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On the existence of solutions for a multi-singular pointwise defined fractional system

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Abstract

One of best ways for increasing our abilities in exact modeling of natural phenomena is working with a singular version of different fractional differential equations. As is well known, multi-singular equations are a modern version of singular equations. In this paper, we investigate the existence of solutions for a multi-singular fractional differential system. We consider some particular boundary value conditions on the system. By using the α-ψ-contractions and locating some control conditions, we prove that the system via infinite singular points has solutions. Finally, we provide an example to illustrate our main result.

MSC: Primary 34A08; secondary 37C25; 46F30

Keywords: Boundary value condition; Multi-singular system; Solution of a pointwise defined equation; The fractional Caputo derivative

1 Introduction

The fractional derivatives have an long history. It is natural that many phenomena could be modeled by using singular fractional integro-differential equations. Due to the emergence of fractional differential equations in some mathematical models of distinct phenomena in the world, fractional calculus is perfectly appealing ([1–12]) for some real modelings ([13–15]). On the other side, much work is conducted in the field of fractional differential equations among which some have a singular point to control these sorts of points ([16–19]) and we have nonlinear delay-fractional differential equations ([20–23]).

In 2011, Feng et al. studied the existence of a solution for the singular system

$$\begin{cases} D^{\alpha}u(t) + f(t, v(t)) = 0, \\ D^{\beta}v(t) + g(t, u(t)) = 0, \end{cases}$$

with boundary conditions $u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = 0$, where $2 < \alpha, \beta \leq 3$, $f, g : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous, $\lim_{t\to 0^+} f(t, \cdot) = +\infty$ and $\lim_{t\to 0^+} g(t, \cdot) = +\infty$ ([24]). In 2014 Jleli et al. proved the existence of a positive solution for the singular fractional boundary value problem $D^{\alpha}u(t) + f(t, u(t)) = 0$ with $u(0) = u'(0) = 0$ and $u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i)$, where $0 < t < 1$, $2 < \alpha \leq 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t,x)$ is singular at $t = 0$ and D^α is the Caputo derivative ([25]). Later, some more systems of fractional differential equations and inclusions were studied ([26–28]). In 2017 Shabibi et al. reviewed the singular fractional integro-differential system

$$\begin{cases} D^{\alpha_1} u_1 + f_1(t, u_1, \dots, u_m, D^{\mu_1} u_1, \dots, D^{\mu_m} u_m) \\ \quad + g_1(t, u_1, \dots, u_m, D^{\mu_1} u_1, \dots, D^{\mu_m} u_m) = 0, \\ \vdots \\ D^{\alpha_m} u_m + f_m(t, u_1, \dots, u_m, D^{\mu_1} u_1, \dots, D^{\mu_m} u_m) \\ \quad + g_m(t, u_1, \dots, u_m, D^{\mu_1} u_1, \dots, D^{\mu_m} u_m) = 0, \end{cases}$$

with boundary conditions $u_i(0) = 0$, $u_i'(1) = 0$ and $\frac{d^k}{dt^k}[u_i(t)]_{t=0} = 0$ for $1 \leq i \leq m$ and $2 \leq k \leq n-1$, where $\alpha_i \geq 2$, $[\alpha_i] = n-1$, $0 < \mu_i < 1$, D is the Caputo fractional derivative, f_i is a Caratheodory function, g_i satisfies the Lipschitz condition and $f_i(t, x_1, \dots, x_{2m})$ is singular at $t = 0$ of for all $1 \leq i \leq m$ ([29]). One of our aims is to generalize this system in a certain sense. In 2020, Talaee et al. studied the existence of solutions for the pointwise defined differential equation $D^\alpha x(t) = f(t, x(t), x'(t), D^\beta x(t), \int_0^t g(\xi)x(\xi)\,d\xi)$ with boundary conditions $x(\mu) = \int_0^1 h(z)x(z)\,dz$ and $x(0) = x^{(j)}(0) = 0$, for $2 \leq j \leq n-1$, where $\alpha \geq 2$, $n = [\alpha] + 1$, $\mu, \beta \in (0,1)$, $g, h : [0,1] \to \mathbb{R}$ are mappings such that $g, h \in L^1[0,1]$ and $f \in L^1$ is singular at some points of $[0,1]$ ([30]).

By using main idea of the literature, we investigate the existence of solutions for the nonlinear fractional differential pointwise defined system

$$\begin{cases} D^{\alpha_1} x_1(t) = f_1(t, x_1(t), x_1'(t), D^{\beta_1} x_1(t), I^{p_1} x_1(t), \\ \qquad \dots, x_m(t), x_m'(t), D^{\beta_m} x_m(t), I^{p_m} x_m(t)), \\ \vdots \\ D^{\alpha_m} x_m(t) = f_m(t, x_1(t), x_1'(t), D^{\beta_1} x_1(t), I^{p_1} x_1(t), \\ \qquad \dots, x_m(t), x_m'(t), D^{\beta_m} x_m(t), I^{p_m} x_m(t)), \end{cases} \quad t \in [0,1], \tag{1}$$

with boundary value conditions $x_k^{(j)}(0) = 0$ for $2 \leq j \leq n_k - 1$ and $k = 1, \dots, m$,

$$x_k(\theta_k) = \sum_{i=1}^{n_0} \lambda_{i,k} D^{\mu_{i,k}} x_k(\gamma_{i,k})$$

and $x_k'(0) = x_k(\eta_k)$ for all $k = 1, 2, \dots, m$, where $\lambda_{i,k} \geq 0$, $\beta_k, \gamma_{i,k}, \mu_{i,k}, \theta_k, \eta_k \in (0,1)$, $p_k > 0$, $m, n_0 \in \mathbb{N}$, $k = 1, 2, \dots, m$, $i = 1, 2, \dots, n_0$, D^{α_k} is the Caputo fractional derivative of order $\alpha_k \geq 2$, $n_k = [\alpha_k] + 1$, $f_k : [0,1] \times X^{4m} \to \mathbb{R}$, is singular at some points $[0,1]$, where $X = C^1[0,1]$. Note that in system (1), we investigate the problem with multi-singular points, while in the mentioned other systems, the problems have no singular points or have almost one singular point (in $t = 0$). In fact, the novelty of this work is that the multi-singular points can be controlled and investigated. Note that system (1) is a generalization for the mentioned systems. Recall that $D^\alpha x(t) = f(t)$ is a pointwise defined equation on $[0,1]$ if there exists a set $E \subset [0,1]$ such that the measure of E^c is zero and the equation holds on E ([30]). Recall that the Riemann–Liouville integral of order p with the lower limit

$a \geq 0$ for a function $f:(a,\infty)\to\mathbb{R}$ is defined by $I^p_{a^+}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-s)^{p-1}f(s)\,ds$, provided that the right-hand side is pointwise defined on (a,∞). We denote $I^p_{0^+}f(t)$ by $I^pf(t)$ ([31]). The Caputo fractional derivative of order $\alpha > 0$ is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\,ds$, where $n = [\alpha]+1$ and $f:(a,\infty)\to\mathbb{R}$ is a function ([31]). Let Ψ be the family of nondecreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < \infty$ for all $t>0$. One can check that $\psi(t) < t$ for all $t>0$ ([32]). Let $T:X\to X$ and $\alpha: X\times X\to[0,\infty)$ be two maps. Then T is called an α-admissible map whenever $\alpha(x,y)\geq 1$ implies $\alpha(Tx,Ty)\geq 1$ ([32]). Let (X,d) be a metric space, $\psi\in\Psi$ and $\alpha:X\times X\to[0,\infty)$ a map. A self-map $T:X\to X$ is called an α-ψ-contraction whenever $\alpha(x,y)d(Tx,Ty)\leq\psi(d(x,y))$ for all $x,y\in X$ ([32]). We need the following results.

Lemma 1.1 ([32]) *Let (X,d) be a complete metric space, $\psi\in\Psi$, $\alpha:X\times X\to[0,\infty)$ a map and $T:X\to X$ an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$, then T has a fixed point.*

Lemma 1.2 ([33]) *Let $n-1\leq\alpha<n$ and $x\in C(0,1)$. Then $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some real constants $c_0,\dots,c_{n-1}$.*

2 Main results

Now, we present our main results.

Lemma 2.1 *Let $\alpha\geq 2$, $[\alpha]=n-1$, $\lambda_i\geq 0$, $\mu_i,\gamma_i,\eta\in(0,1)$ for all $i=1,\dots,n_0$, $\theta\in(0,1)$ and $f\in L^1[0,1]$. Then the solution of the problem $D^\alpha x(t) = f(t)$ with the boundary conditions $x^{(j)}(0)=0$ for $2\leq j\leq n-1$, $x(\theta)=\sum_{i=1}^{n_0}\lambda_i D^{\mu_i}x(\gamma_i)$ and $x'(0)=x(\eta)$ is given by*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds \\ &\quad + \frac{1-\eta+t}{(\Delta_\gamma-\theta-1+\eta)\Gamma(\alpha)}\int_0^\theta(\theta-s)^{\alpha-1}f(s)\,ds \\ &\quad - \frac{1-\eta+t}{(\Delta_\gamma-\theta-1+\eta)\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}f(s)\,ds \\ &\quad - \frac{1-\eta+t}{(\Delta_\gamma-\theta-1+\eta)}\sum_{i=1}^{n_0}\frac{\lambda_i}{\Gamma(\alpha-\mu_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\mu_i-1}f(s)\,ds, \end{aligned}$$

where $\Delta_\gamma := \sum_{i=1}^{n_0}\frac{\lambda_i(\gamma_i)^{1-\mu_i}}{\Gamma(2-\mu_i)}$ and $1-\Delta_\gamma\neq\eta-\theta$.

Proof By using a similar method to [30], we conclude that Lemma 1.2 holds on $L^1[0,1]$. Let x be a solution for the problem. By using Lemma 1.2, we have

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds + c_0 + c_1t+\cdots+c_{n-1}t^{n-1}.$$

Since $x^{(j)}(0)=0$ for $2\leq j\leq n-1$, we conclude that

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds + c_0 + c_1t \tag{2}$$

and so $x(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\,ds + c_0 + c_1\eta$ and $x'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s)\,ds + c_1$. Thus, $x'(0) = c_1$ and by using the boundary condition $x'(0) = x(\eta)$ we get

$$\frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\,ds + c_0 + c_1\eta = c_1.$$

Hence,

$$c_1 = \frac{1}{(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\,ds + \frac{1}{1-\eta} c_0. \tag{3}$$

On the other hand by using (2), for each $i = 1, \ldots, n_0$ we have

$$D^{\mu_i} x(t) = \frac{1}{\Gamma(\alpha - \mu_i)} \int_0^t (t-s)^{\alpha-\mu_i-1} f(s)\,ds + c_1 \frac{t^{1-\mu_i}}{\Gamma(2-\mu_i)}$$

which implies $\lambda_i D^{\mu_i} x(\gamma_i) = \frac{\lambda_i}{\Gamma(\alpha-\mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\,ds + c_1 \frac{\lambda_i(\gamma_i)^{1-\mu_i}}{\Gamma(2-\mu_i)}$. Hence,

$$\sum_{i=1}^{n_0} \lambda_i D^{\mu_i} x(\gamma_i) = \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\alpha-\mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\,ds + c_1 \sum_{i=1}^{n_0} \frac{\lambda_i(\gamma_i)^{1-\mu_i}}{\Gamma(2-\mu_i)}.$$

Since $x(\theta) = \frac{1}{\Gamma(\alpha)} \int_0^\theta (\theta - s)^{\alpha-1} f(s)\,ds + c_0 + c_1\theta$ and $x(\theta) = \sum_{i=1}^{n_0} \lambda_i D^{\mu_i} x(\gamma_i)$, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^\theta (\theta - s)^{\alpha-1} f(s)\,ds + c_0 + c_1\theta \\ &\quad = \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\alpha-\mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\,ds + c_1 \sum_{i=1}^{n_0} \frac{\lambda_i(\gamma_i)^{1-\mu_i}}{\Gamma(2-\mu_i)} \end{aligned}$$

and so by using (3), we have

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^\theta (\theta - s)^{\alpha-1} f(s)\,ds + c_0 + \frac{\theta}{(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\,ds \\ &\quad + \frac{\theta}{1-\eta} c_0 = \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\alpha-\mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\,ds \\ &\quad + \frac{\sum_{i=1}^{n_0} \frac{\lambda_i(\gamma_i)^{1-\mu_i}}{\Gamma(2-\mu_i)}}{(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\,ds + \frac{\sum_{i=1}^{n_0} \frac{\lambda_i(\gamma_i)^{1-\mu_i}}{\Gamma(2-\mu_i)}}{1-\eta} c_0. \end{aligned}$$

If $\Delta_\gamma := \sum_{i=1}^{n_0} \frac{\lambda_i(\gamma_i)^{1-\mu_i}}{\Gamma(2-\mu_i)}$, then

$$\begin{aligned} c_0 \left(\frac{\Delta_\gamma - \theta - 1 + \eta}{1-\eta} \right) = {}& \frac{1}{\Gamma(\alpha)} \int_0^\theta (\theta - s)^{\alpha-1} f(s)\,ds \\ &+ \frac{\theta}{(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\,ds \\ &- \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\alpha-\mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\,ds \end{aligned}$$

$$- \frac{\Delta_\gamma}{(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\, ds$$

so

$$\begin{aligned} c_0 = {} & \frac{1-\eta}{(\Delta_\gamma - \theta - 1 + \eta)\Gamma(\alpha)} \int_0^\theta (\theta - s)^{\alpha-1} f(s)\, ds \\ & - \frac{1-\eta}{(\Delta_\gamma - \theta - 1 + \eta)} \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\alpha - \mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\, ds \\ & + \frac{\theta - \Delta_\gamma}{(\Delta_\gamma - \theta - 1 + \eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\, ds. \end{aligned}$$

Thus, by using (2) and (3) we get

$$\begin{aligned} c_1 = {} & \frac{1}{(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\, ds \\ & + \frac{1}{(\Delta_\gamma - \theta - 1 + \eta)\Gamma(\alpha)} \int_0^\theta (\theta - s)^{\alpha-1} f(s)\, ds \\ & + \frac{\theta - \Delta_\gamma}{(\Delta_\gamma - \theta - 1 + \eta)(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\, ds \\ & - \frac{1}{(\Delta_\gamma - \theta - 1 + \eta)} \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\alpha - \mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\, ds \end{aligned}$$

and

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)\, ds \\ & + \frac{1-\eta}{(\Delta_\gamma - \theta - 1 + \eta)\Gamma(\alpha)} \int_0^\theta (\theta - s)^{\alpha-1} f(s)\, ds \\ & + \frac{\theta - \Delta_\gamma}{(\Delta_\gamma - \theta - 1 + \eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\, ds \\ & - \frac{1-\eta}{(\Delta_\gamma - \theta - 1 + \eta)} \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\alpha - \mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\, ds \\ & + \frac{t}{(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\, ds \\ & + \frac{t}{(\Delta_\gamma - \theta - 1 + \eta)\Gamma(\alpha)} \int_0^\theta (\theta - s)^{\alpha-1} f(s)\, ds \\ & + \frac{(\theta - \Delta_\gamma)t}{(\Delta_\gamma - \theta - 1 + \eta)(1-\eta)\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} f(s)\, ds \\ & - \frac{t}{(\Delta_\gamma - \theta - 1 + \eta)} \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\alpha - \mu_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\mu_i-1} f(s)\, ds. \end{aligned}$$

Hence,

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)\, ds$$

$$
\begin{aligned}
&+\frac{1-\eta+t}{(\Delta_\gamma-\theta-1+\eta)\Gamma(\alpha)}\int_0^\theta(\theta-s)^{\alpha-1}f(s)\,ds\\
&+\frac{-(1-\eta)^2+(\theta-\Delta_\gamma)t+t(\Delta_\gamma-\theta-1+\eta)}{(1-\eta)(\Delta_\gamma-\theta-1+\eta)\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}f(s)\,ds\\
&-\frac{1-\eta+t}{(\Delta_\gamma-\theta-1+\eta)}\sum_{i=1}^{n_0}\frac{\lambda_i}{\Gamma(\alpha-\mu_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\mu_i-1}f(s)\,ds
\end{aligned}
$$

and so

$$
\begin{aligned}
x(t)={}&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds\\
&+\frac{1-\eta+t}{(\Delta_\gamma-\theta-1+\eta)\Gamma(\alpha)}\int_0^\theta(\theta-s)^{\alpha-1}f(s)\,ds\\
&-\frac{1-\eta+t}{(\Delta_\gamma-\theta-1+\eta)\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}f(s)\,ds\\
&-\frac{1-\eta+t}{(\Delta_\gamma-\theta-1+\eta)}\sum_{i=1}^{n_0}\frac{\lambda_i}{\Gamma(\alpha-\mu_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\mu_i-1}f(s)\,ds.
\end{aligned}
$$

This completes the proof. □

Consider the space $X = C^1[0,1]$ with the norm $\|\cdot\|_*$ and the space X^m with the norm $\|\cdot\|_{**}$, where $\|(x_1,\dots,x_m)\|_{**} = \max\{\|x_1\|_*,\dots,\|x_m\|_*\}$, $\|x\|_* = \max\{\|x\|,\|x'\|\}$ and $\|.\|$ is the supremum norm on $C[0,1]$. Let f_k be a map $[0,1]\times X^{4m}$ that is singular at some points of $[0,1]$, for $k=1,\dots,m$. Define $F : X^m \to X^m$ as

$$
F(x_1,\dots,x_m)(t)=\begin{pmatrix}\phi_1(x_1,\dots,x_m)(t)\\ \vdots\\ \phi_m(x_1,\dots,x_m)(t)\end{pmatrix},
$$

where

$$
\begin{aligned}
&\phi_k(x_1,\dots,x_m)(t)\\
&\quad=\frac{1}{\Gamma(\alpha_k)}\int_0^t(t-s)^{\alpha_k-1}f_k\bigl(s,x_1(s),x_1'(s),D^{\beta_1}x_1(s),\\
&\quad I^{p_1}x_1(s),\dots,x_m(s),x_m'(s),D^{\beta_m}x_m(s),I^{p_m}x_m(s)\bigr)\,ds\\
&\quad+\frac{1-\eta_k+t}{(\Delta_\gamma-\theta_k-1+\eta)\Gamma(\alpha)}\int_0^{\theta_k}(\theta_k-s)^{\alpha_k-1}f_k\bigl(s,x_1(s),x_1'(s),D^{\beta_1}x_1(s),\\
&\quad I^{p_1}x_1(s),\dots,x_m(s),x_m'(s),D^{\beta_m}x_m(s),I^{p_m}x_m(s)\bigr)\,ds\\
&\quad-\frac{1-\eta_k+t}{(\Delta_{\gamma_k}-\theta_k-1+\eta_k)\Gamma(\alpha_k)}\int_0^{\eta_k}(\eta_k-s)^{\alpha-1}f_k\bigl(s,x_1(s),x_1'(s),D^{\beta_1}x_1(s),\\
&\quad I^{p_1}x_1(s),\dots,x_m(s),x_m'(s),D^{\beta_m}x_m(s),I^{p_m}x_m(s)\bigr)\,ds\\
&\quad-\frac{1-\eta_k+t}{(\Delta_{\gamma_k}-\theta_k-1+\eta_k)}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\int_0^{\gamma_{i,k}}(\gamma_{i,k}-s)^{\alpha_k-\mu_{i,k}-1}f_k\bigl(s,x_1(s),x_1'(s),
\end{aligned}
$$

$$D^{\beta_1}x_1(s), I^{p_1}x_1(s), \dots, x_m(s), x'_m(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\, ds,$$

for $1 \le k \le m$, where $\Delta_{\gamma_k} := \sum_{i=1}^{n_0} \frac{\lambda_{i,k}(\gamma_{i,k})^{1-\mu_{i,k}}}{\Gamma(2-\mu_{i,k})}$. Then we have

$$F'(x_1, \dots, x_m)(t) = \begin{pmatrix} \phi'_1(x_1, \dots, x_m)(t) \\ \vdots \\ \phi'_m(x_1, \dots, x_m)(t) \end{pmatrix},$$

where for each $1 \le k \le m$ we have

$$\begin{aligned}
&\phi'_k(x_1, \dots, x_m)(t) \\
&= \frac{1}{\Gamma(\alpha_k - 1)} \int_0^t (t-s)^{\alpha_k - 2} f_k\big(s, x_1(s), x'_1(s), D^{\beta_1}x_1(s), \\
&I^{p_1}x_1(s), \dots, x_m(s), x'_m(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\, ds \\
&+ \frac{1}{(\Delta_\gamma - \theta_k - 1 + \eta)\Gamma(\alpha)} \int_0^{\theta_k} (\theta_k - s)^{\alpha_k - 1} f_k\big(s, x_1(s), x'_1(s), D^{\beta_1}x_1(s), \\
&I^{p_1}x_1(s), \dots, x_m(s), x'_m(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\, ds \\
&- \frac{1}{(\Delta_{\gamma_k} - \theta_k - 1 + \eta_k)\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k - s)^{\alpha - 1} f_k\big(s, x_1(s), x'_1(s), D^{\beta_1}x_1(s), \\
&I^{p_1}x_1(s), \dots, x_m(s), x'_m(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\, ds \\
&- \frac{1}{(\Delta_{\gamma_k} - \theta_k - 1 + \eta_k)} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^{\gamma_{i,k}} (\gamma_{i,k} - s)^{\alpha_k - \mu_{i,k} - 1} f_k\big(s, x_1(s), x'_1(s), \\
&D^{\beta_1}x_1(s), I^{p_1}x_1(s), \dots, x_m(s), x'_m(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\, ds.
\end{aligned}$$

It is obvious that the singular pointwise defined equation (1) has a solution u if and only if u is a fixed point of the map F.

Theorem 2.2 *Let m, n and n_0 be natural numbers, $\alpha_k \ge 2$, $[\alpha_k] = n_k - 1$, $\lambda_{i,k} \ge 0$, $\gamma_{i,k}, \mu_{i,k}, \theta_k, \eta_k \in (0,1)$, $p_k > 0$ for $i = 1, \dots, n_0$ and $k = 1, 2, \dots, m$, $f_k : [0,1] \times X^m \to \mathbb{R}$ some singular mappings on some points of $[0,1]$ such that*

$$\big|f_k(t, x_1, \dots, x_{4m}) - f_k(t, y_1, \dots, y_{4m})\big| \le \Phi_k(t) M_k\big(|x_1 - y_1|, \dots, |x_{4m} - y_{4m}|\big)$$

for all $x_1, \dots, x_{4m}, y_1, \dots, y_{4m} \in X$ and almost all $t \in [0,1]$. Assume that

$$\big|f_i(t, x_1, \dots, x_{4m})\big| \le \sum_{j=1}^{4m} T_{k,j}\big(t, |x_k|\big),$$

where $M_k : X^{4m} \to \mathbb{R}^+$ is non-decreasing mapping respect to all components such that $\lim_{z \to 0^+} \frac{M_k(z, \dots, z)}{z} := q_k \in [0, \infty)$ and $T_{k,j} : [0,1] \times X \to \mathbb{R}^+$ is a map with $T_{k,j}(\cdot, z)$ is nondecreasing respect to z and $\lim_{z \to 0^+} \frac{T_{k,j}(t,z)}{z} := b_{k,j}(t)$ for almost all $t \in [0,1]$ and for some $b_{k,j} :\in \mathbb{R}^+[0,1] \to \mathbb{R}^+$ such that $(1-t)^{\alpha_k - 2} b_{k,j}(t) \in L^1[0,1]$ for $1 \le j \le 4m$ and

$1 \leq k \leq m$. *Let* $\Delta = \max\{1, \frac{1}{\Gamma(2-\beta_1)}, \dots, \frac{1}{\Gamma(2-\beta_m)}, \frac{1}{\Gamma(p_1+1)}, \dots, \frac{1}{\Gamma(p_m+1)}\}$ *and* $\hat{b}_{i,k}, \hat{\phi}_k \in L^1[0,1]$, $\Delta_{\gamma_k} := \sum_{i=1}^{n_0} \frac{\lambda_{i,k}(\gamma_{i,k})^{1-\mu_{i,k}}}{\Gamma(2-\mu_{i,k})}$ *and* $1 - \Delta_{\gamma_k} \neq \eta_k - \theta_k$, *where* $\hat{\phi}_k(s) = (1-s)^{\alpha_i-2} a_{i,j}(s)$. *If*

$$\max_{1\leq k\leq m}\left[\frac{1}{\Gamma(\alpha_k-1)} + \frac{2(2-\eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} + \frac{(2-\eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})}\right)\right]\max\left\{\sum_{j=1}^{m}\|\hat{b}_{k,j}\|, q_k\hat{\Phi}_k\right\} \in \left[0, \frac{1}{\Delta}\right),$$

then the pointwise defined system

$$\begin{cases} D^{\alpha_1}x_1(t) = f_1(t, x_1(t), x_1'(t), D^{\beta_1}x_1(t), I^{p_1}x_1(t), \\ \qquad \dots, x_m(t), x_m'(t), D^{\beta_m}x_m(t), I^{p_m}x_m(t)), \\ D^{\alpha_m}x_m(t) + f_m(t, x_1(t), x_1'(t), D^{\beta_1}x_1(t), I^{p_1}x_1(t), \\ \quad \dots, x_m(t), x_m'(t), D^{\beta_m}x_m(t), I^{p_m}x_m(t)), \end{cases}$$

with boundary conditions $x_k^{(j)}(0) = 0$, $x_k(\theta_k) = \sum_{i=1}^{n_0}\lambda_{i,k}D^{\mu_{i,k}}x_k(\gamma_{i,k})$ *and* $x_k'(0) = x_k(\eta_k)$ *for* $2 \leq j \leq n_k - 1$ *and* $1 \leq k \leq m$, *has a solution.*

Proof First, we prove F is continuous on X^m. Let $\epsilon > 0$ and $\|(x_1,\dots,x_m) - (y_1,\dots,y_m)\|_{**} < \epsilon$. Then $\max_{1\leq k\leq m}\|x_k - y_k\|_* < \epsilon$ and so $\|x_k - y_k\|_* < \epsilon$ for all $1 \leq k \leq m$. Thus,

$$\begin{aligned} &\bigl|\phi_k(x_1,\dots,x_n)(t) - \phi_k(y_1,\dots,y_n)(t)\bigr| \\ &\leq \frac{1}{\Gamma(\alpha_k)}\int_0^t (t-s)^{\alpha_k-1}\bigl|f_k\bigl(s, x_1(s), x_1'(s), D^{\beta_1}x_1(s), \\ &\quad I^{p_1}x_1(s), \dots, x_m(s), x_m'(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\bigr) - f_k\bigl(s, y_1(s), y_1'(s), \\ &\quad D^{\beta_1}y_1(s), I^{p_1}y_1(s), \dots, y_m(s), y_m'(s), D^{\beta_m}y_m(s), I^{p_m}y_m(s)\bigr)\bigr|\,ds \\ &\quad + \frac{1-\eta_k+t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)}\int_0^{\theta_k}(\theta_k - s)^{\alpha_k-1}\bigl|f_k\bigl(s, x_1(s), x_1'(s), D^{\beta_1}x_1(s), \\ &\quad I^{p_1}x_1(s), \dots, x_m(s), x_m'(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\bigr) - f_k\bigl(s, y_1(s), y_1'(s), \\ &\quad D^{\beta_1}y_1(s), I^{p_1}y_1(s), \dots, y_m(s), y_m'(s), D^{\beta_m}y_m(s), I^{p_m}y_m(s)\bigr)\bigr|\,ds \\ &\quad + \frac{1-\eta_k+t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)}\int_0^{\eta_k}(\eta_k - s)^{\alpha-1}\bigl|f_k\bigl(s, x_1(s), x_1'(s), D^{\beta_1}x_1(s), \\ &\quad I^{p_1}x_1(s), \dots, x_m(s), x_m'(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\bigr) - f_k\bigl(s, y_1(s), y_1'(s), \\ &\quad D^{\beta_1}y_1(s), I^{p_1}y_1(s), \dots, y_m(s), y_m'(s), D^{\beta_m}y_m(s), I^{p_m}y_m(s)\bigr)\bigr|\,ds \\ &\quad + \frac{1-\eta_k+t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})}\int_0^{\gamma_{i,k}}(\gamma_{i,k} - s)^{\alpha_k - \mu_{i,k} - 1} \\ &\quad \times \bigl|f_k\bigl(s, x_1(s), x_1'(s), D^{\beta_1}x_1(s), I^{p_1}x_1(s), \dots, x_m(s), x_m'(s), D^{\beta_m}x_m(s), \\ &\quad I^{p_m}x_m(s)\bigr) - f_k\bigl(s, y_1(s), y_1'(s), D^{\beta_1}y_1(s), I^{p_1}y_1(s), \\ &\quad \dots, y_m(s), y_m'(s), D^{\beta_m}y_m(s), I^{p_m}y_m(s)\bigr)\bigr|\,ds \end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-s)^{\alpha_k - 1} \Phi_k(s) M_k\big(\big|x_1(s) - y_1(s)\big|, \big|x_1'(s) - y_1'(s)\big|,$$
$$\big|D^{\beta_1}(x_1 - y_1)(s)\big|, I^{p_1}(x_1 - y_1)(s)|, \ldots, \big|x_m(s) - y_m(s)\big|,$$
$$\big|x_m'(s) - y_m'(s)\big|, \big|D^{\beta_m}(x_m - y_m)(s)\big|, I^{p_m}(x_m - y_m)(s)|\big)\, ds$$
$$+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\theta_k} (\theta_k - s)^{\alpha_k - 1} \Phi_k(s) M_k\big(\big|x_1(s) - y_1(s)\big|, \big|x_1'(s) - y_1'(s)\big|,$$
$$\big|D^{\beta_1}(x_1 - y_1)(s)\big|, I^{p_1}(x_1 - y_1)(s)|, \ldots, \big|x_m(s) - y_m(s)\big|,$$
$$\big|x_m'(s) - y_m'(s)\big|, \big|D^{\beta_m}(x_m - y_m)(s)\big|, I^{p_m}(x_m - y_m)(s)|\big)\, ds$$
$$+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k - s)^{\alpha - 1} \Phi_k(s) M_k\big(\big|x_1(s) - y_1(s)\big|, \big|x_1'(s) - y_1'(s)\big|,$$
$$\big|D^{\beta_1}(x_1 - y_1)(s)\big|, I^{p_1}(x_1 - y_1)(s)|, \ldots, \big|x_m(s) - y_m(s)\big|,$$
$$\big|x_m'(s) - y_m'(s)\big|, \big|D^{\beta_m}(x_m - y_m)(s)\big|, I^{p_m}(x_m - y_m)(s)|\big)\, ds$$
$$+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^{\gamma_{i,k}} (\gamma_{i,k} - s)^{\alpha_k - \mu_{i,k} - 1}$$
$$\times \Phi_k(s) M_k\big(\big|x_1(s) - y_1(s)\big|, \big|x_1'(s) - y_1'(s)\big|, \big|D^{\beta_1}(x_1 - y_1)(s)\big|, I^{p_1}(x_1 - y_1)(s)|,$$
$$\ldots, \big|x_m(s) - y_m(s)\big|, \big|x_m'(s) - y_m'(s)\big|, \big|D^{\beta_m}(x_m - y_m)(s)\big|, I^{p_m}(x_m - y_m)(s)|\big)\, ds$$

for all $1 \leq k \leq m$ and $t \in [0,1]$. Now for $\beta \in (0,1)$ and $t \in [0,1]$, we have

$$D^\beta(x - y)(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{\beta - 2}\big(x' - y'\big)(s)\, ds$$

and so $|D^\beta(x-y)(t)| \leq \frac{\|x'-y'\|}{\Gamma(1-\beta)} \int_0^t (t-s)^{\beta-2}\, ds = \frac{\|x'-y'\|}{\Gamma(2-\beta)} t^{\beta-1}$. Hence, $|D^\beta(x-y)(t)| \leq \frac{\|x'-y'\|}{\Gamma(2-\beta)}$ and $|I^p(x-y)(t)| \leq \frac{\|x-y\|}{\Gamma(p)} \int_0^t (t-s)^{p-1}\, ds = \frac{\|x-y\|}{\Gamma(p+1)} t^p$. Thus, $|I^p(x-y)(t)| \leq \frac{\|x-y\|}{\Gamma(p+1)}$ and

$$\big|\phi_k(x_1, \ldots, x_n)(t) - \phi_k(y_1, \ldots, y_n)(t)\big|$$
$$\leq \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-s)^{\alpha_k - 1} \Phi_k(s) M_k\bigg(\|x_1 - y_1\|, \big\|x_1' - y_1'\big\|, \frac{\|x_1' - y_1'\|}{\Gamma(2 - \beta_1)},$$
$$\frac{\|x_1 - y_1\|}{\Gamma(p_1 + 1)}, \ldots, \|x_m - y_m\|, \big\|x_m' - y_m'\big\|, \frac{\|x_m' - y_m'\|}{\Gamma(2 - \beta_m)}, \frac{\|x_m - y_m\|}{\Gamma(p_m + 1)}\bigg)\, ds$$
$$+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\theta_k} (\theta_k - s)^{\alpha_k - 1} \Phi_k(s) M_k\bigg(\|x_1 - y_1\|, \big\|x_1' - y_1'\big\|,$$
$$\frac{\|x_1' - y_1'\|}{\Gamma(2 - \beta_1)}, \frac{\|x_1 - y_1\|}{\Gamma(p_1 + 1)}, \ldots, \|x_m - y_m\|, \big\|x_m' - y_m'\big\|, \frac{\|x_m' - y_m'\|}{\Gamma(2 - \beta_m)}, \frac{\|x_m - y_m\|}{\Gamma(p_m + 1)}\bigg)\, ds$$
$$+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k - s)^{\alpha - 1} \Phi_k(s) M_k\bigg(\|x_1 - y_1\|, \big\|x_1' - y_1'\big\|,$$
$$\frac{\|x_1' - y_1'\|}{\Gamma(2 - \beta_1)}, \frac{\|x_1 - y_1\|}{\Gamma(p_1 + 1)}, \ldots, \|x_m - y_m\|, \big\|x_m' - y_m'\big\|, \frac{\|x_m' - y_m'\|}{\Gamma(2 - \beta_m)}, \frac{\|x_m - y_m\|}{\Gamma(p_m + 1)}\bigg)\, ds$$
$$+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^{\gamma_{i,k}} (\gamma_{i,k} - s)^{\alpha_k - \mu_{i,k} - 1}$$

$$\begin{aligned}
&\times \Phi_k(s) M_k\biggl(\|x_1 - y_1\|, \bigl\|x_1' - y_1'\bigr\|, \frac{\|x_1' - y_1'\|}{\Gamma(2-\beta_1)}, \frac{\|x_1 - y_1\|}{\Gamma(p_1+1)}, \dots, \\
&\|x_m - y_m\|, \bigl\|x_m' - y_m'\bigr\|, \frac{\|x_m' - y_m'\|}{\Gamma(2-\beta_m)}, \frac{\|x_m - y_m\|}{\Gamma(p_m+1)} \biggr) \, ds \\
\le{}& \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-s)^{\alpha_k - 1} \Phi_k(s) M_k\bigl(\Delta_1 \|x_1 - y_1\|_*, \dots, \Delta_1 \|x_1 - y_1\|_*, \\
&\dots, \Delta_m \|x_m - y_m\|_*, \dots, \Delta_m \|x_m - y_m\|_*, \bigr) \, ds \\
&+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k| \Gamma(\alpha_k)} \int_0^{\theta_k} (\theta_k - s)^{\alpha_k - 1} \Phi_k(s) M_k\bigl(\Delta_1 \|x_1 - y_1\|_*, \\
&\dots, \Delta_1 \|x_1 - y_1\|_*, \dots, \Delta_m \|x_m - y_m\|_*, \dots, \Delta_m \|x_m - y_m\|_*\bigr) \, ds \\
&+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k| \Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k - s)^{\alpha - 1} \Phi_k(s) M_k\bigl(\Delta_1 \|x_1 - y_1\|_*, \\
&\dots, \Delta_1 \|x_1 - y_1\|_*, \dots, \Delta_m \|x_m - y_m\|_*, \dots, \Delta_m \|x_m - y_m\|_*\bigr) \, ds \\
&+ \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^{\gamma_{i,k}} (\gamma_{i,k} - s)^{\alpha_k - \mu_{i,k} - 1} \\
&\times \Phi_k(s) M_k\bigl(\Delta_1 \|x_1 - y_1\|_*, \dots, \Delta_1 \|x_1 - y_1\|_*, \\
&\dots, \Delta_m \|x_m - y_m\|_*, \dots, \Delta_m \|x_m - y_m\|_*\bigr) \, ds,
\end{aligned}$$

where for $1 \le j \le m$ $\Delta_j = \max\{1, \frac{1}{\Gamma(2-\beta_j)}, \frac{1}{\Gamma(p_j+1)}\}$ and $\|x_j - y_j\|_* = \max\{\|x_j - y_j\|, \|x_j' - y_j'\|\}$. Let $\Delta = \max_{1 \le j \le m} \Delta_j$ and $\|(x_1, \dots, x_n) - (y_1, \dots, y_n)\|_{**} = \max_{1 \le j \le m}\{\|x_j - y_j\|_*\}$. Then, for each $t \in [0,1]$ and $1 \le j \le m$, we have

$$\begin{aligned}
&\bigl|\phi_k(x_1, \dots, x_n)(t) - \phi_k(y_1, \dots, y_n)(t)\bigr| \\
&\quad \le \frac{M_k(\Delta \|(x_1, \dots, x_n) - (y_1, \dots, y_n)\|_{**}, \dots, \Delta \|(x_1, \dots, x_n) - (y_1, \dots, y_n)\|_{**})}{\Gamma(\alpha_k)} \\
&\qquad \times \int_0^1 (1-s)^{\alpha_k - 1} \Phi_k(s) \, ds \\
&\qquad + (1 - \eta_k + t) M_k\bigl(\Delta \bigl\|(x_1, \dots, x_n) - (y_1, \dots, y_n)\bigr\|_{**}, \dots, \\
&\qquad \Delta \bigl\|(x_1, \dots, x_n) - (y_1, \dots, y_n)\bigr\|_{**}\bigr) / \bigl(|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k| \Gamma(\alpha_k)\bigr) \\
&\qquad \times \int_0^1 (1-s)^{\alpha_k - 1} \Phi_k(s) \, ds \\
&\qquad + (1 - \eta_k + t) M_k\bigl(\Delta \bigl\|(x_1, \dots, x_n) - (y_1, \dots, y_n)\bigr\|_{**}, \dots, \\
&\qquad \Delta \bigl\|(x_1, \dots, x_n) - (y_1, \dots, y_n)\bigr\|_{**}\bigr) / \bigl(|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k| \Gamma(\alpha_k)\bigr) \\
&\qquad \times \int_0^1 (1-s)^{\alpha - 1} \Phi_k(s) \, ds \\
&\qquad + (1 - \eta_k + t) M_k\bigl(\Delta \bigl\|(x_1, \dots, x_n) - (y_1, \dots, y_n)\bigr\|_{**}, \dots, \\
&\qquad \Delta \bigl\|(x_1, \dots, x_n) - (y_1, \dots, y_n)\bigr\|_{**}\bigr) / \bigl(|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\bigr) \\
&\qquad \times \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^1 (1-s)^{\alpha_k - \mu_{i,k} - 1} \Phi_k(s) \, ds.
\end{aligned} \tag{4}$$

Since $\lim_{z\to 0^+} \frac{M_k(\Delta z,\dots,\Delta z)}{\Delta z} = q_k$, for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $0 < z < \delta(\epsilon)$ implies $\frac{M_k(\Delta z,\dots,\Delta z)}{\Delta z} < q_k + \epsilon$ for all $1 \leq k \leq m$. Thus,

$$M_k(\Delta z,\dots,\Delta z) < (q_k + \epsilon)\Delta z \tag{5}$$

for $0 < z < \delta(\epsilon)$. Put $\delta_M(\epsilon) = \min\{\delta(\epsilon), \epsilon\}$. Then, for each $0 < z < \delta_M(\epsilon)$, we have

$$M_k(\Delta z,\dots,\Delta z) < (q_k + \epsilon)\Delta \epsilon$$

for all $1 \leq k \leq m$. Let $\|(x_1,\dots,x_n) - (y_1,\dots,y_n)\|_{**} < \delta_M(\epsilon)$. Then we have

$$M_k\bigl(\Delta \bigl\|(x_1,\dots,x_n) - (y_1,\dots,y_n)\bigr\|_{**},\dots,\Delta \bigl\|(x_1,\dots,x_n) - (y_1,\dots,y_n)\bigr\|_{**}\bigr) < (q_k + \epsilon)\Delta \epsilon$$

for all $1 \leq k \leq m$ and so

$$\begin{aligned}
&\bigl|\phi_k(x_1,\dots,x_n)(t) - \phi_k(y_1,\dots,y_n)(t)\bigr| \\
&\quad\leq \frac{(q_k + \epsilon)\Delta \epsilon}{\Gamma(\alpha_k)} \int_0^1 (1-s)^{\alpha_k - 1}\Phi_k(s)\,ds \\
&\qquad + \frac{(1-\eta_k + t)(q_k + \epsilon)\Delta \epsilon}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^1 (1-s)^{\alpha_k - 1}\Phi_k(s)\,ds \\
&\qquad + \frac{(1-\eta_k + t)(q_k + \epsilon)\Delta \epsilon}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^1 (1-s)^{\alpha - 1}\Phi_k(s)\,ds \\
&\qquad + \frac{(1-\eta_k + t)(q_k + \epsilon)\Delta \epsilon}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^1 (1-s)^{\alpha_k - \mu_{i,k} - 1}\Phi_k(s)\,ds \\
&\quad\leq \frac{(q_k + \epsilon)\Delta \epsilon}{\Gamma(\alpha_k)} \|\hat{\Phi}_k\|_{[0,1]} + \frac{(1-\eta_k + t)(q_k + \epsilon)\Delta \epsilon}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \|\hat{\Phi}_k\|_{[0,1]} \\
&\qquad + \frac{(1-\eta_k + t)(q_k + \epsilon)\Delta \epsilon}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \|\hat{\Phi}_k\|_{[0,1]} \\
&\qquad + \frac{(1-\eta_k + t)(q_k + \epsilon)\Delta \epsilon}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \|\hat{\Phi}_k\|_{[0,1]} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigl\|\phi_k(x_1,\dots,x_n) - \phi_k(y_1,\dots,y_n)\bigr\| \\
&\quad\leq \Biggl(\frac{1}{\Gamma(\alpha_k)} + \frac{2(2-\eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \\
&\qquad + \frac{(2-\eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})}\Biggr)(q_k + \epsilon)\Delta \|\hat{\Phi}_k\|_{[0,1]}\epsilon.
\end{aligned}$$

If $\|(x_1,\dots,x_m) - (y_1,\dots,y_m)\|_{**} < \epsilon$ for all $t \in [0,1]$ and $k = 1,\dots,m$, then we get

$$\begin{aligned}
&\bigl|\phi'_k(x_1,\dots,x_n)(t) - \phi'_k(y_1,\dots,y_n)(t)\bigr| \\
&\quad\leq \frac{1}{\Gamma(\alpha_k - 1)} \int_0^t (t-s)^{\alpha_k - 2}\bigl|f_k\bigl(s, x_1(s), x'_1(s), D^{\beta_1}x_1(s),
\end{aligned}$$

$$
\begin{aligned}
&I^{p_1}x_1(s),\dots,x_m(s),x'_m(s),D^{\beta_1}x_m(s),I^{p_m}x_m(s)\big)-f_k\big(s,y_1(s),y'_1(s),\\
&D^{\beta_1}y_1(s),I^{p_1}y_1(s),\dots,y_m(s),y'_m(s),D^{\beta_1}y_m(s),I^{p_m}y_m(s)\big)\big|\,ds\\
&+\frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\theta_k}(\theta_k-s)^{\alpha_k-1}\big|f_k\big(s,x_1(s),x'_1(s),D^{\beta_1}x_1(s),\\
&I^{p_1}x_1(s),\dots,x_m(s),x'_m(s),D^{\beta_1}x_m(s),I^{p_m}x_m(s)\big)-f_k\big(s,y_1(s),y'_1(s),\\
&D^{\beta_1}y_1(s),I^{p_1}y_1(s),\dots,y_m(s),y'_m(s),D^{\beta_1}y_m(s),I^{p_m}y_m(s)\big)\big|\,ds\\
&+\frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\eta_k}(\eta_k-s)^{\alpha-1}\big|f_k\big(s,x_1(s),x'_1(s),D^{\beta_1}x_1(s),\\
&I^{p_1}x_1(s),\dots,x_m(s),x'_m(s),D^{\beta_1}x_m(s),I^{p_m}x_m(s)\big)-f_k\big(s,y_1(s),y'_1(s),\\
&D^{\beta_1}y_1(s),I^{p_1}y_1(s),\dots,y_m(s),y'_m(s),D^{\beta_1}y_m(s),I^{p_m}y_m(s)\big)\big|\,ds\\
&+\frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\\
&\times\int_0^{\gamma_{i,k}}(\gamma_{i,k}-s)^{\alpha_k-\mu_{i,k}-1}\big|f_k\big(s,x_1(s),x'_1(s),D^{\beta_1}x_1(s),\\
&I^{p_1}x_1(s),\dots,x_m(s),x'_m(s),D^{\beta_1}x_m(s),I^{p_m}x_m(s)\big)-f_k\big(s,y_1(s),y'_1(s),\\
&D^{\beta_1}y_1(s),I^{p_1}y_1(s),\dots,y_m(s),y'_m(s),D^{\beta_1}y_m(s),I^{p_m}y_m(s)\big)\big|\,ds\\
\le{}&\frac{1}{\Gamma(\alpha_k-1)}\int_0^t(t-s)^{\alpha_k-2}\Phi_k(s)M_k\big(\big|x_1(s)-y_1(s)\big|,\big|x'_1(s)-y_1(s)\big|,\\
&\big|D^{\beta_1}(x_1-y_1)(s)\big|,I^{p_1}(x_1-y_1)(s)|,\dots,\big|x_m(s)-y_m(s)\big|,\\
&\big|x'_m(s)-y_m(s)\big|,\big|D^{\beta_m}(x_m-y_m)(s)\big|,I^{p_m}(x_m-y_m)(s)|\big)\,ds\\
&+\frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\theta_k}(\theta_k-s)^{\alpha_k-1}\Phi_k(s)M_k\big(\big|x_1(s)-y_1(s)\big|,\big|x'_1(s)-y_1(s)\big|,\\
&\big|D^{\beta_1}(x_1-y_1)(s)\big|,I^{p_1}(x_1-y_1)(s)|,\dots,\big|x_m(s)-y_m(s)\big|,\\
&\big|x'_m(s)-y_m(s)\big|,\big|D^{\beta_m}(x_m-y_m)(s)\big|,I^{p_m}(x_m-y_m)(s)|\big)\,ds\\
&+\frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\eta_k}(\eta_k-s)^{\alpha-1}\Phi_k(s)M_k\big(\big|x_1(s)-y_1(s)\big|,\big|x'_1(s)-y_1(s)\big|,\\
&\big|D^{\beta_1}(x_1-y_1)(s)\big|,I^{p_1}(x_1-y_1)(s)|,\dots,\big|x_m(s)-y_m(s)\big|,\\
&\big|x'_m(s)-y_m(s)\big|,\big|D^{\beta_m}(x_m-y_m)(s)\big|,I^{p_m}(x_m-y_m)(s)|\big)\,ds\\
&+\frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\int_0^{\gamma_{i,k}}(\gamma_{i,k}-s)^{\alpha_k-\mu_{i,k}-1}\\
&\times\Phi_k(s)M_k\big(\big|x_1(s)-y_1(s)\big|,\big|x'_1(s)-y_1(s)\big|,\big|D^{\beta_1}(x_1-y_1)(s)\big|,I^{p_1}(x_1-y_1)(s)|\\
&,\dots,\big|x_m(s)-y_m(s)\big|,\big|x'_m(s)-y_m(s)\big|,\big|D^{\beta_m}(x_m-y_m)(s)\big|,I^{p_m}(x_m-y_m)(s)|\big)\,ds\\
\le{}&\frac{M_k(\Delta\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\|_{**},\dots,\Delta\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\|_{**})}{\Gamma(\alpha_k-1)}\\
&\times\int_0^1(1-s)^{\alpha_k-2}\Phi_k(s)\,ds
\end{aligned}
$$

$$\begin{aligned}
&+ \frac{M_k(\Delta\|(x_1,\ldots,x_n)-(y_1,\ldots,y_n)\|_{**},\ldots,\Delta\|(x_1,\ldots,x_n)-(y_1,\ldots,y_n)\|_{**})}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \\
&\times \int_0^1 (1-s)^{\alpha_k-1}\Phi_k(s)\,ds \\
&+ \frac{M_k(\Delta\|(x_1,\ldots,x_n)-(y_1,\ldots,y_n)\|_{**},\ldots,\Delta\|(x_1,\ldots,x_n)-(y_1,\ldots,y_n)\|_{**})}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \\
&\times \int_0^1 (1-s)^{\alpha-1}\Phi_k(s)\,ds \\
&+ \frac{M_k(\Delta\|(x_1,\ldots,x_n)-(y_1,\ldots,y_n)\|_{**},\ldots,\Delta\|(x_1,\ldots,x_n)-(y_1,\ldots,y_n)\|_{**})}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|} \\
&\times \sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\int_0^1 (1-s)^{\alpha_k-\mu_{i,k}-1}\Phi_k(s)\,ds \\
\leq{}& \frac{(q_k+\epsilon)\Delta\epsilon}{\Gamma(\alpha_k-1)}\|\hat{\Phi}_k\|_{[0,1]} + \frac{(q_k+\epsilon)\Delta\epsilon}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\|\hat{\Phi}_k\|_{[0,1]} \\
&+ \frac{(q_k+\epsilon)\Delta\epsilon}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\|\hat{\Phi}_k\|_{[0,1]} \\
&+ \frac{(q_k+\epsilon)\Delta\epsilon}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\|\hat{\Phi}_k\|_{[0,1]}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|\phi_k'(x_1,\ldots,x_n)-\phi_k'(y_1,\ldots,y_n)\| \\
&\leq \biggl(\frac{1}{\Gamma(\alpha_k-1)} + \frac{2}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \\
&\quad + \frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\biggr)(q_k+\epsilon)\Delta\|\hat{\Phi}_k\|_{[0,1]}\epsilon
\end{aligned}$$

and so

$$\begin{aligned}
&\|\phi_k(x_1,\ldots,x_n)-\phi_k(y_1,\ldots,y_n)\|_* \\
&= \max\bigl\{\|\phi_k(x_1,\ldots,x_n)-\phi_k(y_1,\ldots,y_n)\|, \|\phi_k'(x_1,\ldots,x_n)-\phi_k'(y_1,\ldots,y_n)\|\bigr\} \\
&\leq \biggl(\frac{1}{\Gamma(\alpha_k-1)} + \frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \\
&\quad + \frac{(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\biggr)(q_k+\epsilon)\Delta\|\hat{\Phi}_k\|_{[0,1]}\epsilon.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\|F(x_1,\ldots,x_n)-F(y_1,\ldots,y_n)\|_{**} \\
&= \max_{1\leq k\leq m}\|\phi_k(x_1,\ldots,x_n)-\phi_k(y_1,\ldots,y_n)\|_*
\end{aligned}$$

$$\le \max_{1\le k\le m}\left\{\left(\frac{1}{\Gamma(\alpha_k-1)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\right.\right.$$
$$\left.\left.+\frac{(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)(q_k+\epsilon)\Delta\|\hat{\Phi}_k\|_{[0,1]}\right\}\epsilon.$$

This implies $F(x_1,\dots,x_n)\to F(y_1,\dots,y_n)$ in X^m when $(x_1,\dots,x_n)\to(y_1,\dots,y_n)$. Hence, F is continuous on X^m. Since $\lim_{z\to 0^+}\frac{T_{k,j}(t,\Delta z)}{\Delta z}=b_{k,j}(t)$ for $1\le k\le m$ and $1\le j\le 4m$, for every $\epsilon>0$ we can choose $\delta(\epsilon)$ such that $z\in(0,\delta(\epsilon)]$ implies $\frac{T_{k,j}(t,\Delta z)}{\Delta z}\le b_{k,j}(t)+\epsilon$ for almost all $t\in[0,1]$. Thus,

$$T_{k,j}(t,\Delta z)\le\big(b_{k,j}(t)+\epsilon\big)\Delta z \tag{6}$$

for $z\in(0,\delta(\epsilon)]$ and almost all $t\in[0,1]$. On the other hand, by using the assumptions we have

$$\max_{1\le k\le m}\left[\frac{1}{\Gamma(\alpha_k-1)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\right.$$
$$\left.+\frac{(2-\eta_k)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)\right]\sum_{j=1}^{m}\|\hat{b}_{k,j}\|\Delta<1.$$

Choose $\epsilon_0>0$ such that

$$\max_{1\le k\le m}\left(\left[\frac{1}{\Gamma(\alpha_k-1)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\right.\right.$$
$$\left.+\frac{(2-\eta_k)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)\right]\sum_{j=1}^{m}\|\hat{b}_{k,j}\|$$
$$+m\epsilon_0\left[\frac{1}{\Gamma^2(\alpha_k-1)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma^2(\alpha_k)}\right.$$
$$\left.\left.+\frac{(2-\eta_k)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma^2(\alpha_k-\mu_{i,k})}\right)\right]\right)\Delta<1.$$

Since

$$\max_{1\le k\le m}\left\{\left(\frac{1}{\Gamma(\alpha_k-1)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\right.\right.$$
$$\left.\left.+\frac{(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)q_k\|\hat{\Phi}_k\|_{[0,1]}\right\}\in\left[0,\frac{1}{\Delta}\right),$$

so we can choose $\epsilon_1>0$ such that

$$\max_{1\le k\le m}\left\{\left(\frac{1}{\Gamma(\alpha_k-1)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\right.\right.$$
$$\left.\left.+\frac{(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)(q_k+\epsilon_1)\|\hat{\Phi}_k\|_{[0,1]}\right\}\in\left[0,\frac{1}{\Delta}\right).$$

Let $\delta_0 = \delta(\epsilon_0)$ and put $r := \min\{\delta_0, \epsilon_0, \frac{\delta_M(\epsilon_1)}{2}\}$. By using (6), $z \in (0, r]$ implies

$$T_{k,j}(t, \Delta z) \leq \big(b_{k,j}(t) + \epsilon_0\big)\Delta r$$

and specially for $z = r$ we have

$$T_{k,j}(t, \Delta r) \leq \big(b_{k,j}(t) + \epsilon_0\big)\Delta r \tag{7}$$

for almost all $t \in [0, 1]$. Let $C = \{(x_1, \dots, x_m) \in X^m : \|(x_1, \dots, x_m)\|_{**} \leq r\}$. Define the mapping $\alpha : X^{2m} \to \mathbb{R}$ by $\alpha((x_1, \dots, x_m), (y_1, \dots, y_m)) = 1$ when $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ both are in C and $\alpha((x_1, \dots, x_m), (y_1, \dots, y_m)) = 0$ otherwise. If

$$\alpha\big((x_1, \dots, x_m), (y_1, \dots, y_m)\big) \geq 1,$$

then $\|(x_1, \dots, x_m)\|_{**} \leq r$ and $\|y_1, \dots, y_m)\|_{**} \leq r$ and so $\|x_k\|_* \leq r$ and $\|y_k\|_* \leq r$ for all $1 \leq k \leq m$. Thus, for each $t \in [0, 1]$, we have

$$\begin{aligned}
&\big|\phi_k(x_1, \dots, x_n)(t)\big| \\
&\quad\leq \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-s)^{\alpha_k - 1} \big|f_k\big(s, x_1(s), x_1'(s), D^{\beta_1}x_1(s), \\
&\qquad I^{p_1}x_1(s), \dots, x_m(s), x_m'(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\big|\, ds \\
&\qquad + \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\theta_k} (\theta_k - s)^{\alpha_k - 1} \big|f_k\big(s, x_1(s), x_1'(s), D^{\beta_1}x_1(s), \\
&\qquad I^{p_1}x_1(s), \dots, x_m(s), x_m'(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\big|\, ds \\
&\qquad + \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k - s)^{\alpha - 1} \big|f_k\big(s, x_1(s), x_1'(s), D^{\beta_1}x_1(s), \\
&\qquad I^{p_1}x_1(s), \dots, x_m(s), x_m'(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\big|\, ds \\
&\qquad + \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^{\gamma_{i,k}} (\gamma_{i,k} - s)^{\alpha_k - \mu_{i,k} - 1} \\
&\qquad \times \big|f_k\big(s, x_1(s), x_1'(s), D^{\beta_1}x_1(s), I^{p_1}x_1(s), \dots, x_m(s), x_m'(s), D^{\beta_m}x_m(s), I^{p_m}x_m(s)\big)\big|\, ds \\
&\quad\leq \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-s)^{\alpha_k - 1} \big[T_{k,1}\big(s, |x_1(s)|\big) + T_{k,2}\big(s, |x_1'(s)|\big) + T_{k,3}\big(s, |D^{\beta_1}x_1(s)|\big) \\
&\qquad + T_{k,4}\big(s, |I^{p_1}x_1(s)|\big) + \dots + T_{k,4m-3}\big(s, |x_m(s)|\big) + T_{k,4m-2}\big(s, |x_m'(s)|\big) \\
&\qquad + T_{k,4m-1}\big(s, |D^{\beta_m}x_m(s)|\big) + T_{k,4m}\big(s, |I^{p_m}x_m(s)|\big)\big]\, ds \\
&\qquad + \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\theta_k} (\theta_k - s)^{\alpha_k - 1} \big[T_{k,1}\big(s, |x_1(s)|\big) + T_{k,2}\big(s, |x_1'(s)|\big) \\
&\qquad + T_{k,3}\big(s, |D^{\beta_1}x_1(s)|\big) + T_{k,4}\big(s, |I^{p_1}x_1(s)|\big) + \dots + T_{k,4m-3}\big(s, |x_m(s)|\big) \\
&\qquad + T_{k,4m-2}\big(s, |x_m'(s)|\big) + T_{k,4m-1}\big(s, |D^{\beta_m}x_m(s)|\big) + T_{k,4m}\big(s, |I^{p_m}x_m(s)|\big)\big]\, ds \\
&\qquad + \frac{1 - \eta_k + t}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k - s)^{\alpha - 1} \big[T_{k,1}\big(s, |x_1(s)|\big) + T_{k,2}\big(s, |x_1'(s)|\big)
\end{aligned}$$

$$
\begin{aligned}
&+ T_{k,3}\big(s, \big|D^{\beta_1}x_1(s)\big|\big) + T_{k,4}\big(s, \big|I^{p_1}x_1(s)\big|\big) + \cdots + T_{k,4m-3}\big(s, \big|x_m(s)\big|\big) \\
&+ T_{k,4m-2}\big(s, \big|x_m'(s)\big|\big) + T_{k,4m-1}\big(s, \big|D^{\beta_m}x_m(s)\big|\big) + T_{k,4m}\big(s, \big|I^{p_m}x_m(s)\big|\big)\big]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\int_0^{\gamma_{i,k}}(\gamma_{i,k}-s)^{\alpha_k-\mu_{i,k}-1} \\
&\times \big[T_{k,1}\big(s, \big|x_1(s)\big|\big) + T_{k,2}\big(s, \big|x_1'(s)\big|\big) + T_{k,3}\big(s, \big|D^{\beta_1}x_1(s)\big|\big) \\
&+ T_{k,4}\big(s, \big|I^{p_1}x_1(s)\big|\big) + \cdots + T_{k,4m-3}\big(s, \big|x_m(s)\big|\big) + T_{k,4m-2}\big(s, \big|x_m'(s)\big|\big) \\
&+ T_{k,4m-1}\big(s, \big|D^{\beta_m}x_m(s)\big|\big) + T_{k,4m}\big(s, \big|I^{p_m}x_m(s)\big|\big)\big]\,ds \\
\le{}& \frac{1}{\Gamma(\alpha_k)}\int_0^t (t-s)^{\alpha_k-1}\bigg[T_{k,1}\big(s, \|x_1\|\big) + T_{k,2}\big(s, \|x_1'\|\big) + T_{k,3}\bigg(s, \frac{\|x_1\|}{\Gamma(2-\beta_1)}\bigg) \\
&+ T_{k,4}\bigg(s, \frac{\|x_1\|}{\Gamma(p_1+1)}\bigg) + \cdots + T_{k,4m-3}\big(s, \|x_m\|\big) + T_{k,4m-2}\big(s, \|x_m'\|\big) \\
&+ T_{k,4m-1}\bigg(s, \frac{\|x_m\|}{\Gamma(2-\beta_m)}\bigg) + T_{k,4m}\bigg(s, \frac{\|x_m\|}{\Gamma(p_m+1)}\bigg)\bigg]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\theta_k}(\theta_k-s)^{\alpha_k-1}\bigg[T_{k,1}\big(s, \|x_1\|\big) + T_{k,2}\big(s, \|x_1'\|\big) \\
&+ T_{k,3}\bigg(s, \frac{\|x_1\|}{\Gamma(2-\beta_1)}\bigg) + T_{k,4}\bigg(s, \frac{\|x_1\|}{\Gamma(p_1+1)}\bigg) + \cdots + T_{k,4m-3}\big(s, \|x_m\|\big) \\
&+ T_{k,4m-2}\big(s, \|x_m'\|\big) + T_{k,4m-1}\bigg(s, \frac{\|x_m\|}{\Gamma(2-\beta_m)}\bigg) + T_{k,4m}\bigg(s, \frac{\|x_m\|}{\Gamma(p_m+1)}\bigg)\bigg]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\eta_k}(\eta_k-s)^{\alpha-1}\bigg[T_{k,1}\big(s, \|x_1\|\big) + T_{k,2}\big(s, \|x_1'\|\big) \\
&+ T_{k,3}\bigg(s, \frac{\|x_1\|}{\Gamma(2-\beta_1)}\bigg) + T_{k,4}\bigg(s, \frac{\|x_1\|}{\Gamma(p_1+1)}\bigg) + \cdots + T_{k,4m-3}\big(s, \|x_m\|\big) \\
&+ T_{k,4m-2}\big(s, \|x_m'\|\big) + T_{k,4m-1}\bigg(s, \frac{\|x_m\|}{\Gamma(2-\beta_m)}\bigg) + T_{k,4m}\bigg(s, \frac{\|x_m\|}{\Gamma(p_m+1)}\bigg)\bigg]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\int_0^{\gamma_{i,k}}(\gamma_{i,k}-s)^{\alpha_k-\mu_{i,k}-1} \\
&\times \bigg[T_{k,1}\big(s, \|x_1\|\big) + T_{k,2}\big(s, \|x_1'\|\big) + T_{k,3}\bigg(s, \frac{\|x_1\|}{\Gamma(2-\beta_1)}\bigg) \\
&+ T_{k,4}\bigg(s, \frac{\|x_1\|}{\Gamma(p_1+1)}\bigg) + \cdots + T_{k,4m-3}\big(s, \|x_m\|\big) + T_{k,4m-2}\big(s, \|x_m'\|\big) \\
&+ T_{k,4m-1}\bigg(s, \frac{\|x_m\|}{\Gamma(2-\beta_m)}\bigg) + T_{k,4m}\bigg(s, \frac{\|x_m\|}{\Gamma(p_m+1)}\bigg)\bigg]\,ds \\
\le{}& \frac{1}{\Gamma(\alpha_k)}\int_0^t (t-s)^{\alpha_k-1}\big[T_{k,1}\big(s, \Delta\|x_1\|_*\big) + \cdots + T_{k,4}\big(s, \Delta\|x_1\|_*\big) \\
&+ \cdots + T_{k,4m-3}\big(s, \|x_m\|_*\big) + \cdots + T_{k,4m}\big(s, \|x_m\|_*\big)\big]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\theta_k}(\theta_k-s)^{\alpha_k-1}\big[T_{k,1}\big(s, \Delta\|x_1\|_*\big) \\
&+ \cdots + T_{k,4}\big(s, \Delta\|x_1\|_*\big) + \cdots + T_{k,4m-3}\big(s, \|x_m\|_*\big) + \cdots + T_{k,4m}\big(s, \|x_m\|_*\big)\big]\,ds
\end{aligned}
$$

$$\begin{aligned}
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k-s)^{\alpha-1}\big[T_{k,1}\big(s,\Delta\|x_1\|_*\big) \\
&+ \cdots + T_{k,4}\big(s,\Delta\|x_1\|_*\big) + \cdots + T_{k,4m-3}\big(s,\|x_m\|_*\big) + \cdots + T_{k,4m}\big(s,\|x_m\|_*\big)\big]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})} \int_0^{\gamma_{i,k}} (\gamma_{i,k}-s)^{\alpha_k-\mu_{i,k}-1} \\
&\times \big[T_{k,1}\big(s,\Delta\|x_1\|_*\big) + \cdots + T_{k,4}\big(s,\Delta\|x_1\|_*\big) \\
&+ \cdots + T_{k,4m-3}\big(s,\|x_m\|_*\big) + \cdots + T_{k,4m}\big(s,\|x_m\|_*\big)\big]\,ds \\
\le{}& \frac{1}{\Gamma(\alpha_k)} \int_0^t (t-s)^{\alpha_k-1}\big[T_{k,1}(s,\Delta r) + \cdots + T_{k,4m}(s,\Delta r)\big]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \int_0^{\theta_k} (\theta_k-s)^{\alpha_k-1}\big[T_{k,1}(s,\Delta r) + \cdots + T_{k,4m}(s,\Delta r)\big]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k-s)^{\alpha-1}\big[T_{k,1}(s,\Delta r) + \cdots + T_{k,4m}(s,\Delta r)\big]\,ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})} \int_0^{\gamma_{i,k}} (\gamma_{i,k}-s)^{\alpha_k-\mu_{i,k}-1} \\
&\times \big[T_{k,1}(s,\Delta r) + \cdots + T_{k,4m}(s,\Delta r)\big]\,ds.
\end{aligned}$$

Now by using (5), we obtain

$$\begin{aligned}
&\big|\phi_k(x_1,\ldots,x_n)(t)\big| \\
&\le \frac{1}{\Gamma(\alpha_k)} \int_0^1 (1-s)^{\alpha_k-1}\left[\sum_{j=1}^m \big(b_{k,j}(s)+\epsilon_0\big)\Delta r\right] ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \int_0^1 (1-s)^{\alpha-1}\left[\sum_{j=1}^m \big(b_{k,j}(s)+\epsilon_0\big)\Delta r\right] ds \\
&+ \frac{1-\eta_k+t}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})} \int_0^1 (1-s)^{\alpha_k-\mu_{i,k}-1} \\
&\times \left[\sum_{j=1}^m \big(b_{k,j}(s)+\epsilon_0\big)\Delta r\right] ds \\
&\le \frac{\Delta r}{\Gamma(\alpha_k)} \sum_{j=1}^m \left[\int_0^1 (1-s)^{\alpha_k-2} b_{k,j}(s)\,ds + \epsilon_0 \int_0^1 (1-s)^{\alpha_k-1}\,ds\right] \\
&+ \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \sum_{j=1}^m \left[\int_0^1 (1-s)^{\alpha_k-2} b_{k,j}(s)\,ds + \epsilon_0 \int_0^1 (1-s)^{\alpha_k-1}\,ds\right] \\
&+ \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \sum_{j=1}^m \left[\int_0^1 (1-s)^{\alpha_k-2} b_{k,j}(s)\,ds + \epsilon_0 \int_0^1 (1-s)^{\alpha_k-1}\,ds\right] \\
&+ \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\sum_{j=1}^{m} \left[\int_0^1 (1-s)^{\alpha_k - 2} b_{k,j}(s)\, ds + \epsilon_0 \int_0^1 (1-s)^{\alpha_k - \mu_{i,k} - 1}\, ds \right] \right) \\
&\leq \frac{\Delta r}{\Gamma(\alpha_k)} \sum_{j=1}^{m} \left(\|\hat{b}_{k,j}\| + \frac{\epsilon_0}{\Gamma(\alpha_k)} \right) \\
&\quad + \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \sum_{j=1}^{m} \left(\|\hat{b}_{k,j}\| + \frac{\epsilon_0}{\Gamma(\alpha_k)} \right) \\
&\quad + \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \sum_{j=1}^{m} \left(\|\hat{b}_{k,j}\| + \frac{\epsilon_0}{\Gamma(\alpha_k)} \right) \\
&\quad + \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \left[\frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \sum_{j=1}^{m} \left(\|\hat{b}_{k,j}\| + \frac{\epsilon_0}{\Gamma(\alpha_k - \mu_{i,k})} \right) \right] \\
&= \frac{\Delta r}{\Gamma(\alpha_k)} \sum_{j=1}^{m} \|\hat{b}_{k,j}\| + \frac{m\epsilon_0}{\Gamma^2(\alpha_k)} \Delta r \\
&\quad + \frac{2(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \sum_{j=1}^{m} \|\hat{b}_{k,j}\| + \frac{2(1-\eta_k+t)m\epsilon_0}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma^2(\alpha_k)} \Delta r \\
&\quad + \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \left(\sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \right) \left(\sum_{j=1}^{m} \|\hat{b}_{k,j}\| \right) \\
&\quad + \frac{(1-\eta_k+t)m\epsilon_0}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma^2(\alpha_k - \mu_{i,k})} \Delta r \\
&= \left(\left[\frac{1}{\Gamma(\alpha_k)} + \frac{2(1-\eta_k+t)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \right. \right. \\
&\quad \left. + \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \left(\sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \right) \right] \sum_{j=1}^{m} \|\hat{b}_{k,j}\| \\
&\quad + m\epsilon_0 \left[\frac{1}{\Gamma^2(\alpha_k)} + \frac{2(1-\eta_k+t)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma^2(\alpha_k)} \right. \\
&\quad \left. \left. + \frac{(1-\eta_k+t)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \left(\sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma^2(\alpha_k - \mu_{i,k})} \right) \right] \right) \Delta r
\end{aligned}
$$

and so

$$
\begin{aligned}
&\left\| \phi_k(x_1, \dots, x_n) \right\| \\
&\quad \leq \left(\left[\frac{1}{\Gamma(\alpha_k)} + \frac{2(2-\eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \right. \right. \\
&\qquad \left. + \frac{(2-\eta_k)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \left(\sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \right) \right] \sum_{j=1}^{m} \|\hat{b}_{k,j}\| \\
&\qquad + m\epsilon_0 \left[\frac{1}{\Gamma^2(\alpha_k)} + \frac{2(2-\eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma^2(\alpha_k)} \right.
\end{aligned}
$$

$$+ \frac{(2-\eta_k)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma^2(\alpha_k-\mu_{i,k})}\right)\Bigg]\Bigg)\Delta r.$$

Hence,

$$\begin{aligned}
&\left\|\phi_k(x_1,\dots,x_n)\right\| \\
&\leq \Bigg(\Bigg[\frac{1}{\Gamma(\alpha_k-1)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)} \\
&\quad + \frac{(2-\eta_k)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)\Bigg]\sum_{j=1}^{m}\|\hat{b}_{k,j}\| \\
&\quad + m\epsilon_0\Bigg[\frac{1}{\Gamma^2(\alpha_k-1)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma^2(\alpha_k)} \\
&\quad + \frac{(2-\eta_k)\Delta r}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma^2(\alpha_k-\mu_{i,k})}\right)\Bigg]\Bigg)\Delta r \\
&\leq r.
\end{aligned}$$

Let $t\in[0,1]$, $1\leq k\leq m$ and $(x_1,\dots,x_n)\in C$. Then we have

$$\begin{aligned}
&\left|\phi_k'(x_1,\dots,x_n)(t)\right| \\
&\leq \frac{1}{\Gamma(\alpha_k-1)}\int_0^t (t-s)^{\alpha_k-2}\big|f_k\big(s,x_1(s),x_1'(s),D^{\beta_1}x_1(s), \\
&\quad I^{p_1}x_1(s),\dots,x_m(s),x_m'(s),D^{\beta_1}x_m(s),I^{p_m}x_m(s)\big)\big|\,ds \\
&\quad + \frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\theta_k}(\theta_k-s)^{\alpha_k-1}\big|f_k\big(s,x_1(s),x_1'(s),D^{\beta_1}x_1(s), \\
&\quad I^{p_1}x_1(s),\dots,x_m(s),x_m'(s),D^{\beta_1}x_m(s),I^{p_m}x_m(s)\big)\big|\,ds \\
&\quad + \frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\eta_k}(\eta_k-s)^{\alpha-1}\big|f_k\big(s,x_1(s),x_1'(s),D^{\beta_1}x_1(s), \\
&\quad I^{p_1}x_1(s),\dots,x_m(s),x_m'(s),D^{\beta_m}x_m(s),I^{p_m}x_m(s)\big)\big|\,ds \\
&\quad + \frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\int_0^{\gamma_{i,k}}(\gamma_{i,k}-s)^{\alpha_k-\mu_{i,k}-1} \\
&\quad \times \big|f_k\big(s,x_1(s),x_1'(s),D^{\beta_1}x_1(s),I^{p_1}x_1(s),\dots,x_m(s),x_m'(s),D^{\beta_m}x_m(s),I^{p_m}x_m(s)\big)\big|\,ds \\
&\leq \frac{1}{\Gamma(\alpha_k-1)}\int_0^t (t-s)^{\alpha_k-2}\Bigg[T_{k,1}\big(s,\|x_1\|\big)+T_{k,2}\big(s,\|x_1'\|\big)+T_{k,3}\left(s,\frac{\|x_1\|}{\Gamma(2-\beta_1)}\right) \\
&\quad + T_{k,4}\left(s,\frac{\|x_1\|}{\Gamma(p_1+1)}\right)+\cdots+T_{k,4m-3}\big(s,\|x_m\|\big)+T_{k,4m-2}\big(s,\|x_m'\|\big) \\
&\quad + T_{k,4m-1}\left(s,\frac{\|x_m\|}{\Gamma(2-\beta_m)}\right)+T_{k,4m}\left(s,\frac{\|x_m\|}{\Gamma(p_m+1)}\right)\Bigg]\,ds \\
&\quad + \frac{1}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\int_0^{\theta_k}(\theta_k-s)^{\alpha_k-1}\Big[T_{k,1}\big(s,\|x_1\|\big)+T_{k,2}\big(s,\|x_1'\|\big)
\end{aligned}$$

$$
\begin{aligned}
&+ T_{k,3}\left(s, \frac{\|x_1\|}{\Gamma(2-\beta_1)}\right) + T_{k,4}\left(s, \frac{\|x_1\|}{\Gamma(p_1+1)}\right) + \cdots + T_{k,4m-3}\big(s, \|x_m\|\big) \\
&+ T_{k,4m-2}\big(s, \|x'_m\|\big) + T_{k,4m-1}\left(s, \frac{\|x_m\|}{\Gamma(2-\beta_m)}\right) + T_{k,4m}\left(s, \frac{\|x_m\|}{\Gamma(p_m+1)}\right)\Bigg]\,ds \\
&+ \frac{1}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k - s)^{\alpha-1}\Big[T_{k,1}\big(s, \|x_1\|\big) + T_{k,2}\big(s, \|x'_1\|\big) \\
&+ T_{k,3}\left(s, \frac{\|x_1\|}{\Gamma(2-\beta_1)}\right) + T_{k,4}\left(s, \frac{\|x_1\|}{\Gamma(p_1+1)}\right) + \cdots + T_{k,4m-3}\big(s, \|x_m\|\big) \\
&+ T_{k,4m-2}\big(s, \|x'_m\|\big) + T_{k,4m-1}\left(s, \frac{\|x_m\|}{\Gamma(2-\beta_m)}\right) + T_{k,4m}\left(s, \frac{\|x_m\|}{\Gamma(p_m+1)}\right)\Bigg]\,ds \\
\le{}& \frac{1}{\Gamma(\alpha_k-1)} \int_0^t (t-s)^{\alpha_k-2}\big[T_{k,1}(s,\Delta r) + \cdots + T_{k,4m}(s,\Delta r)\big]\,ds \\
&+ \frac{1}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\theta_k} (\theta_k - s)^{\alpha_k-1}\big[T_{k,1}(s,\Delta r) + \cdots + T_{k,4m}(s,\Delta r)\big]\,ds \\
&+ \frac{1}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^{\eta_k} (\eta_k - s)^{\alpha-1}\big[T_{k,1}(s,\Delta r) + \cdots + T_{k,4m}(s,\Delta r)\big]\,ds \\
&+ \frac{1}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^{\gamma_{i,k}} (\gamma_{i,k} - s)^{\alpha_k - \mu_{i,k} - 1} \\
&\times \big[T_{k,1}(s,\Delta r) + \cdots + T_{k,4m}(s,\Delta r)\big]\,ds \\
\le{}& \frac{1}{\Gamma(\alpha_k-1)} \int_0^1 (1-s)^{\alpha_k-2}\left[\sum_{j=1}^{m}\big(b_{k,j}(s) + \epsilon_0\big)\Delta r\right] ds \\
&+ \frac{2}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \int_0^1 (1-s)^{\alpha-1}\left[\sum_{j=1}^{m}\big(b_{k,j}(s) + \epsilon_0\big)\Delta r\right] ds \\
&+ \frac{1}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \int_0^1 (1-s)^{\alpha_k - \mu_{i,k} - 1} \\
&\times \left[\sum_{j=1}^{m}\big(b_{k,j}(s) + \epsilon_0\big)\Delta r\right] ds \\
\le{}& \frac{\Delta r}{\Gamma(\alpha_k-2)} \sum_{j=1}^{m}\left[\int_0^1 (1-s)^{\alpha_k-2} b_{k,j}(s)\,ds + \epsilon_0 \int_0^1 (1-s)^{\alpha_k-2}\,ds\right] \\
&+ \frac{2\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \sum_{j=1}^{m}\left[\int_0^1 (1-s)^{\alpha_k-2} b_{k,j}(s)\,ds + \epsilon_0 \int_0^1 (1-s)^{\alpha_k-1}\,ds\right] \\
&+ \frac{\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \\
&\times \left(\sum_{j=1}^{m}\left[\int_0^1 (1-s)^{\alpha_k-2} b_{k,j}(s)\,ds + \epsilon_0 \int_0^1 (1-s)^{\alpha_k - \mu_{i,k} - 1}\,ds\right]\right) \\
={}& \frac{\Delta r}{\Gamma(\alpha_k-1)} \sum_{j=1}^{m} \|\hat{b}_{k,j}\| + \frac{m\epsilon_0}{\Gamma^2(\alpha_k-1)}\Delta r
\end{aligned}
$$

$$+ \frac{2\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \sum_{j=1}^{m} \|\hat{b}_{k,j}\| + \frac{2m\epsilon_0}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma^2(\alpha_k)} \Delta r$$

$$+ \frac{\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \left(\sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \right) \left(\sum_{j=1}^{m} \|\hat{b}_{k,j}\| \right)$$

$$+ \frac{m\epsilon_0}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma^2(\alpha_k - \mu_{i,k})} \Delta r.$$

Thus, we get

$$\begin{aligned}
&\left\| \phi_k'(x_1, \ldots, x_n) \right\| \\
&\quad \leq \left(\left[\frac{1}{\Gamma(\alpha_k - 1)} + \frac{2(2 - \eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \right.\right. \\
&\qquad \left. + \frac{(2 - \eta_k)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \left(\sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \right) \right] \sum_{j=1}^{m} \|\hat{b}_{k,j}\| \\
&\qquad + m\epsilon_0 \left[\frac{1}{\Gamma^2(\alpha_k - 1)} + \frac{2(2 - \eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma^2(\alpha_k)} \right. \\
&\qquad \left.\left. + \frac{(2 - \eta_k)\Delta r}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \left(\sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma^2(\alpha_k - \mu_{i,k})} \right) \right] \right) \Delta r \\
&\quad \leq r
\end{aligned}$$

and so $\|\phi_k(x_1, \ldots, x_n)\|_* \leq r$ for all $1 \leq k \leq m$. Hence,

$$\left\| F(x_1, \ldots, x_n) \right\|_{**} = \max_{1 \leq k \leq m} \left\| \phi_k(x_1, \ldots, x_n) \right\|_* \leq r$$

and so $F(x_1, \ldots, x_n) \in C$. For similar reasons, we find $F(y_1, \ldots, y_n) \in C$ and so

$$\alpha\big(F(x_1, \ldots, x_n), F(y_1, \ldots, y_n)\big) \geq 1.$$

Since $C \neq \phi$, for each $(x_1, \ldots, x_n) \in C$, $F(x_1, \ldots, x_n) \in C$ and so

$$\alpha\big((x_1, \ldots, x_n), F(x_1, \ldots, x_n)\big) \geq 1.$$

Let

$$\begin{aligned}
\lambda &:= \max_{1 \leq k \leq m} \left\{ \left(\frac{1}{\Gamma(\alpha_k - 1)} + \frac{2(2 - \eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)} \right.\right. \\
&\qquad \left.\left. + \frac{(2 - \eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|} \sum_{i=1}^{n_0} \frac{\lambda_{i,k}}{\Gamma(\alpha_k - \mu_{i,k})} \right) q_k \|\hat{\Phi}_k\|_{[0,1]} \right\} \Delta \\
&< 1
\end{aligned}$$

and $(x_1,\dots,x_n),(y_1,\dots,y_n)\in C$, Then $\alpha((x_1,\dots,x_n),(y_1,\dots,y_n))=1$. On the other hand by using (4), for each $(x_1,\dots,x_n),(y_1,\dots,y_n)\in X^m$, $t\in[0,1]$ and $1\le k\le m$ we have

$$\begin{aligned}
&\bigl|\phi_k(x_1,\dots,x_n)(t)-\phi_k(y_1,\dots,y_n)(t)\bigr| \\
&\quad\le \frac{M_k(\Delta\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\|_{**},\dots,\Delta\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\|_{**})}{\Gamma(\alpha_k)}\|\hat{\Phi}_k\| \\
&\qquad+\bigl((1-\eta_k+t)M_k\bigl(\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**},\dots, \\
&\qquad\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**}\bigr)/\bigl(|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)\bigr)\bigr)\|\hat{\Phi}_k\| \\
&\qquad+\bigl((1-\eta_k+t)M_k\bigl(\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**},\dots, \\
&\qquad\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**}\bigr)/\bigl(|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)\bigr)\bigr)\|\hat{\Phi}_k\| \\
&\qquad+(1-\eta_k+t)M_k\bigl(\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**},\dots, \\
&\qquad\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**}\bigr)/\bigl(\bigl|\Delta_{\gamma_k}-\theta_k-1+\eta_k\bigr|\bigr) \\
&\qquad\times\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\|\hat{\Phi}_k\|.
\end{aligned}$$

Since $(x_1,\dots,x_n),(y_1,\dots,y_n)\in C$, $\|(x_1,\dots,x_n)\|_{**}\le r$ and $\|(y_1,\dots,y_n)\|_{**}\le r$. Thus,

$$\begin{aligned}
&\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**} \\
&\quad\le\bigl\|(x_1,\dots,x_n)\bigr\|_{**}+\bigl\|(y_1,\dots,y_n)\bigr\|_{**}\le r+r\le\frac{\delta_M}{2}+\frac{\delta_M}{2}=\delta_M.
\end{aligned}$$

By using (5), for each $1\le k\le m$ we get

$$\begin{aligned}
&M_k\bigl(\Delta\bigl\|(x_1,\dots,x_n)\bigr\|_{**}-\bigl\|(y_1,\dots,y_n)\bigr\|_{**},\dots,\Delta\bigl\|(x_1,\dots,x_n)\bigr\|_{**}-\bigl\|(y_1,\dots,y_n)\bigr\|_{**}\bigr) \\
&\quad<(q_k+\epsilon_1)\Delta\bigl\|(x_1,\dots,x_n)\bigr\|_{**}-\bigl\|(y_1,\dots,y_n)\bigr\|_{**}
\end{aligned}$$

and so

$$\begin{aligned}
&\bigl|\phi_k(x_1,\dots,x_n)(t)-\phi_k(y_1,\dots,y_n)(t)\bigr| \\
&\quad\le\frac{\|\hat{\Phi}_k\|}{\Gamma(\alpha_k)}(q_k+\epsilon_1)\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**} \\
&\qquad+\frac{(1-\eta_k+t)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\|\hat{\Phi}_k\|(q_k+\epsilon_1)\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**} \\
&\qquad+\frac{(1-\eta_k+t)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}\|\hat{\Phi}_k\|(q_k+\epsilon_1)\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**} \\
&\qquad+\frac{(1-\eta_k+t)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|} \\
&\qquad\times\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)\|\hat{\Phi}_k\|(q_k+\epsilon_1)\Delta\bigl\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\bigr\|_{**}.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\left\|\phi_k(x_1,\dots,x_n)-\phi_k(y_1,\dots,y_n)\right\| \\
&\quad\le\left(\frac{1}{\Gamma(\alpha_k)}+\frac{2(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|\Gamma(\alpha_k)}+\frac{(1-\eta_k+t)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)\right) \\
&\qquad\times(q_k+\epsilon_1)\Delta\left\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\right\|_{**} \\
&\quad\le\lambda\left\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\right\|_{**}
\end{aligned}$$

and so

$$\left\|\phi_k(x_1,\dots,x_n)-\phi_k(y_1,\dots,y_n)\right\|\le\lambda\left\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\right\|_{**}.$$

Similarly, we can find

$$\left\|\phi_k'(x_1,\dots,x_n)-\phi_k'(y_1,\dots,y_n)\right\|\le\lambda\left\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\right\|_{**}$$

and so $\|\phi_k(x_1,\dots,x_n)-\phi_k(y_1,\dots,y_n)\|_*\le\lambda\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\|_{**}$. Hence,

$$\begin{aligned}
\left\|F(x_1,\dots,x_n)-F(y_1,\dots,y_n)\right\|_{**}&=\max_{1\le k\le m}\left\|\phi_k(x_1,\dots,x_n)-\phi_k(y_1,\dots,y_n)\right\|_* \\
&\le\lambda\left\|(x_1,\dots,x_n)-(y_1,\dots,y_n)\right\|_{**}.
\end{aligned}\tag{8}$$

Now, consider the map $\psi:[0,\infty)\to[0,\infty)$ defined by $\psi(t)=\lambda t$. If $(x_1,\dots,x_n)\notin C$ or $(y_1,\dots,y_n)\notin C$, then $\alpha((x_1,\dots,x_n),(y_1,\dots,y_n))=0$ and so

$$\begin{aligned}
&\alpha\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big)\,d\big(F(x_1,\dots,x_n),F(y_1,\dots,y_n)\big) \\
&\quad\le\psi\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big).
\end{aligned}$$

If $(x_1,\dots,x_n)$, $(y_1,\dots,y_n)\notin C$, then $\alpha((x_1,\dots,x_n),(y_1,\dots,y_n))=1$ and so by using (8), we obtain

$$\begin{aligned}
&\alpha\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big)\,d\big(F(x_1,\dots,x_n),F(y_1,\dots,y_n)\big) \\
&\quad\le\psi\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big).
\end{aligned}$$

Now by using Lemma 1.1, we conclude that F has a fixed point in X^m which is a solution for the problem. □

Here, we present an example to illustrate our main result.

Example 2.3 Consider the pointwise defined problem

$$\begin{cases}D^{\frac{5}{2}}x(t)=f_1(t,x(t),x'(t),D^{\frac{1}{2}}x(t),I^{\frac{1}{3}}x(t),y(t),y'(t),D^{\frac{1}{3}}y(t),I^{\frac{1}{2}}y(t)),\\ D^{\frac{7}{3}}y(t)=f_2(t,x(t),x'(t),D^{\frac{1}{2}}x(t),I^{\frac{1}{3}}x(t),y(t),y'(t),D^{\frac{1}{3}}y(t),I^{\frac{1}{2}}y(t)),\end{cases}\tag{9}$$

with boundary conditions $x''(0) = y''(0) = 0$, $x(\frac{1}{2}) = y(\frac{1}{2}) = D^{\frac{1}{2}}x(\frac{1}{3})$, $x'(0) = x(\frac{1}{4})$ and $y'(0) = y(\frac{1}{3})$, where $f_1(t,x_1,\dots,x_8) = \sum_{j=1}^{8} \frac{1}{t^{\sigma_j}}|x_j|$, $f_2(t,x_1,\dots,x_8) = \frac{c(t)}{p(t)}\sum_{j=1}^{8}|x_j|$, $c(t) = 1$ and $p(t) = 0$ whenever $t \in [0,1] \cap \mathbb{Q}$, $c(t) = 0$ and $p(t) = 1$ whenever $t \in [0,1] \cap \mathbb{Q}^c$ and $\sigma_1,\dots,\sigma_8 \in (0,\frac{1}{2})$. Note that

$$\bigl|f_1(t,x_1,\dots x_8) - f_1(t,y_1,\dots,y_8)\bigr| \le \sum_{k=1}^{8} \frac{1}{50t^{\sigma_i}}|x_k - y_k|,$$

$$\bigl|f_2(t,x_1,\dots x_8) - f_2(t,y_1,\dots,y_8)\bigr| \le \frac{c(t)}{40p(t)} \sum_{k=1}^{8} |x_k - y_k|,$$

$$\bigl|f_1(t,x_1,\dots,x_8) - f_1(t,y_1,\dots,y_8)\bigr| \le \Phi_1(t)M_1\bigl(|x_1 - y_1|,\dots,|x_8 - y_8|\bigr)$$

and $|f_2(t,x_1,\dots,x_{4m}) - f_1(t,y_1,\dots,y_{4m})| \le \Phi_2(t)M_2(|x_1 - y_1|,\dots,|x_8 - y_8|)$, where $\Phi_1(t) = \frac{1}{50t^{\sigma}}$, $\Phi_2(t) = \frac{c(t)}{40p(t)}$, $\sigma := \min\{\sigma_1,\dots,\sigma_8\}$ $M_1(x_1,\dots,x_8) = M_2(x_1,\dots,x_8) = \sum_{k=1}^{8}|x_k|$. Also,

$$\bigl|f_k(t,x_1,\dots,x_8)\bigr| \le \sum_{j=1}^{8} T_{k,j}\bigl(t,|x_k|\bigr)$$

for $k = 1,2$, where $T_{1,j}(t,|x_k|) = \frac{1}{50t^{\sigma_j}}|x_j|$ and $T_{2,j}(t,|x_k|) = \frac{c(t)}{40p(t)}|x_j|$ for $j = 1,\dots,8$. Then $M_k : X^8 \to \mathbb{R}^+$ is nondecreasing with respect to all components, $\lim_{z\to 0^+} \frac{M_k(z,\dots,z)}{z} = 8 := q_k \in [0,\infty)$ for $k = 1,2$, $T_{k,j}(\cdot,z)$ is nondecreasing with respect to z, $\lim_{z\to 0^+} \frac{T_{1,j}(t,z)}{z} = \frac{1}{50t^{\sigma_j}} := b_{1,j}(s)$,

$$\lim_{z\to 0^+} \frac{T_{2,j}(t,z)}{z} = \frac{c(t)}{40p(t)} := b_{1,j}(s)$$

for $j = 1,\dots,8$ and almost all $t \in [0,1]$, $\|\hat{\phi}_1\| \le \frac{1}{50(1-\sigma)}$, $\|\hat{\phi}_2\| \le \frac{1}{60}$ $\|\hat{b}_{1,j}\| \le \frac{1}{50(1-\sigma_j)}$, $\|\hat{b}_{2,j}\| \le \frac{1}{60}$,

$$\begin{aligned}\Delta_{\gamma_1} = \Delta_{\gamma_2} &= \sum_{i=1}^{n_0} \frac{\lambda_{i,k}(\gamma_{i,k})^{1-\mu_{i,k}}}{\Gamma(2-\mu_{i,k})} \\ &= \frac{\lambda_1(\gamma_1)^{1-\mu_1}}{\Gamma(2-\mu_1)} = \frac{(\frac{1}{3})^{\frac{1}{2}}\Gamma(\frac{3}{2})}{\Gamma(2-\mu_1)} \frac{2}{\sqrt{3\pi}}\end{aligned}$$

and $1 - \Delta_{\gamma_k} \ne \eta_k - \theta_k$. Put

$$\begin{aligned}\Delta &= \max\left\{1, \frac{1}{\Gamma(2-\beta_1)}, \frac{1}{\Gamma(2-\beta_2)}, \frac{1}{\Gamma(p_1+1)}, \frac{1}{\Gamma(p_m+1)}\right\} \\ &= \max\left\{1, \frac{1}{\Gamma(\frac{3}{2})}, \frac{1}{\Gamma(\frac{5}{3})}, \frac{1}{\Gamma(\frac{5}{3})}, \frac{1}{\Gamma(\frac{1}{2})}\right\} = \frac{2}{\sqrt{\pi}}.\end{aligned}$$

Then we have

$$\max_{1\le k\le m}\left[\frac{1}{\Gamma(\alpha_k - 1)} + \frac{2(2-\eta_k)}{|\Delta_{\gamma_k} - \theta_k - 1 + \eta_k|\Gamma(\alpha_k)}\right.$$

$$+\frac{(2-\eta_k)}{|\Delta_{\gamma_k}-\theta_k-1+\eta_k|}\left(\sum_{i=1}^{n_0}\frac{\lambda_{i,k}}{\Gamma(\alpha_k-\mu_{i,k})}\right)\Bigg]\max\left\{\sum_{j=1}^{m}\|\hat{b}_{k,j}\|,q_k\hat{\Phi}_k\right\}$$

$$\max\left\{\left[\frac{1}{\Gamma(\frac{3}{2})}+\frac{2(\frac{7}{4})}{|\frac{2}{\sqrt{3\pi}}-\theta_k-1+\frac{1}{4}|\Gamma(\frac{5}{2})}+\frac{\frac{7}{4}}{|\frac{2}{\sqrt{3\pi}}-\frac{1}{2}-1+\frac{7}{4}|}\left(\frac{1}{\Gamma(2)}\right)\right]\times\frac{16}{50},\right.$$

$$\left.\left[\frac{1}{\Gamma(\frac{5}{2})}+\frac{2(\frac{5}{3})}{|\frac{2}{\sqrt{3\pi}}-\frac{1}{2}-1+\frac{1}{3}|\Gamma(\frac{7}{2})}+\frac{\frac{5}{3}}{|\frac{2}{\sqrt{3\pi}}-\frac{1}{2}-1+\frac{7}{4}|}\left(\frac{1}{\Gamma(3)}\right)\right]\times\frac{2}{15}\right\}\in\left[0,\frac{1}{\Delta}\right).$$

Now by using Theorem 2.2, problem (9) has a solution.

3 Conclusion

Some phenomena could be modeled by singular fractional differential equations. By studying multi-singular fractional differential equations we like to increase our abilities in modeling complicated phenomena in the world. In this work by using α-ψ-contractions and locating some control conditions, we investigate the existence of solutions for a multi-singular fractional differential system. Also, we present an example to illustrate our main result.

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