Awan et al. *Advances in Difference Equations* (2020) 2020:575
https://doi.org/10.1186/s13662-020-03036-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A new q-integral identity and estimation of its bounds involving generalized exponentially μ-preinvex functions

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Abstract

In the article, we introduce the generalized exponentially μ-preinvex function, derive a new q-integral identity for second order q-differentiable function, and establish several new q-trapezoidal type integral inequalities for the function whose absolute value of second q-derivative is exponentially μ-preinvex.

MSC: 26A51; 26D15; 05A30

Keywords: Hermite–Hadamard inequality; Hölder inequality; Power mean inequality; Convexity; Generalized exponentially μ-preinvex function; Quantum calculus

1 Introduction and preliminaries

Convexity [1–5] is a very simple and natural notion which plays a pivotal role in different fields of pure and applied sciences [6–9] such as optimization theory [10], engineering and management sciences [11, 12]. In recent past the classical concept of convexity has been extended and generalized in different directions [13–25]. A significant generalization of convexity is the preinvexity, which was introduced and studied by Weir and Mond [26]. Recently, Awan et al. [27] introduced and studied another extension of classical convexity which is called exponentially convex functions.

Another important aspect which makes the theory of convexity more charming is its relation with the theory of inequalities. Many inequalities can be obtained using the theory of convex functions [28–36]. One of the most inequalities in convex functions is Hermite–Hadamard inequality [37–39], which provides us a necessary and sufficient condition for a function to be convex. In recent years many new generalizations, improvements, and variants of the Hermite–Hadamard inequality have been obtained in the literature [40–48] by use of the ordinary, quantum, and fractional calculus.

The main purpose of the article is to introduce the class of generalized exponentially μ-preinvex functions. We derive a new q-integral identity and then some new estimates of bounds for it essentially utilizing the concepts of quantum calculus.

2 Preliminaries

In this section, we introduce the new definition of exponentially μ-preinvex function, establish a new q-integral identity, and obtain new associated q-bounds.

First of all, let $\mathcal{K} \subset \Re^n$ be a nonempty set, $\Lambda : \mathcal{K} \to \Re$ be a continuous function, and $\mu : \mathcal{K} \times \mathcal{K} \to \Re \setminus \{0\}$ and $\theta : \mathcal{K} \times \mathcal{K} \to \Re^n$ be two continuous bifunctions.

Definition 2.1 A set $\mathcal{K} \subseteq \Re^n$ is said to be μ-invex with respect to the bifunctions $\mu(\cdot,\cdot)$ and $\theta(\cdot,\cdot)$ if

$$a + k\mu(b,a)\theta(b,a) \in \mathcal{K}$$

for all $a, b \in \mathcal{K}$ and $k \in [0,1]$.

Note that the convex set with $\mu(b,a) = 1$ and $\theta(b,a) = b - a$ is an invex set, but the converse is not true. For example, the set $\mathcal{K} = \Re \setminus (-1/2, 1/2)$ is an invex set with respect to θ and $\mu(b,a) = 1$, where

$$\theta(b,a) = \begin{cases} b - a, & \text{for } b > 0, a > 0 \text{ or } b < 0, a < 0, \\ a - b, & \text{for } b < 0, a > 0 \text{ or } b < 0, a < 0. \end{cases}$$

It is clear that $\mathcal{K}$ is not a convex set.

Definition 2.2 A function $\Lambda : \mathcal{K} \to \Re$ is said to be generalized exponentially μ-preinvex function if there exist bifunctions $\mu(\cdot,\cdot)$ and $\theta(\cdot,\cdot)$, $\chi \geq 1$ and nonpositive α such that

$$\Lambda\bigl(a + k\mu(b,a)\theta(b,a)\bigr) \leq (1-k)^s \frac{\Lambda(a)}{\chi^{\alpha a}} + k^s \frac{\Lambda(b)}{\chi^{\alpha b}}$$

for all $a, b \in \mathcal{K}$, $k \in [0,1]$, and $s \in (0,1]$.

Note that, if $\alpha = 0$ or $\chi = 1$, then the class of generalized exponentially μ-preinvex functions reduces to the class of generalized μ-preinvex functions. The class of generalized exponentially μ-preinvex function includes the class of of preinvexity for $\alpha = 0$ and $\mu(b,a) = 1$. Also note that if we take $\chi = e$, then we have the class of exponentially μ-preinvex functions, which is defined as follows.

Definition 2.3 A function $\Lambda : \mathcal{K} \to \Re$ is said to be exponentially μ-preinvex if there exist bifunctions $\mu(\cdot,\cdot)$ and $\theta(\cdot,\cdot)$ and nonpositive α such that

$$\Lambda\bigl(a + k\mu(b,a)\theta(b,a)\bigr) \leq (1-k)^s \frac{\Lambda(a)}{e^{\alpha a}} + k^s \frac{\Lambda(b)}{e^{\alpha b}}$$

for all $a, b \in \mathcal{K}$, $k \in [0,1]$, and $s \in (0,1]$.

Example 2.1 The function $\Lambda : \Re \to \Re$ defined by $\Lambda(k) = k^2$ is exponentially μ-preinvex for all $\alpha < 0$ and $\mu(b,a) = 1$.

Next, we recall some previously known concepts and results, which will be helpful in obtaining the quantum analogues of the main results of the article.

Definition 2.4 (see [49, 50]) Let $0 < q < 1$ and $\Lambda : J = [a,b] \to \Re$ be an arbitrary function. Then the q-derivative of Λ on J at t is defined as follows:

$$ {}_aD_q\Lambda(t) = \frac{\Lambda(t) - \Lambda(qt + (1-q)a)}{(1-q)(t-a)} \quad (t \neq a) \quad \text{and} \quad {}_aD_q\Lambda(a) = \lim_{t\to a_a} D_q\Lambda(t). $$

We note that $\lim_{q\to 1} {}_aD_q\Lambda(t) = d\Lambda(t)/dt$ is just the classical derivative if Λ is differentiable.

Definition 2.5 (see [49, 50]) Let $\Lambda : J = [a,b] \to \Re$ be an arbitrary function. Then the second-order q-derivative on the interval J is defined by

$$ {}_aD_q^2\Lambda(t) = {}_aD_q\bigl({}_aD_q\Lambda(t)\bigr) $$

provided ${}_aD_q$ is q-differentiable on J. Similarly, the higher order q-derivative on J can be defined by

$$ {}_aD_q^n\Lambda(t) ={}_a D_q\bigl({}_aD_q^{n-1}\Lambda(t)\bigr). $$

Definition 2.6 (see [49, 50]) Let $0 < q < 1$ and $\Lambda : J = [a,b] \to \Re$ be an arbitrary function. Then the q-integral on J is defined by

$$ \int_a^x \Lambda(k)\,\mathrm{d}_qk = (1-q)(x-a)\sum_{n=0}^{\infty} q^n\Lambda\bigl(q^nx + \bigl(1-q^n\bigr)a\bigr) $$

for $x \in J$.

Note that if $a = 0$, then we have the classical q-integral, which is defined as follows:

$$ \int_0^x \Lambda(k)\,\mathrm{d}_qk = (1-q)x\sum_{n=0}^{\infty} q^n\Lambda\bigl(q^nx\bigr). $$

Lemma 2.2 (see [49, 50]) *Let* $\alpha \in \Re \setminus \{-1\}$. *Then*

$$ \int_a^x (k-a)^\alpha\,\mathrm{d}_qk = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}. $$

Definition 2.7 (see [51]) Let $a \in \Re$ and $n \in \mathbb{N}$. Then the q-analogue of a is defined by

$$ [a]_q = \frac{1-q^n}{1-q}. $$

Definition 2.8 (see [51]) Let $k, p > 0$. Then $\mathbf{B}_q(k,p)$ is defined by

$$ \mathbf{B}_q(k,p) = \int_0^1 x^{k-1}(1-qx)_q^{p-1}\,\mathrm{d}_qx. $$

For more details for q-calculus, we recommend the literature [52–55] to the readers.

3 Results and discussions

In this section, we present our main results of the article.

Lemma 3.1 *Let $0 < q < 1$ and $\Lambda : \mathcal{K} \to \Re$ be an arbitrary function such that $\mathrm{D}_q^2\Lambda$ is q-integrable on $\mathcal{K}$. Then one has*

$$\begin{aligned}
&\frac{q\Lambda(a) + \Lambda(a + \mu(b,a)\theta(b,a))}{q+1} - \frac{1}{\mu(b,a)\theta(b,a)} \int_a^{a+\mu(b,a)\theta(b,a)} \Lambda(x)\,\mathrm{d}_q x \\
&\quad = \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1} \int_0^1 k(1-qk)\mathrm{D}_q^2\Lambda\bigl(a + k\mu(b,a)\theta(b,a)\bigr)\,\mathrm{d}_q k. \qquad (3.1)
\end{aligned}$$

Proof We clearly see that

$$\begin{aligned}
&\int_0^1 k(1-qk)\mathrm{D}_q^2\Lambda\bigl(a + k\mu(b,a)\theta(b,a)\bigr)\,\mathrm{d}_q k \\
&\quad = \int_0^1 \bigl(k(1-qk)\bigl[q\Lambda\bigl(a + k\mu(b,a)\theta(b,a)\bigr) - (1+q)\Lambda\bigl(a + qk\mu(b,a)\theta(b,a)\bigr) \\
&\qquad + \Lambda\bigl(a + q^2k\mu(b,a)\theta(b,a)\bigr)\bigr]\bigr) \\
&\qquad / \bigl(k^2 q(1-q)^2\mu^2(b,a)\theta^2(b,a)\bigr)\,\mathrm{d}_q k \\
&\quad = \left(q\sum_{n=0}^{\infty}\Lambda\bigl(a + q^n\mu(b,a)\theta(b,a)\bigr) - (1+q)\sum_{n=0}^{\infty}\Lambda\bigl(a + q^{n+1}\mu(b,a)\theta(b,a)\bigr)\right. \\
&\qquad \left. + \sum_{n=0}^{\infty}\Lambda\bigl(a + q^{n+2}\mu(b,a)\theta(b,a)\bigr)\right) \\
&\qquad / \bigl(q(1-q)\mu^2(b,a)\theta^2(b,a)\bigr) \\
&\qquad - q\left\{\frac{q(1-q)\mu(b,a)\theta(b,a)\sum_{n=0}^{\infty}q^n\Lambda(a + q^n\mu(b,a)\theta(b,a))}{q(1-q)^2\mu^3(b,a)\theta^3(b,a)}\right. \\
&\qquad - (1+q)(1-q)\mu(b,a)\theta(b,a) \\
&\qquad \times \frac{\sum_{n=0}^{\infty}q^{n+1}\Lambda(a + q^{n+1}\mu(b,a)\theta(b,a))}{q^2(1-q)^2\mu^3(b,a)\theta^3(b,a)} \\
&\qquad \left. + \frac{(1-q)\mu(b,a)\theta(b,a)\sum_{n=0}^{\infty}q^{n+2}\Lambda(a + q^{n+2}\mu(b,a)\theta(b,a))}{q^3(1-q)^2\mu^3(b,a)\theta^3(b,a)}\right\} \\
&\quad = \frac{q(\Lambda(a + \mu(b,a)\theta(b,a)) - \Lambda(a)) - \Lambda(a + q\mu(b,a)\theta(b,a)) + \Lambda(a)}{q(1-q)\mu^2(b,a)\theta^2(b,a)} \\
&\qquad - \frac{1+q}{q^2\mu^3(b,a)\theta^3(b,a)} \\
&\qquad \times \int_a^{a+\mu(b,a)\theta(b,a)} \Lambda(x)\,\mathrm{d}_q k - \frac{q^2+q-1}{q^2(1-q)\mu^2(b,a)\theta^2(b,a)}\Lambda\bigl(a + \mu(b,a)\theta(b,a)\bigr) \\
&\qquad + \frac{\Lambda(a + q\mu(b,a)\theta(b,a))}{q(1-q)\mu^2(b,a)\theta^2(b,a)} \\
&\quad = \frac{q\Lambda(a) + \Lambda(a + \mu(b,a)\theta(b,a))}{q^2\mu^2(b,a)\theta^2(b,a)} - \frac{1+q}{q^2\mu^3(b,a)\theta^3(b,a)} \int_a^{a+\mu(b,a)\theta(b,a)} \Lambda(x)\,\mathrm{d}_q x.
\end{aligned}$$

Multiplying both sides of the above equality by $q^2\mu^2(b,a)\theta^2(b,a)/(1+q)$, we get the required result. □

Theorem 3.2 *Let $\Lambda : \mathcal{K} \to \Re$ be an arbitrary function, $\mu(b,a)\theta(b,a) > 0$ with $\mathrm{D}_q^2\Lambda$ be q-integrable on $\mathcal{K}$, where $0 < q < 1$ is a constant. If $|\mathrm{D}_q^2\Lambda|^r$ is a generalized exponentially μ-preinvex function with $\chi \geq 1$ and non-positive α, then for $s \in (0,1]$ and $r > 1$ we have*

$$\left|\frac{q\Lambda(a) + \Lambda(a+\mu(b,a)\theta(b,a))}{q+1} - \frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)} \Lambda(x)\,\mathrm{d}_q x\right|$$
$$\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\psi_1\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r + \psi_2\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right)^{\frac{1}{r}}, \tag{3.2}$$

where

$$\psi_1 = 2^{1-s}\mathbf{B}_q(2,r+1) - \mathbf{B}_q(s+2,r+1)$$

and

$$\psi_2 = \mathbf{B}_q(s+2,r+1).$$

Proof Using Lemma 3.1, the well-known power mean inequality, and the given hypothesis of the theorem, we get

$$\left|\frac{q\Lambda(a) + \Lambda(a+\mu(b,a)\theta(b,a))}{q+1} - \frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)} \Lambda(x)\,\mathrm{d}_q x\right|$$
$$= \left|\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\int_0^1 k(1-qk)\mathrm{D}_q^2\Lambda\big(a+k\mu(b,a)\theta(b,a)\big)\,\mathrm{d}_q k\right|$$
$$\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\int_0^1 k(1-qk)\left|\mathrm{D}_q^2\Lambda\big(a+k\mu(b,a)\theta(b,a)\big)\,\mathrm{d}_q k\right|$$
$$\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\int_0^1 k\,\mathrm{d}_q k\right)^{1-\frac{1}{r}}$$
$$\times\left(\int_0^1 k(1-qk)^r\left|\mathrm{D}_q^2\Lambda\big(a+k\mu(b,a)\theta(b,a)\big)\right|^r\mathrm{d}_q k\right)^{\frac{1}{r}}$$
$$\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}$$
$$\times\left(\int_0^1 k(1-qk)^r\left[(1-k)^s\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r + k^s\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right]\mathrm{d}_q k\right)^{\frac{1}{r}}$$
$$\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}$$
$$\times\left(\int_0^1 k(1-qk)_q^r\left[(1-k)^s\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r + k^s\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right]\mathrm{d}_q k\right)^{\frac{1}{r}}$$
$$\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1 k(1-qk)_q^r\left[\left(2^{1-s}-k^s\right)\left|\frac{D_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r+k^s\left|\frac{D_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right]d_qk\right)^{\frac{1}{r}}$$
$$=\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\psi_1\left|\frac{D_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r+\psi_2\left|\frac{D_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right)^{\frac{1}{r}},$$

where

$$\psi_1=\int_0^1 k\left(2^{1-s}-k^s\right)(1-qk)_q^r\,d_qk=2^{1-s}\mathbf{B}_q(2,r+1)-\mathbf{B}_q(s+2,r+1)\geq 0$$

due to $2^{1-s}-k^s\geq 0$ for all $k\in[0,1]$ and $s\in(0,1]$, and

$$\psi_2=\int_0^1 k^{s+1}(1-qk)_q^r\,d_qk=\mathbf{B}_q(s+2,r+1).$$

This proof is completed. □

Theorem 3.3 *Let $\Lambda:\mathcal{K}\to\Re$ be an arbitrary function, $\mu(b,a)\theta(b,a)>0$ with $D_q^2\Lambda$ be q-integrable on $\mathcal{K}$, where $0<q<1$ is a constant. If $|D_q^2\Lambda|^r$ is a generalized exponentially μ-preinvex function with $\chi\geq 1$ and non-positive α, then for $s\in(0,1]$, $r>1$, and $1/p+1/r=1$ we obtain*

$$\left|\frac{q\Lambda(a)+\Lambda(a+\mu(b,a)\theta(b,a))}{q+1}-\frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)}\Lambda(x)\,d_qx\right|$$
$$\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)\mathbf{B}_q^{\frac{1}{p}}(2,p+1)}{q+1}$$
$$\times\left(\frac{(2^{1-s}[s+2]_q-1-q)\left|\frac{D_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r+(1+q)\left|\frac{D_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r}{(1+q)[s+2]_q}\right)^{\frac{1}{r}}. \tag{3.3}$$

Proof Using Lemma 3.1, Hölder's inequality, and the given hypothesis of the theorem, we have

$$\left|\frac{q\Lambda(a)+\Lambda(a+\mu(b,a)\theta(b,a))}{q+1}-\frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)}\Lambda(x)\,d_qx\right|$$
$$=\left|\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\int_0^1 k(1-qk)D_q^2\Lambda\left(a+k\mu(b,a)\theta(b,a)\right)d_qk\right|$$
$$\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\int_0^1 k(1-qk)^p\,d_qk\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 k\left|D_q^2\Lambda\left(a+k\mu(b,a)\theta(b,a)\right)\right|^r d_qk\right)^{\frac{1}{r}}$$
$$\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\int_0^1 k(1-qk)_q^p\,d_qk\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 k\left[(1-k)^s\left|\frac{D_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r+k^s\left|\frac{D_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right]d_qk\right)^{\frac{1}{r}}$$
$$\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)\mathbf{B}_q^{\frac{1}{p}}(2,p+1)}{q+1}$$

$$\times\left(\frac{(2^{1-s}[s+2]_q-1-q)|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}|^r+(1+q)|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}|^r}{[s+2]_q}\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 3.4 *Let $\Lambda:\mathcal{K}\to\Re$ be an arbitrary function, $\mu(b,a)\theta(b,a)>0$ with $\mathrm{D}_q^2\Lambda$ be q-integrable on $\mathcal{K}$, where $0<q<1$ is a constant. If $|\mathrm{D}_q^2\Lambda|^r$ is a generalized exponentially μ-preinvex function with $\chi\geq 1$ and non-positive α, then for $s\in(0,1]$, $r>1$, and $1/p+1/r=1$ we get*

$$\begin{aligned}&\left|\frac{q\Lambda(a)+\Lambda(a+\mu(b,a)\theta(b,a))}{q+1}-\frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)}\Lambda(x)\,\mathrm{d}_qx\right|\\&\quad\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\\&\qquad\times\left(\left(2^{1-s}\mathbf{B}_q(r+1,r+1)-\mathbf{B}_q(r+s+1,r+1)\right)\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r\right.\\&\qquad\left.+\mathbf{B}_q(r+s+1,r+1)\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right)^{\frac{1}{r}}.\end{aligned}\tag{3.4}$$

Proof Using Lemma 3.1, Hölder's inequality, and the given hypothesis of the theorem, we obtain

$$\begin{aligned}&\left|\frac{q\Lambda(a)+\Lambda(a+\mu(b,a)\theta(b,a))}{q+1}-\frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)}\Lambda(x)\,\mathrm{d}_qx\right|\\&\quad=\left|\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\int_0^1k(1-qk)\mathrm{D}_q^2\Lambda\big(a+k\mu(b,a)\theta(b,a)\big)\,\mathrm{d}_qk\right|\\&\quad\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\int_0^1 1\,\mathrm{d}_qk\right)^{\frac{1}{p}}\\&\qquad\times\left(\int_0^1k^r(1-qk)^r\left|\mathrm{D}_q^2\Lambda\big(a+k\mu(b,a)\theta(b,a)\big)\right|^r\mathrm{d}_qk\right)^{\frac{1}{r}}\\&\quad\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\int_0^1k^r(1-qk)_q^r\left[(1-k)^s\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r+k^s\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right]\mathrm{d}_qk\right)^{\frac{1}{r}}\\&\quad\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\left(2^{1-s}\mathbf{B}_q(r+1,r+1)-\mathbf{B}_q(r+s+1,r+1)\right)\right.\\&\qquad\left.\times\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r+\mathbf{B}_q(r+s+1,r+1)\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right)^{\frac{1}{r}},\end{aligned}$$

where

$$2^{1-s}\mathbf{B}_q(r+1,r+1)-\mathbf{B}_q(r+s+1,r+1)\geq 0.$$

Using the same idea as in the proof of Theorem 3.2, we can complete the proof. □

Theorem 3.5 *Let $\Lambda:\mathcal{K}\to\Re$ be an arbitrary function, $\mu(b,a)\theta(b,a)>0$ with $\mathrm{D}_q^2\Lambda$ be q-integrable on $\mathcal{K}$, where $0<q<1$ is a constant. If $|\mathrm{D}_q^2\Lambda|^r$ is a generalized exponentially μ-*

preinvex function with $\chi \geq 1$ *and non-positive* α, *then for* $s \in (0,1]$, $r > 1$, *and* $1/p + 1/r = 1$ *we have*

$$\begin{aligned}
&\left|\frac{q\Lambda(a) + \Lambda(a+\mu(b,a)\theta(b,a))}{q+1} - \frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)} \Lambda(x)\,\mathrm{d}_q x\right| \\
&\quad\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)\mathbf{B}_q^{\frac{1}{p}}(p+1,p+1)}{q+1} \\
&\qquad\times \left(\frac{(2^{1-s}[s+1]_q - 1)|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}|^r + |\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}|^r}{[s+1]_q}\right)^{\frac{1}{r}}.
\end{aligned} \tag{3.5}$$

Proof Using Lemma 3.1, Hölder's inequality, and the given hypothesis of the theorem, we get

$$\begin{aligned}
&\left|\frac{q\Lambda(a) + \Lambda(a+\mu(b,a)\theta(b,a))}{q+1} - \frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)} \Lambda(x)\,\mathrm{d}_q x\right| \\
&\quad= \left|\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\int_0^1 k(1-qk)\mathrm{D}_q^2\Lambda\big(a+k\mu(b,a)\theta(b,a)\big)\,\mathrm{d}_q k\right| \\
&\quad\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\int_0^1 k^p(1-qk)^p\,\mathrm{d}_q k\right)^{\frac{1}{p}} \\
&\qquad\times \left(\int_0^1 \left|\mathrm{D}_q^2\Lambda\big(a+k\mu(b,a)\theta(b,a)\big)\right|^r\,\mathrm{d}_q k\right)^{\frac{1}{r}} \\
&\quad\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)\mathbf{B}_q^{\frac{1}{p}}(p+1,p+1)}{q+1} \\
&\qquad\times \left(\int_0^1 \left[(1-k)^s\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r + k^s\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right]\mathrm{d}_q k\right)^{\frac{1}{r}} \\
&\quad\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)\mathbf{B}_q^{\frac{1}{p}}(p+1,p+1)}{q+1}\left(\frac{(2^{1-s}[s+1]_q - 1)|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}|^r + |\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}|^r}{[s+1]_q}\right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Theorem 3.6 *Let* $\Lambda : \mathcal{K} \to \Re$ *be an arbitrary function,* $\mu(b,a)\theta(b,a) > 0$ *with* $\mathrm{D}_q^2\Lambda$ *be q-integrable on* $\mathcal{K}$, *where* $0 < q < 1$ *is a constant. If* $|\mathrm{D}_q^2\Lambda|^r$ *is a generalized exponentially* μ*-preinvex function with* $\chi \geq 1$ *and non-positive* α, *then for* $s \in (0,1]$, $r > 1$, *and* $1/p + 1/r = 1$ *we obtain*

$$\begin{aligned}
&\left|\frac{q\Lambda(a) + \Lambda(a+\mu(b,a)\theta(b,a))}{q+1} - \frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)} \Lambda(x)\,\mathrm{d}_q x\right| \\
&\quad\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1} \\
&\qquad\times \left(\frac{1}{[p+1]_q}\right)^{\frac{1}{p}}\left(\big(2^{1-s}\mathbf{B}_q(1,r+1) - \mathbf{B}_q(s+1,r+1)\big)\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r\right. \\
&\qquad\left.+ \mathbf{B}_q(s+1,r+1)\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right)^{\frac{1}{r}}.
\end{aligned} \tag{3.6}$$

Proof Using Lemma 3.1, Hölder's inequality, and the given hypothesis of the theorem, we have

$$\begin{aligned}
&\left|\frac{q\Lambda(a)+\Lambda(a+\mu(b,a)\theta(b,a))}{q+1}-\frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)}\Lambda(x)\,\mathrm{d}_qx\right| \\
&\quad=\left|\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\int_0^1 k(1-qk)\mathrm{D}_q^2\Lambda\bigl(a+k\mu(b,a)\theta(b,a)\bigr)\,\mathrm{d}_qk\right| \\
&\quad\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\int_0^1 k^p\,\mathrm{d}_qk\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_0^1(1-qk)^r\bigl|\mathrm{D}_q^2\Lambda\bigl(a+k\mu(b,a)\theta(b,a)\bigr)\bigr|^r\,\mathrm{d}_qk\right)^{\frac{1}{r}} \\
&\quad\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\frac{1}{[p+1]_q}\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_0^1(1-qk)^r\left[(1-k)^s\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r+k^s\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right]\mathrm{d}_qk\right)^{\frac{1}{r}} \\
&\quad\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1}\left(\frac{1}{[p+1]_q}\right)^{\frac{1}{p}}\Bigl(\bigl(2^{1-s}\mathbf{B}_q(1,r+1)-\mathbf{B}_q(s+1,r+1)\bigr) \\
&\qquad\times\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r+\mathbf{B}_q(s+1,r+1)\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\Bigr)^{\frac{1}{r}},
\end{aligned}$$

where

$$2^{1-s}\mathbf{B}_q(1,r+1)-\mathbf{B}_q(s+1,r+1)\geq 0.$$

Using the same idea as in the proof of Theorem 3.2, we complete the proof. □

Theorem 3.7 *Let $\Lambda:\mathcal{K}\to\Re$ be an arbitrary function, $\mu(b,a)\theta(b,a)>0$ with $\mathrm{D}_q^2\Lambda$ be q-integrable on $\mathcal{K}$, where $0<q<1$ is a constant. If $|\mathrm{D}_q^2\Lambda|^r$ is a generalized exponentially μ-preinvex function with $\chi\geq 1$ and non-positive α, then for $s\in(0,1]$, $r>1$, and $1/p+1/r=1$ we get*

$$\begin{aligned}
&\left|\frac{q\Lambda(a)+\Lambda(a+\mu(b,a)\theta(b,a))}{q+1}-\frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)}\Lambda(x)\,\mathrm{d}_qx\right| \\
&\quad\leq\frac{q^2\mu^2(b,a)\theta^2(b,a)\mathbf{B}_q^{\frac{1}{p}}(1,p+1)}{q+1}\left(\left(\frac{2^{1-s}}{[r+1]_q}-\frac{1}{[r+s+1]_q}\right)\left|\frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}}\right|^r\right. \\
&\qquad\left.+\left(\frac{1}{[r+s+1]_q}\right)\left|\frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}}\right|^r\right)^{\frac{1}{r}}. \qquad (3.7)
\end{aligned}$$

Proof Using Lemma 3.1, Hölder's inequality, and the given hypothesis of the theorem, we obtain

$$\left|\frac{q\Lambda(a)+\Lambda(a+\mu(b,a)\theta(b,a))}{q+1}-\frac{1}{\mu(b,a)\theta(b,a)}\int_a^{a+\mu(b,a)\theta(b,a)}\Lambda(x)\,\mathrm{d}_qx\right|$$

$$
\begin{aligned}
&= \left| \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1} \int_0^1 k(1-qk) \mathrm{D}_q^2 \Lambda\big(a+k\mu(b,a)\theta(b,a)\big)\, \mathrm{d}_q k \right| \\
&\leq \frac{q^2\mu^2(b,a)\theta^2(b,a)}{q+1} \left(\int_0^1 (1-qk)^p \, \mathrm{d}_q k \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 k^r \left| \mathrm{D}_q^2 \Lambda\big(a+k\mu(b,a)\theta(b,a)\big) \right|^r \mathrm{d}_q k \right)^{\frac{1}{r}} \\
&\leq \frac{q^2\mu^2(b,a)\theta^2(b,a) \mathbf{B}_q^{\frac{1}{p}}(1,p+1)}{q+1} \\
&\quad \times \left(\int_0^1 k^r \left[(1-k)^s \left| \frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}} \right|^r + k^s \left| \frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}} \right|^r \right] \mathrm{d}_q k \right)^{\frac{1}{r}} \\
&\leq \frac{q^2\mu^2(b,a)\theta^2(b,a) \mathbf{B}_q^{\frac{1}{p}}(1,p+1)}{q+1} \\
&\quad \times \left(\left(\frac{2^{1-s}}{[r+1]_q} - \frac{1}{[r+s+1]_q} \right) \left| \frac{\mathrm{D}_q^2\Lambda(a)}{\chi^{\alpha a}} \right|^r + \left(\frac{1}{[r+s+1]_q} \right) \left| \frac{\mathrm{D}_q^2\Lambda(b)}{\chi^{\alpha b}} \right|^r \right)^{\frac{1}{r}}.
\end{aligned}
$$

This completes the proof. □

4 Conclusion

In this paper, we have defined the class of generalized exponentially μ-preinvex functions and derived a new generalized quantum integral identity. With the help of this auxiliary result, we have obtained some new estimates of the quantum bounds essentially using the class of generalized exponentially μ-preinvex functions. It is worth to mention here that if we take $\chi = e$, then all of the main results reduce to the results for exponentially μ-preinvex functions. To the best of our knowledge, these results are new in the literature. Since the quantum calculus has wide applications in many mathematical areas, this new class of functions can be applied to obtain more results in convex analysis, special functions, quantum mechanics, optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

Acknowledgements

Authors are thankful to the editor and anonymous referee for their valuable comments and suggestions. These suggestions helped us a lot in improving the standard of the paper. First and second authors are thankful to Higher Education Commission Pakistan.

Funding

The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485) and the Natural Science Foundation of Huzhou City (Grant No. 2018YZ07).

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 5 May 2020 Accepted: 6 October 2020 Published online: 14 October 2020

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