convex functional, let $f_k : C \times C \to R$ be a bifunction satisfying* (C1)-(C4) *and $T_i : C \to C$, $\forall i \in N$ be an infinite family of closed and uniformly Bregman totally quasi-D-asymptotically nonexpansive mappings with nonnegative real sequences $\{\nu_n^{(i)}\}$, $\{u_n^{(i)}\}$ and a strictly increasing and continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$. If $\lim_{n\to+\infty} \sup_{i\in N^*} \{\nu_n^{(i)}\} = 0$ and $\lim_{n\to+\infty} \sup_{i\in N^*} \{u_n^{(i)}\} = 0$. Assume that T_i is uniformly asymptotically regular on C for all $i \geq 1$, i.e., $\lim_{n\to+\infty} \sup_{x\in K} \|T_i^{n+1}x - T_i^n x\| = 0$ holds for any bounded subset K of C and $F = [\bigcap_{i=1}^{+\infty} F(T_i)] \cap [\bigcap_{k=1}^{m} \mathrm{GMEP}(f_k, \varphi_k)] \neq \emptyset$. For all $z, y \in C$, $G_k(z, y) = f_k(z, y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z\rangle$, $T^{G_k}_{r_{k,n}}(x) = \{z \in C : G_k(z, y) + \frac{1}{r_{k,n}} \langle y - z, \nabla g(z) - \nabla g(x)\rangle \geq 0, \forall y \in C\}$. For an initial point $x_1 \in E$, let $C_1^i = C$ for each $i \geq 1$ and $C_1 = \bigcap_{i=1}^{\infty} C_1^i$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n^i = \nabla g^*[\alpha_n \nabla g(x_n) + (1 - \alpha_n)\nabla g(z_n^i)], \\ z_n^i = \nabla g^*[\beta_n \nabla g(x_n) + (1 - \beta_n)\nabla g(T_i^n x_n)], \\ u_n^i = T^{G_m}_{r_{m,n}} T^{G_{m-1}}_{r_{m-1,n}} \cdots T^{G_2}_{r_{2,n}} T^{G_1}_{r_{1,n}} y_n^i, \\ C_{n+1}^i = \{z \in C_n : D(z, u_n^i) \leq \alpha_n D(z, x_n) + (1 - \alpha_n) D(z, z_n^i) \leq D(z, x_n) + \zeta_n^i\}, \\ C_{n+1} = \bigcap_{i=1}^{+\infty} C_{n+1}^i, \\ x_{n+1} = \mathrm{Proj}^g_{C_{n+1}}(x_1), \end{cases}$$

where the sequences $\{\zeta_n^i\}$, $\{r_{k,n}\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(1) $\zeta_n^i = \nu_n^{(i)} \cdot \sup_{p\in C} \zeta[D(p, x_n)] + u_n^{(i)}$ *for each $n \geq 1$ and $i \geq 1$.*

(2) *For each $k = 1, 2, \ldots, m$, $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0, +\infty)$ satisfying* $\liminf_{n\to+\infty} r_{k,n} > 0$.

(3) $\{\alpha_n\}$, $\{\beta_n\}$ *are real sequences in* $[0, 1]$ *such that* $\liminf_{n\to\infty}(1 - \alpha_n)(1 - \beta_n)\beta_n > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}^g_F(x_1)$.

Remark 2.2 Corollary 2.1 improves Theorem 3.1 in [43], in the following aspects:

(1) For the structure of Banach spaces, we extend the normalized duality mapping to a more general case, that is, a convex, continuous, and strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets.

(2) For the mappings, we extend the mapping from two quasi-nonexpansive mappings to a countable family of Bregman totally quasi-D-asymptotically nonexpansive mappings.

(3) For generalized mixed equilibrium problems, we extend the problems from one to a finite family.

Setting $\zeta(t) = t$, $v_n^{(i)} = k_n^i - 1$, $\lim_{n\to+\infty} k_n^i = 1$, $u_n^{(i)} = e_n^i \equiv 0$ for each $i \geq 1$ in Theorem 2.1, we have the following.

Corollary 2.2 *Let E be a reflexive Banach space and $g : E \to R$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E. For each $k = 1,2,\dots,m$, let $A_k : C \to E^*$ be a continuous and monotone mapping, $\varphi_k : C \to R$ be a lower semicontinuous and convex functional, let $f_k : C \times C \to R$ be a bifunction satisfying* (C1)-(C4) *and $T_i : C \to C$, $\forall i \in N$ be an infinite family of closed and Bregman quasi-D-asymptotically nonexpansive mappings with nonnegative real sequences $\{k_n^i\}$. If $\lim_{n\to+\infty} \sup_{i\in N^*}\{k_n^i\} = 1$. Assume that T_i is uniformly asymptotically regular on C for all $i \geq 1$, i.e., $\lim_{n\to+\infty} \sup_{x\in K} \|T_i^{n+1}x - T_i^n x\| = 0$ holds for any bounded subset K of C and $F = [\bigcap_{i=1}^{+\infty} F(T_i)] \cap [\bigcap_{k=1}^m \mathrm{GMEP}(f_k, \varphi_k)] \neq \emptyset$. For all $z, y \in C$, $G_k(z,y) = f_k(z,y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z\rangle$, $T_{r_{k,n}}^{G_k}(x) = \{z \in C : G_k(z,y) + \frac{1}{r_{k,n}}\langle y - z, \nabla g(z) - \nabla g(x)\rangle \geq 0, \forall y \in C\}$. For an initial point $x_1 \in E$, let $C_1^i = C$ for each $i \geq 1$ and $C_1 = \bigcap_{i=1}^\infty C_1^i$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n^i = \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(z_n^i)], \\ z_n^i = \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(T_i^n x_n)], \\ u_n^i = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} y_n^i, \\ C_{n+1}^i = \{z \in C_n : D(z, u_n^i) \leq \alpha_n D(z, x_n) + (1-\alpha_n) D(z, z_n^i) \leq D(z, x_n) + \zeta_n^i\}, \\ C_{n+1} = \bigcap_{i=1}^{+\infty} C_{n+1}^i, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}}^g(x_1), \end{cases}$$

where the sequences $\{\zeta_n^i\}$, $\{r_{k,n}\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(1) $\zeta_n^i = (k_n^i - 1) \cdot \sup_{p\in C} D(p, x_n)$.

(2) *For each $k = 1,2,\dots,m$, $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0,+\infty)$ satisfying $\liminf_{n\to+\infty} r_{k,n} > 0$.*

(3) *$\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1]$ such that $\liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n)\beta_n > 0$.*

Then the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_F^g(x_1)$.

Setting $v_n^{(i)} = u_n^{(i)} = e_n^i \equiv 0$ for each $i \geq 1$ in Theorem 2.1, we have Corollary 2.3.

Corollary 2.3 *Let E be a reflexive Banach space and $g : E \to R$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E. For each $k = 1,2,\dots,m$, let $A_k : C \to E^*$ be a continuous and monotone mapping, $\varphi_k : C \to R$ be a lower semicontinuous and convex functional, let $f_k : C \times C \to R$ be a bifunction satisfying* (C1)-(C4) *and $T_i : C \to C$, $\forall i \in N$ be an infinite family of closed and Bregman quasi-D-nonexpansive mappings. Assume that T_i is uniformly asymptotically regular on C for all $i \geq 1$, i.e., $\lim_{n\to+\infty} \sup_{x\in K} \|T_i^{n+1}x - T_i^n x\| = 0$ holds for any bounded subset K of C and $F = [\bigcap_{i=1}^{+\infty} F(T_i)] \cap [\bigcap_{k=1}^m \mathrm{GMEP}(f_k, \varphi_k)] \neq \emptyset$. For all $z, y \in C$, $G_k(z,y) = f_k(z,y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z\rangle$, $T_{r_{k,n}}^{G_k}(x) = \{z \in C : G_k(z,y) + \frac{1}{r_{k,n}}\langle y - z, \nabla g(z) - \nabla g(x)\rangle \geq 0, \forall y \in C\}$. For an initial point $x_1 \in E$, let $C_1^i = C$ for each $i \geq 1$ and $C_1 = \bigcap_{i=1}^\infty C_1^i$ and define the sequence*

$\{x_n\}$ by

$$\begin{cases} y_n^i = \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(z_n^i)], \\ z_n^i = \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(T_i^n x_n)], \\ u_n^i = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} y_n^i, \\ C_{n+1}^i = \{z \in C_n : D(z,u_n^i) \le \alpha_n D(z,x_n) + (1-\alpha_n) D(z,z_n^i) \le D(z,x_n)\}, \\ C_{n+1} = \bigcap_{i=1}^{+\infty} C_{n+1}^i, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}}^g (x_1). \end{cases}$$

For each $k = 1,2,\ldots,m$, $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0,+\infty)$ satisfying $\liminf_{n\to+\infty} r_{k,n} > 0$. $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1]$ such that $\liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n)\beta_n > 0$. Then the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_F^g(x_1)$.

Setting $i = 1$, $T_i = T$, $e_n^i = e_n$, and $g(x) = \|x\|^2$ in Theorem 2.1, we have Corollary 2.4.

Corollary 2.4 *Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E. For each $k = 1,2,\ldots,m$, let $A_k : C \to E^*$ be a continuous and monotone mapping, $\varphi_k : C \to R$ be a lower semicontinuous and convex functional, let $f_k : C \times C \to R$ be a bifunction satisfying (C1)-(C4) and $T : E \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{\nu_n\}$, $\{u_n\}$ and a strictly increasing and continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$. If $\nu_n, u_n \to 0$ (as $n \to +\infty$). Assume that T is uniformly asymptotically regular on E, i.e., $\lim_{n\to+\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0$ holds for any bounded subset K of E and $F = F(T) \cap [\bigcap_{k=1}^m \mathrm{GMEP}(f_k,\varphi_k)] \neq \emptyset$. For all $z, y \in C$, $G_k(z,y) = f_k(z,y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z\rangle$, $T_{r_{k,n}}^{G_k}(x) = \{z \in C : G_k(z,y) + \frac{2}{r_{k,n}}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in C\}$. For an initial point $x_1 \in E$, let $C_1 = C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}[\alpha_n J x_n + (1-\alpha_n) J z_n], \\ z_n = J^{-1}[\beta_n J x_n + (1-\beta_n) J(T^n(x_n + e_n))], \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z,u_n) \le \alpha_n \phi(z,x_n) + (1-\alpha_n)\phi(z,z_n) \le \phi(z,x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \end{cases}$$

where the sequences $\{\zeta_n^i\}$, $\{r_{k,n}\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(1) *$\zeta_n = \phi(x_n, x_n + e_n) + \sup_{p\in C}\langle x_n - p, J(x_n + e_n) - Jx_n\rangle + \nu_n \cdot \sup_{p\in C} \zeta[\phi(p, x_n + e_n)] + u_n$, $e_n \in E$ satisfying $\lim_{n\to+\infty} \|e_n\| = 0$ for each $n \ge 1$.*

(2) *For each $k = 1,2,\ldots,m$, $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0,+\infty)$ satisfying $\liminf_{n\to+\infty} r_{k,n} > 0$.*

(3) *$\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1]$ such that $\liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n)\beta_n > 0$.*

Then the sequence $\{x_n\}$ converges strongly to $\Pi_C(x_1)$.

Letting E be a Hilbert space in Theorem 2.1, we have Corollary 2.5.

Corollary 2.5 *Let C be a nonempty, closed, and convex subset of real Hilbert space E. For each $k = 1,2,\ldots,m$, let $A_k : C \to E^*$ be a continuous and monotone mapping, $\varphi_k : C \to R$ be a lower semicontinuous and convex functional, let $f_k : C \times C \to R$ be a bifunction satisfying (C1)-(C4) and $T_i : E \to C$, $\forall i \in N$ be an infinite family of closed and totally quasi-asymptotically nonexpansive mappings with nonnegative real sequences*

$\{v_n^{(i)}\}$, $\{u_n^{(i)}\}$ *and a strictly increasing and continuous function* $\zeta : R^+ \to R^+$ *with* $\zeta(0) = 0$. *If* $\lim_{n\to+\infty}\sup_{i\in N^*}\{v_n^{(i)}\} = 0$ *and* $\lim_{n\to+\infty}\sup_{i\in N^*}\{u_n^{(i)}\} = 0$. *Assume that* T_i *is uniformly asymptotically regular on* E *for all* $i \ge 1$, *i.e.*, $\lim_{n\to+\infty}\sup_{x\in K}\|T_i^{n+1}x - T_i^n x\| = 0$ *holds for any bounded subset* K *of* E *and* $F = [\bigcap_{i=1}^{+\infty}F(T_i)] \cap [\bigcap_{k=1}^{m}\mathrm{GMEP}(f_k,\varphi_k)] \ne \emptyset$. *For all* $z, y \in C$, $G_k(z,y) = f_k(z,y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z\rangle$, $T_{r_{k,n}}^{G_k}(x) = \{z \in C : G_k(z,y) + \frac{1}{r_{k,n}}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}$. *For an initial point* $x_1 \in E$, *let* $C_1^i = C$ *for each* $i \ge 1$ *and* $C_1 = \bigcap_{i=1}^{\infty} C_1^i$ *and define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n^i = \alpha_n x_n + (1-\alpha_n)z_n^i, \\ z_n^i = \beta_n x_n + (1-\beta_n)T_i^n(x_n + e_n^i), \\ u_n^i = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \cdots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} y_n^i, \\ C_{n+1}^i = \{z \in C_n : \|z - u_n^i\|^2 \le \alpha_n\|z - x_n\|^2 + (1-\alpha_n)\|z - z_n^i\|^2 \le \|z - x_n\|^2 + \zeta_n^i\}, \\ C_{n+1} = \bigcap_{i=1}^{+\infty} C_{n+1}^i, \\ x_{n+1} = P_{C_{n+1}}(x_1), \end{cases}$$

where the sequences $\{\zeta_n^i\}$, $\{r_{k,n}\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy the following conditions*:

(1) $\zeta_n^i = \|e_n^i\|^2 + \sup_{p\in C}\langle x_n - p, e_n^i\rangle + v_n^{(i)} \cdot \sup_{p\in C}\zeta[\|p - (x_n + e_n^i)\|^2] + u_n^{(i)}$, $e_n^i \in E$ *satisfying* $\lim_{n\to+\infty}\sup_{i\in N^*}\{\|e_n^i\|\} = 0$ *for each* $n \ge 1$ *and* $i \ge 1$.

(2) *For each* $k = 1, 2, \ldots, m$, $\{r_{k,n}\}_{n=1}^{+\infty} \subset (0, +\infty)$ *satisfying* $\liminf_{n\to+\infty} r_{k,n} > 0$.

(3) $\{\alpha_n\}$, $\{\beta_n\}$ *are real sequences in* $[0,1]$ *such that* $\liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n)\beta_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $P_F(x_1)$.

Remark 2.3 Corollary 2.5 improves Theorem 2.1 of Martinez-Yanes and Xu [44] in the following aspects:

(1) From a nonexpansive mapping to a countable family of Bregman totally quasi-D-asymptotically nonexpansive mappings.

(2) Our algorithms take into account computational errors term.

(3) Considering the generalized mixed equilibrium problems from zero to a finite family.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1] Department of Mathematics, Shaoxing University, Zhejiang, 312000, China. [2] Center for General Education, China Medical University, Taichung, 40402, Taiwan. [3] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

Acknowledgements
The research of the first author was partially supported by the National Natural Science Foundation of China (Grant No. 10971194) and Scientific Research Project of Educational Commission of Zhejiang Province of China (Grant No. Y20112300). The second author was partially supported by the Grant MOST 103-2923-E-037-001-MY3.

Received: 10 August 2014 Accepted: 10 February 2015 Published online: 28 February 2015

References

1. Bauschke, HH, Borwein, JM, Combettes, PL: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. Commun. Contemp. Math. **3**, 615-647 (2001)
2. Bonnans, JF, Shapiro, A: Perturbation Analysis of Optimization Problem. Springer, New York (2000)
3. Kassay, G, Reich, S, Sabach, S: Iterative methods for solving systems of variational inequalities in reflexive Banach spaces. SIAM J. Optim. **21**, 1319-1344 (2011)

4. Agarwal, RP, Chen, JW, Cho, YJ: Strong convergence theorems for equilibrium problems and weak Bregman relatively nonexpansive mappings in Banach spaces. J. Inequal. Appl. **2013**, 119 (2013). doi:10.1186/1029-242X-2013-119
5. Pang, CT, Naraghirad, E: Approximating common fixed points of Bregman weakly relatively nonexpansive mappings in Banach spaces. J. Funct. Spaces **2014**, Article ID 743279 (2014)
6. Alber, Y: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, A (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15-50. Dekker, New York (1996)
7. Naraghirad, E, Yao, JC: Bregman weak relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2013**, 141 (2013)
8. Zhu, J, Chang, SS: Halpern-Mann's iterations for Bregman strongly nonexpansive mappings in reflexive Banach spaces with applications. J. Inequal. Appl. **2013**, 146 (2013)
9. Reich, S, Sabach, S: Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numer. Funct. Anal. Optim. **31**, 22-44 (2010)
10. Chang, SS, Wang, L, Wan, XR, Chan, CK: Strong convergence theorems for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. Appl. Math. Comput. **228**, 38-48 (2014)
11. Qin, X, Su, Y: Strong convergence theorems for relatively nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 1958-1965 (2007)
12. Plubtieng, S, Ungchittrakool, K: Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space. J. Approx. Theory **149**, 103-115 (2007)
13. Qin, XL, Cho, YJ, Kang, SM, Zhou, HY: Convergence of a hybrid projection algorithm in Banach spaces. Acta Appl. Math. **108**, 299-313 (2009)
14. Wangkeeree, R, Wangkeeree, R: The shrinking projection method for solving variational inequality problems and fixed point problems in Banach spaces. Abstr. Appl. Anal. **2009**, Article ID 624798 (2009)
15. Alber, Y, Chidume, CE, Zegeye, H: Approximating fixed points of total asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2006**, Article ID 10673 (2006)
16. Yang, L, Zhao, F, Kim, JK: Hybrid projection method for generalized mixed equilibrium problem and fixed point problem of infinite family of asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces. Appl. Math. Comput. **218**, 6072-6082 (2012)
17. Bauschke, HH, Combettes, PL: Construction of best Bregman approximations in reflexive Banach spaces. Proc. Am. Math. Soc. **131**(12), 3757-3766 (2003)
18. Alber, Y: The Young-Fenchel transformation and some new characteristics of Banach spaces. In: Jarosz, K (ed.) Functional Spaces. Contemporary Mathematics, vol. 435, pp. 1-19. Am. Math. Soc., Providence (2007)
19. Bauschke, HH, Borwein, JM, Combettes, PL: Bregman monotone optimization algorithms. SIAM J. Control Optim. **42**, 596-636 (2003)
20. Borwein, MJ, Reich, S, Sabach, S: A characterization of Bregman firmly nonexpansive operators using a new monotonicity concept. J. Nonlinear Convex Anal. **12**(1), 161-184 (2011)
21. Hussain, N, Naraghirad, E, Alotaibi, A: Existence of common fixed points using Bregman nonexpansive retracts and Bregman functions in Banach spaces. Fixed Point Theory Appl. **2013**, 113 (2013)
22. Reich, S, Sabach, S: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. **73**, 122-135 (2010)
23. Martin-Marquez, V, Reich, S, Sabach, S: Right Bregman nonexpansive operators in Banach spaces. Nonlinear Anal. **75**, 5448-5465 (2012)
24. Naraghirad, E, Takahashi, W, Yao, JC: Generalized retraction and fixed point theorems using Bregman functions in Banach spaces. J. Nonlinear Convex Anal. **13**(1), 141-156 (2012)
25. Borwein, MJ, Reich, S, Sabach, S: A characterization of Bregman firmly nonexpansive operators using a new monotonicity concept. J. Nonlinear Convex Anal. **12**(1), 161-184 (2011)
26. Reich, S, Sabach, S: A projection method for solving nonlinear problems in reflexive Banach spaces. J. Fixed Point Theory Appl. **9**, 101-116 (2011)
27. Reich, S, Sabach, S: Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces. In: Fixed-Point Algorithms for Inverse Problems in Science and Engineering. Springer Optimization and Its Applications, vol. 49, pp. 301-316. Springer, New York (2011)
28. Reich, S, Sabach, S: Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. Nonlinear Anal. **73**, 122-135 (2010)
29. Sabach, S: Products of finitely many resolvents of maximal monotone mappings in reflexive Banach spaces. SIAM J. Optim. **21**, 1289-1308 (2011)
30. Chang, SS, Chan, CK, Lee, HWJ: Modified block iterative algorithm for quasi-ϕ-asymptotically nonexpansive mappings and equilibrium problem in Banach spaces. Appl. Math. Comput. **217**(18), 7520-7530 (2011)
31. Ofoedu, EU, Malonza, DM: Hybrid approximation of solutions of nonlinear operator equations and applications to equation of Hammerstein-type. Appl. Math. Comput. **217**, 6019-6030 (2011)
32. Butnariu, D, Resmerita, E: Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal. **2006**, Article ID 84919 (2006)
33. Chen, JW, Wan, Z, Yuan, L: Approximation of fixed points of weak Bregman relatively nonexpansive mappings in Banach spaces. Int. J. Math. Math. Sci. **2011**, Article ID 420192 (2011)
34. Bregman, LM: The relaxation method for finding common points of convex sets and its application to the solution of problems in convex programming. USSR Comput. Math. Math. Phys. **7**, 200-217 (1967)
35. Butnariu, D, Iusem, AN: Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization. Kluwer Academic, Dordrecht (2000)
36. Reich, S, Sabach, S: A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. J. Nonlinear Convex Anal. **10**, 471-485 (2009)
37. Alber, Y, Butnariu, D: Convergence of Bregman-projection methods for solving consistent convex feasibility problems in reflexive Banach spaces. J. Optim. Theory Appl. **92**(1), 33-61 (1997)

38. Saewan, S, Kumam, P, Wattanawitoon, K: Convergence theorem based on a new hybrid projection method for finding a common solution of generalized equilibrium and variational inequality problems in Banach spaces. Abstr. Appl. Anal. **2010**, Article ID 734126 (2010)
39. Zhu, JH, Chang, SS, Liu, M: Strong convergence theorem for Bregman strongly nonexpansive mappings and equilibrium problems in reflexive Banach spaces. J. Appl. Math. **2013**, Article ID 962691 (2013)
40. Saewan, S, Kumam, P: Modified hybrid block iterative algorithm for convex feasibility problems and generalized equilibrium problems for uniformly quasi-ϕ-asymptotically nonexpansive mappings. Abstr. Appl. Anal. **2010**, Article ID 357120 (2010)
41. Saewan, S, Kumam, P: A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems. Abstr. Appl. Anal. **2010**, Article ID 123027 (2010)
42. Wattanawitoon, K, Kumam, P: Generalized mixed equilibrium problems for maximal monotone operators and two relatively quasi-nonexpansive mappings. Thai J. Math. **9**(1), 165-189 (2011)
43. Petrot, N, Wattanawitoon, K, Kumam, P: A hybrid projection method for generalized mixed equilibrium problems and fixed point problems in Banach spaces. Nonlinear Anal. Hybrid Syst. **4**(4), 631-643 (2010)
44. Martinez-Yanes, C, Xu, HK: Strong convergence of the CQ method for fixed point iteration processes. Nonlinear Anal. **64**, 2400-2411 (2006)
45. Cai, G, Hu, CS: On the strong convergence of the implicit iterative processes for a finite family of relatively weak quasinonexpansive mappings. Appl. Math. Lett. **23**(1), 73-78 (2010)
46. Solodov, MV, Svaiter, BF: A hybrid projection-proximal point algorithm. J. Convex Anal. **6**, 59-70 (1999)
47. Takahashi, W, Nakajo, K: Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. J. Math. Anal. Appl. **279**, 372-379 (2003)
48. Matsushita, S, Takahashi, W: A strong convergence theorem for relatively nonexpansive mappings in a Banach space. J. Approx. Theory **134**, 257-266 (2005)