Rashid et al. *Advances in Difference Equations* (2020) 2020:571
https://doi.org/10.1186/s13662-020-03026-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Multidimensional fixed points in generalized distance spaces

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Abstract

The main aim of this paper is to study distance spaces, to provide some useful remarks with examples regarding distance spaces, and to establish multiple fixed point results for a C-distance space in the presence of different contractive conditions. This concept allows us to reduce the multidimensional case to a one-dimensional case.

MSC: 47H10; 54H25

Keywords: Multiple fixed point; Distance spaces; Contractive conditions; Inequality

1 Introduction

We start a survey with a coupled fixed point for the conviction of *multidimensional fixed point* because its notion emerged naturally from coupled fixed point. Opoitsev [12–15] introduced coupled fixed point and published a number of papers in the period 1975–1986. In 2011, the concept of *tripled fixed point* was introduced by Berinde and Borcut [1], which included three variables. Samet and Vetro [16] generalized this concept to a fixed point of m-order in 2010. In 2012, *quadruple fixed point* was studied by Karapinar and Berinde [5] for nonlinear contractions in the presence of partially ordered metric spaces. After this beginning, a number of articles were devoted to the study of tripled, quadruple, and also multiple fixed points (also known as "a multidimensional fixed point" or "an m-tuplet fixed point").

In 2016, Choban [2] generalized metric spaces as distance spaces. In 2017, Choban and Berinde [3, 4] established multidimensional fixed point results for distance spaces in the presence of certain contractive condition. In this paper we provide some useful remarks with examples regarding distance spaces and establish multidimensional fixed point results for C-distance spaces with generalized contractive conditions. This point of view allows us to reduce the multiple case of fixed point theorems to a one-dimensional case. In the last section of this article an application of our results is added, which provides a unique solution to a specified class of integral equations.

2 Preliminaries

Let us recall some fundamental concepts regarding distance spaces, which can be found in [2] and [4].

Definition 2.1 Let $\mathcal{M}$ be a nonempty set and $\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$, then σ is a distance on $\mathcal{M}$, if

1. $\sigma(\xi, \eta) \geq 0$ for all $\xi, \eta \in \mathcal{M}$;
2. if $\sigma(\xi, \eta) + \sigma(\eta, \xi) = 0$, then $\xi = \eta$ for all $\xi, \eta \in \mathcal{M}$;
3. if $\xi = \eta$, then $\sigma(\xi, \eta) = 0$ for all $\xi, \eta \in \mathcal{M}$.

Let σ be a distance on $\mathcal{M}$ and the open ball with center ξ and radius $r > 0$ is denoted and defined as $B(\xi, \sigma, r) = \{\eta \in \mathcal{M} : \sigma(\xi, \eta) < r\}$.

Consider a distance space $(\mathcal{M}, \sigma)$ and a sequence $\{\xi_\vartheta : \vartheta \in \mathbb{N}\}$ in $\mathcal{M}$ and $\xi \in \mathcal{M}$. Then $\{\xi_\vartheta : \vartheta \in \mathbb{N}\}$:

1. *converges to* ξ if and only if $\lim_{\vartheta \to \infty} \sigma(\xi, \xi_\vartheta) = 0$;
2. is *Cauchy* if $\lim_{r,\vartheta \to \infty} \sigma(\xi_\vartheta, \xi_r) = 0$.

If every Cauchy sequence in the distance space $\mathcal{M}$ converges to some ξ in $\mathcal{M}$, then $(\mathcal{M}, \sigma)$ is called complete distance space.

Note: Every metric space is a distance space but converse is not true in general.

Example 2.1 Let $\mathcal{M} = \mathbb{N}$, define

$$\begin{aligned} &\sigma(r, \vartheta) = r - \vartheta \quad \text{for all } r, \vartheta \in \mathbb{N}, \text{where } r \geq \vartheta, \\ &\sigma(\vartheta, r) = \vartheta^{-1} - r^{-1}. \end{aligned}$$

Clearly, $r - \vartheta \geq 0$ and $\vartheta^{-1} - r^{-1} = \frac{r-\vartheta}{r\vartheta} \geq 0$. Then

$$\begin{aligned} \sigma(r, \vartheta) + \sigma(\vartheta, r) = 0 \quad &\Leftrightarrow \quad r - \vartheta + \frac{r - \vartheta}{r\vartheta} = 0 \\ &\Leftrightarrow \quad r - \vartheta = 0 \quad \text{and} \quad \frac{r - \vartheta}{r\vartheta} = 0 \\ &\Leftrightarrow \quad r = \vartheta. \end{aligned}$$

Hence σ is a distance on $\mathcal{M}$ but it is not a metric on $\mathcal{M}$.

Now we include some important remarks with examples.

Remark In general, distance σ is not a continuous function.

Example 2.2 Let $\mathcal{M} = \{\beta^{-\vartheta} : \vartheta \in \mathbb{N}\} \cup \{0\}$, and for all $\xi \in \mathcal{M}$, $\beta \geq 1$, define

$$\sigma(\xi, \eta) = \begin{cases} 0, & \xi = \eta, \\ \beta^{-\vartheta}, & \xi = \beta^{-\vartheta} \text{ and } \eta = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, in all cases $\sigma(\xi, \eta) \geq 0$ and $\sigma(\xi, \eta) + \sigma(\eta, \xi) = 0$ if and only if $\xi = \eta$. Thus σ is a distance on $\mathcal{M}$. Let $\{\xi_\vartheta : \vartheta \in \mathbb{N}\}$ be a null sequence, *i.e.*, $(\xi_\vartheta)_{\vartheta \in \mathbb{N}} = (\beta^{-\vartheta}) \to 0$ and $(\eta_\vartheta)_{\vartheta \in \mathbb{N}} = (0, 0, 0, \ldots) \to 0$. Then

$$\sigma(\xi_\vartheta, \eta_\vartheta) = \sigma\big(\beta^{-\vartheta}, 0\big) = 1, \qquad \sigma(\xi, \eta) = \sigma(0, 0) = 0.$$

Consider

$$\left|\sigma(\xi_\vartheta,\eta_\vartheta)-\sigma(\xi,\eta)\right| = |1-0| = 1 \not< \epsilon.$$

So σ is not a continuous function.

Remark A convergent sequence need not be Cauchy in a distance space.

Example 2.3 If we consider the distance space in Example 2.2, then clearly the sequence $\{\beta^{-\vartheta} : \vartheta \in \mathbb{N}\}$ is a convergent sequence that converges to 0, but it is not Cauchy because

$$\lim_{\vartheta,r\to\infty} \sigma\left(\beta^{-\vartheta},\beta^{-r}\right) = 1 \not< \epsilon \quad \text{for all } r,\vartheta \in \mathbb{N}.$$

Remark Convergent sequence in a distance space may not have a unique limit.

Example 2.4 Let

$$\mathcal{M} = \left\{\vartheta^{-1} : \vartheta \in \mathbb{N}\right\} \cup \{0,a\}, \quad \text{where } a \geq 1.$$

Now we define

$$\begin{aligned}
&\sigma\left(\vartheta^{-1},r^{-1}\right) = \left|\vartheta^{-1}-r^{-1}\right| \quad \text{for all } \vartheta,r \in \mathbb{N},\\
&\sigma\left(0,\vartheta^{-1}\right) = \vartheta^{-1}, \qquad \sigma(0,a) = 2, \qquad \sigma(a,0) = 3,\\
&\sigma\left(a,\vartheta^{-1}\right) = \frac{1}{\vartheta a^2}, \qquad \sigma\left(\vartheta^{-1},a\right) = \frac{1}{\vartheta^2 a}, \qquad \sigma\left(\vartheta^{-1},0\right) = 2\vartheta^{-1}.
\end{aligned}$$

Then

$$\lim_{\vartheta\to\infty} \sigma\left(0,\vartheta^{-1}\right) = \lim_{\vartheta\to\infty} \vartheta^{-1} = 0 = \lim_{\vartheta\to\infty} \sigma\left(\vartheta^{-1},0\right) = \lim_{\vartheta\to\infty} 2\vartheta^{-1}.$$

Also

$$\lim_{\vartheta\to\infty} \sigma\left(a,\vartheta^{-1}\right) = \lim_{\vartheta\to\infty} \frac{1}{\vartheta a^2} = 0 = \lim_{\vartheta\to\infty} \sigma\left(\vartheta^{-1},a\right) = \lim_{\vartheta\to\infty} \frac{1}{\vartheta^2 a}.$$

The sequence $(\vartheta^{-1})_{\vartheta\in\mathbb{N}}$ converges to 0 and a which belong to $\mathcal{M}$, which implies that 0 and a both are the limits of the sequence $(\vartheta^{-1})_{\vartheta\in\mathbb{N}}$. Hence the convergent sequence in a distance space may not have a unique limit point.

Definition 2.2 Let $\mathcal{M} \neq \phi$ and σ be a distance on $\mathcal{M}$. Then $(\mathcal{M},\sigma)$ is called symmetric space if $\sigma(\xi,\eta) = \sigma(\eta,\xi)$ for all $\xi,\eta \in \mathcal{M}$.

Definition 2.3 A distance space $(\mathcal{M},\sigma)$ is called C-distance space if any Cauchy sequence that converges has a unique limit point.

Definition 2.4 The function $\Psi : [0,\infty) \to [0,\infty)$ is called altering distance function if it is semicontinuous, nondecreasing and $\Psi(0) = 0$.

Defining distance on the Cartesian product of distance spaces

Consider $(\mathcal{M},\sigma)$ as a distance space, $r \in \mathbb{N} = \{1,2,\ldots\}$. On $\mathcal{M}^r$ define σ^r as

$$\sigma^r\big((\xi_1,\ldots,\xi_r),(\eta_1,\ldots,\eta_r)\big) = \sup\big\{\sigma(\xi_{\acute{\imath}},\eta_{\acute{\imath}}) : \acute{\imath} \le r\big\}.$$

Obviously, $(\mathcal{M}^r,\sigma^r)$ is also a distance space.

Proposition 1 *The distance space $(\mathcal{M}^r,\sigma^r)$ inherits the properties of the distance space $(\mathcal{M},\sigma)$.*

Fix $r \in \mathbb{N}$ and $\Gamma = (\Gamma_1,\ldots,\Gamma_r)$ a collection of mappings

$$\big\{\Gamma_{\acute{\imath}} : \{1,2,\ldots,r\} \to \{1,2,\ldots,r\} : 1 \le \acute{\imath} \le r\big\}.$$

Consider a distance space $(\mathcal{M},\sigma)$ and a mapping $G : \mathcal{M}^r \to \mathcal{M}$. The operator $\Gamma G : \mathcal{M}^r \to \mathcal{M}^r$, which is a composition of G and Γ, is defined as

$$\Gamma G(\xi_1,\ldots,\xi_r) = (\eta_1,\ldots,\eta_r),$$

where

$$\eta_{\acute{\imath}} = G(\xi_{\Gamma_{\acute{\imath}}(1)},\ldots,\xi_{\Gamma_{\acute{\imath}}(r)})$$

for any point $(\xi_1,\ldots,\xi_r) \in \mathcal{M}^r$ and $\acute{\imath} \in \{1,2,\ldots,r\}$. A point $\varkappa = (\varkappa_1,\ldots,\varkappa_r) \in \mathcal{M}^r$ is called Γ-*multiple fixed point* of G if it is a fixed point of ΓG [4], *i.e.*, if $\varkappa = \Gamma G(\varkappa)$ then

$$\varkappa_{\acute{\imath}} = G(\varkappa_{\Gamma_{\acute{\imath}}(1)},\ldots,\varkappa_{\Gamma_{\acute{\imath}}(r)}) \quad \text{for any } \acute{\imath} \in \{1,2,\ldots,r\}.$$

Consider $(\mathcal{M},\sigma)$ as a distance space, $r \in \mathbb{N}$, with mapping $G : \mathcal{M}^r \to \mathcal{M}$, let $\Gamma = \{\Gamma_{\acute{\imath}} : \{1,2,\ldots,r\} \to \{1,2,\ldots,r\} : 1 \le \acute{\imath} \le r\}$ be a collection of mappings. For any $\varkappa = (\varkappa_1,\ldots,\varkappa_r) \in \mathcal{M}^r$, $\varkappa(1) = \Gamma G(\varkappa)$ and $\varkappa(\vartheta + 1) = \Gamma G(\varkappa(\vartheta))$ for each $\vartheta \in \mathbb{N}$. Then $O(G,\Gamma,\varkappa) = \{\varkappa(\vartheta) : \vartheta \in \mathbb{N}\}$ is a Picard sequence.

Proposition 2 *Consider $(\mathcal{M},\sigma)$ as a C-distance space. Then*:

1. $\sigma(\xi,\eta) = 0$ *if and only if* $\xi = \eta$;
2. *If, for $\varkappa \in \mathcal{M}^r$, the Picard sequence $O(G,\Gamma,\varkappa) = \{\varkappa(\vartheta) : \vartheta \in \mathbb{N}\}$ is convergent Cauchy and its limit is a multiple fixed point of G with respect to the mappings Γ, i.e.*,

$$\omega_{\acute{\imath}} = G(\omega_{\Gamma_{\acute{\imath}}(1)},\ldots,\omega_{\Gamma_{\acute{\imath}}(r)}) \quad \textit{for each } \acute{\imath} \in \{1,2,\ldots,r\}.$$

We define contractive conditions which are similar to Kannan and Chatterjea contractions. The mapping G is called:

- Γ-*Kannan type contraction* if there exists $\delta \in [0,\frac{1}{2})$ such that

$$\begin{aligned}&\sigma\big(G(\vartheta_1,\ldots,\vartheta_r),G(\omega_1,\ldots,\omega_r)\big)\\ &\quad\le \delta\Big[\sup_{\acute{\imath}\le r}\big\{\sigma\big(\vartheta_{\acute{\imath}},G(\vartheta_1,\ldots,\vartheta_r)\big)\big\} + \sup_{\acute{\imath}\le r}\big\{\sigma\big(\omega_{\acute{\imath}},G(\omega_1,\ldots,\omega_r)\big)\big\}\Big]\end{aligned}$$

  for all $(\vartheta_1,\ldots,\vartheta_r),(\omega_1,\ldots,\omega_r) \in \mathcal{M}^r$ and $1 \le \acute{\imath} \le r$.

- Γ-*Chatterjea type contraction* if there exists $\delta \in [0, \frac{1}{2})$ such that

$$\begin{aligned}&\sigma\big(G(\vartheta_1,\ldots,\vartheta_r),G(\omega_1,\ldots,\omega_r)\big)\\&\quad\le\delta\Big[\sup_{\acute{\imath}\le r}\big\{\sigma\big(\vartheta_{\acute{\imath}},G(\omega_1,\ldots,\omega_r)\big)\big\}+\sup_{\acute{\imath}\le r}\big\{\sigma\big(\omega_{\acute{\imath}},G(\vartheta_1,\ldots,\vartheta_r)\big)\big\}\Big]\end{aligned}$$

for all $(\vartheta_1,\ldots,\vartheta_r),(\omega_1,\ldots,\omega_r)\in\mathcal{M}^r$ and $1\le\acute{\imath}\le r$.

3 Main results

Proposition 3 *Consider $(\mathcal{M},\sigma)$ as a distance space, $r\in N$, with a mapping $G:\mathcal{M}^r\to\mathcal{M}$, and let $\Gamma=\{\Gamma_{\acute{\imath}}:\{1,2,\ldots,r\}\to\{1,2,\ldots,r\}:1\le\acute{\imath}\le r\}$ be a collection of mappings, $\vartheta=(\vartheta_1,\ldots,\vartheta_r)\in\mathcal{M}^r$, $\omega=(\omega_1,\omega_2,\ldots,\omega_r)\in\mathcal{M}^r$. If there exists $\delta\in[0,\frac{1}{2})$ such that*

$$\begin{aligned}&\sigma\big(G(\vartheta_{\Gamma_{\acute{\imath}}(1)},\ldots,\vartheta_{\Gamma_{\acute{\imath}}(r)}),G(\omega_{\Gamma_{\acute{\imath}}(1)},\ldots,\omega_{\Gamma_{\acute{\imath}}(r)})\big)\\&\quad\le\delta\Big[\sup_{\acute{\imath}\le r}\big\{\sigma\big(\vartheta_{\acute{\imath}},G(\vartheta_{\Gamma_{\acute{\imath}}(1)},\ldots,\vartheta_{\Gamma_{\acute{\imath}}(r)})\big)\big\}+\sup_{\acute{\imath}\le r}\big\{\sigma\big(\omega_{\acute{\imath}},G(\omega_{\Gamma_{\acute{\imath}}(1)},\ldots,\omega_{\Gamma_{\acute{\imath}}(r)})\big)\big\}\Big]\end{aligned}$$

for each $1\le\acute{\imath}\le r$, then

$$\sigma^r\big(\Gamma G(\vartheta),\Gamma G(\omega)\big)\le\delta\big[\sigma^r\big(\vartheta,\Gamma G(\vartheta)\big)+\sigma^r\big(\omega,\Gamma G(\omega)\big)\big].$$

Proof Let us consider, for any $\acute{\imath}\le r$,

$$u_{\acute{\imath}}=G(\vartheta_{\Gamma_{\acute{\imath}}(1)},\ldots,\vartheta_{\Gamma_{\acute{\imath}}(r)})\quad\text{and}\quad v_{\acute{\imath}}=G(\omega_{\Gamma_{\acute{\imath}}(1)},\ldots,\omega_{\Gamma_{\acute{\imath}}(r)}),$$

then

$$u=(u_1,\ldots,u_r)=\Gamma G(\vartheta),\qquad v=(v_1,\ldots,v_r)=\Gamma G(\omega).$$

Now we have

$$\begin{aligned}\sigma^r\big(\Gamma G(\vartheta),\Gamma G(\omega)\big)&=\sigma^r(u,v)=\sup_{1\le\acute{\imath}\le r}\big\{\sigma(u_{\acute{\imath}},v_{\acute{\imath}})\big\}\\&=\sup_{1\le\acute{\imath}\le r}\big\{\sigma\big(G(\vartheta_{\Gamma_{\acute{\imath}}(1)},\ldots,\vartheta_{\Gamma_{\acute{\imath}}(r)}),G(\omega_{\Gamma_{\acute{\imath}}(1)},\ldots,\omega_{\Gamma_{\acute{\imath}}(r)})\big)\big\}\\&\le\sup_{\acute{\imath}\le r}\Big\{\delta\sup_{1\le j\le r}\big\{\sigma\big(\vartheta_j,G(\vartheta_{\Gamma_j(1)},\ldots,\vartheta_{\Gamma_j(r)})\big)\big\}\\&\quad+\delta\sup_{1\le j\le r}\big\{\sigma\big(\omega_j,G(\omega_{\Gamma_j(1)},\ldots,\omega_{\Gamma_j(r)})\big)\big\}\Big\}\\&=\delta\sup_{1\le\acute{\imath}\le r}\big\{\sigma(\vartheta_{\acute{\imath}},u_{\acute{\imath}})\big\}+\delta\sup_{1\le\acute{\imath}\le r}\big\{\sigma(\omega_{\acute{\imath}},v_{\acute{\imath}})\big\}\\&=\delta\sigma^r(\vartheta,u)+\delta\sigma^r(\omega,v)\\&=\delta\big[\sigma^r\big(\vartheta,\Gamma G(\vartheta)\big)+\sigma^r\big(\omega,\Gamma G(\omega)\big)\big].\end{aligned}$$

□

Remark Consider a distance space $(\mathcal{M},\sigma)$. If G is a Γ-Kannan type contraction, then ΓG is a Kannan contraction on the distance space $(\mathcal{M}^r,\sigma^r)$.

Proposition 4 *Consider* $(\mathcal{M},\sigma)$ *as a distance space,* $r \in \mathbb{N}$*, let* $G : \mathcal{M}^r \to \mathcal{M}$ *be an operator,* $\Gamma = \{\Gamma_{\acute{\imath}} : \{1,2,\ldots,r\} \to \{1,2,\ldots,r\} : 1 \le \acute{\imath} \le r\}$ *be a collection of mappings,* $\vartheta = (\vartheta_1,\ldots,\vartheta_r) \in \mathcal{M}^r$*,* $\omega = (\omega_1,\omega_2,\ldots,\omega_r) \in \mathcal{M}^r$*. If there exists* $\delta \in [0,\frac{1}{2})$ *such that*

$$\begin{aligned}&\sigma\bigl(G(\vartheta_{\Gamma_{\acute{\imath}}(1)},\ldots,\vartheta_{\Gamma_{\acute{\imath}}(r)}),G(\omega_{\Gamma_{\acute{\imath}}(1)},\ldots,\omega_{\Gamma_{\acute{\imath}}(r)})\bigr)\\&\quad\le\delta\Bigl[\sup_{\acute{\imath}\le r}\bigl\{\sigma\bigl(\vartheta_{\acute{\imath}},G(\omega_{\Gamma_{\acute{\imath}}(1)},\ldots,\omega_{\Gamma_{\acute{\imath}}(r)})\bigr)\bigr\}+\sup_{\acute{\imath}\le r}\bigl\{\sigma\bigl(\omega_{\acute{\imath}},G(\vartheta_{\Gamma_{\acute{\imath}}(1)},\ldots,\vartheta_{\Gamma_{\acute{\imath}}(r)})\bigr)\bigr\}\Bigr]\end{aligned}$$

for each $1 \le \acute{\imath} \le r$*, then*

$$\sigma^r\bigl(\Gamma G(\vartheta),\Gamma G(\omega)\bigr)\le\delta\bigl[\sigma^r\bigl(\vartheta,\Gamma G(\omega)\bigr)+\sigma^r\bigl(\omega,\Gamma G(\vartheta)\bigr)\bigr].$$

Proof The proof is similar to that of the above proposition. □

Remark Consider a distance space $(\mathcal{M},\sigma)$. If G is a Γ-Chatterjea type contraction, then ΓG is a Chatterjea contraction on the distance space $(\mathcal{M}^r,\sigma^r)$.

Now we give an example of C-distance space.

Example 3.1 Let $\mathcal{M} = \{\beta - \frac{1}{3\vartheta} : \vartheta \in \mathbb{N}\} \cup \{\beta\}$, where $\beta \ge 1$. For all $\xi,\eta \in Z$, where $Z = \{\beta - \frac{1}{3\vartheta} : \vartheta \in \mathbb{N}\}$, consider

$$\sigma(\xi,\beta)=\frac{1}{\vartheta^3+3\beta^2},\qquad\sigma(\beta,\xi)=\frac{1}{3\vartheta\beta^3},\qquad\sigma(\xi,\eta)=|\xi-\eta|.$$

Define a sequence $(\xi_\vartheta)_{\vartheta\in\mathbb{N}} = \beta - \frac{1}{3\vartheta}$ such that

$$\sigma(\beta,\xi_\vartheta)=\frac{1}{3\vartheta\beta^3},\qquad\sigma(\xi_\vartheta,\beta)=\frac{1}{\vartheta^3+3\beta^2},\qquad\sigma(\xi_\vartheta,\xi_l)=|\xi_\vartheta-\xi_l|$$

for all $l,\vartheta \in \mathbb{N}$. Now to show $(\xi_\vartheta)_{\vartheta\in\mathbb{N}}$ is convergent, consider

$$\sigma(\beta,\xi_\vartheta)+\sigma(\xi_\vartheta,\beta)=\frac{1}{3\vartheta\beta^3}+\frac{1}{\vartheta^3+3\beta^2}.$$

Applying limit $\vartheta \to \infty$, we get

$$\lim_{\vartheta\to\infty}\bigl[\sigma(\beta,\xi_\vartheta)+\sigma(\xi_\vartheta,\beta)\bigr]=0,$$

that is,

$$\lim_{\vartheta\to\infty}\sigma(\beta,\xi_\vartheta)\to0,\qquad\lim_{\vartheta\to\infty}\sigma(\xi_\vartheta,\beta)\to0.$$

This implies $(\xi_\vartheta)_{\vartheta\in\mathbb{N}}$ converges to β and

$$\lim_{l,\vartheta\to\infty}\sigma(\xi_\vartheta,\xi_l)=\lim_{l,\vartheta\to\infty}|\xi_\vartheta-\xi_l|=0$$

ensures that it is also Cauchy and the limit of a convergent Cauchy sequence is unique. Hence $(\mathcal{M},\sigma)$ is a C-distance space.

Remark

1. Every metric space is C-distance space but not conversely as already shown in the above example.
2. In a C-distance space, distance σ is not necessarily a continuous function.
3. In a C-distance space, a convergent sequence need not be Cauchy.

Theorem 3.1 *Consider $(\mathcal{M},\sigma)$ as a complete C-distance space, with a mapping $G:\mathcal{M}^r\to\mathcal{M}$. If G is a Γ- Kannan type contraction, then any Picard sequence of a self-mapping ΓG on $\mathcal{M}^r$ is Cauchy and G possesses a unique multidimensional fixed point.*

Proof Since G is a Γ-Kannan type contraction, then ΓG is a Kannan contraction on $(\mathcal{M}^r,\sigma^r)$ and $(\mathcal{M},\sigma)$ is a complete C-distance space, so $(\mathcal{M}^r,\sigma^r)$ is also a complete C-distance space. Let $\varkappa\in\mathcal{M}^r$ and

$$\varkappa(1)=\Gamma G(\varkappa),\qquad \ldots,\qquad \varkappa(\vartheta+1)=\Gamma G\bigl(\varkappa(\vartheta)\bigr).$$

Firstly, we need to show that $(\varkappa(\vartheta))_{\vartheta\in\mathbb{N}}$ is Cauchy, *i.e.*, $\lim_{\vartheta,l\to\infty}\sigma^r(\varkappa(\vartheta),\varkappa(l))\to 0$ and $\lim_{\vartheta,l\to\infty}\sigma^r(\varkappa(l),\varkappa(\vartheta))\to 0$.

For this, consider

$$\begin{aligned}\sigma^r\bigl(\varkappa(\vartheta-1),\varkappa(\vartheta)\bigr)&=\sigma^r\bigl(\Gamma G\bigl(\varkappa(\vartheta-2)\bigr),\Gamma G\bigl(\varkappa(\vartheta-1)\bigr)\bigr)\\&\le\delta\bigl[\sigma^r\bigl(\varkappa(\vartheta-2),\Gamma G\bigl(\varkappa(\vartheta-2)\bigr)\bigr)+\sigma^r\bigl(\varkappa(\vartheta-1),\Gamma G\bigl(\varkappa(\vartheta-1)\bigr)\bigr)\bigr],\\&\quad\text{where }\delta\in\left[0,\frac{1}{2}\right)\\&\le\delta\bigl[\sigma^r\bigl(\varkappa(\vartheta-2),\varkappa(\vartheta-1)\bigr)+\sigma^r\bigl(\varkappa(\vartheta-1),\varkappa(\vartheta)\bigr)\bigr].\end{aligned}$$

This implies

$$\begin{aligned}\sigma^r\bigl(\varkappa(\vartheta-1),\varkappa(\vartheta)\bigr)&\le\frac{\delta}{1-\delta}\sigma^r\bigl(\varkappa(\vartheta-2),\varkappa(\vartheta-1)\bigr)\\&=\frac{\delta}{1-\delta}\sigma^r\bigl(\Gamma G\bigl(\varkappa(\vartheta-3)\bigr),\Gamma G\bigl(\varkappa(\vartheta-2)\bigr)\bigr)\\&\le\delta\times\frac{\delta}{1-\delta}\bigl[\sigma^r(\varkappa(\vartheta-3),\Gamma G\bigl(\varkappa(\vartheta-3)\bigr)\\&\quad+\sigma^r\bigl(\varkappa(\vartheta-2),\Gamma G\bigl(\varkappa(\vartheta-2)\bigr)\bigr)\bigr]\\&\le\left(\frac{\delta}{1-\delta}\right)^2\sigma^r\bigl(\varkappa(\vartheta-3),\varkappa(\vartheta-2)\bigr)\\&\quad\vdots\\&\le\left(\frac{\delta}{1-\delta}\right)^{\vartheta-2}\sigma^r\bigl(\varkappa(1),\varkappa(2)\bigr).\end{aligned}$$

Applying limit $\vartheta\to\infty$, we get

$$\lim_{\vartheta\to\infty}\sigma^r\bigl(\varkappa(\vartheta-1),\varkappa(\vartheta)\bigr)=0.$$

Similarly, we can show that

$$\lim_{\vartheta\to\infty} \sigma^r\big(\varkappa(\vartheta), \varkappa(\vartheta-1)\big) = 0.$$

For $\vartheta > l$, consider

$$\begin{aligned}
&\sigma^r\big(\varkappa(\vartheta), \varkappa(l)\big) + \sigma^r\big(\varkappa(l), \varkappa(\vartheta)\big) \\
&\quad = \sigma^r\big(\Gamma G\big(\varkappa(\vartheta-1)\big), \Gamma G\big(\varkappa(l-1)\big)\big) + \sigma^r\big(\Gamma G\big(\varkappa(l-1)\big), \Gamma G\big(\varkappa(\vartheta-1)\big)\big) \\
&\quad \leq \delta\big[\sigma^r\big(\varkappa(\vartheta-1), \Gamma G\big(\varkappa(\vartheta-1)\big)\big) + \sigma^r\big(\varkappa(l-1), \Gamma G\big(\varkappa(l-1)\big)\big)\big] \\
&\qquad + \delta\big[\sigma^r\big(\varkappa(l-1), \Gamma G\big(\varkappa(l-1)\big)\big) + \sigma^r\big(\varkappa(\vartheta-1), \Gamma G\big(\varkappa(\vartheta-1)\big)\big)\big] \\
&\quad \leq \delta\big[\sigma^r\big(\varkappa(\vartheta-1), \varkappa(\vartheta)\big) + \sigma^r\big(\varkappa(l-1), \varkappa(l)\big)\big] \\
&\qquad + \delta\big[\sigma^r\big(\varkappa(l-1), \varkappa(l)\big) + \sigma^r\big(\varkappa(\vartheta-1), \varkappa(\vartheta)\big)\big].
\end{aligned}$$

Applying limit $l, \vartheta \to \infty$ over the above expression, we get

$$\lim_{\vartheta, l\to\infty} \big[\sigma^r\big(\varkappa(\vartheta), \varkappa(l)\big) + \sigma^r\big(\varkappa(l), \varkappa(\vartheta)\big)\big] = 0,$$

which implies $(\varkappa(\vartheta))_{\vartheta\in\mathbb{N}}$ is Cauchy. Since $(\mathcal{M}^r, \sigma^r)$ is complete, so a Cauchy sequence will converge and the limit of that sequence in a C-distance space is the multidimensional fixed point of G. Since that limit is unique, so a fixed point of the operator G will be unique. □

Example 3.2 Let $\mathcal{M} = \{\frac{1}{\vartheta} : \vartheta \in \mathbb{N}\} \cup \{0\}$. Define, for all $\vartheta, l \in \mathbb{N}$,

$$\sigma\left(0, \frac{1}{\vartheta}\right) = \frac{1}{3\vartheta}, \qquad \sigma\left(\frac{1}{\vartheta}, 0\right) = \frac{1}{4\vartheta}, \qquad \sigma(\xi_\vartheta, \xi_l) = |\xi_\vartheta - \xi_l|,$$

then $(\mathcal{M}, \sigma)$ is a C-distance space. Now

$$\mathcal{M} \times \mathcal{M} = \big\{(\xi, \eta) : \xi, \eta \in \mathcal{M}\big\}$$

and

$$\sigma^2(\xi, \eta) = \sup_{\acute{\imath}\leq 2}\big\{\sigma(\xi_{\acute{\imath}}, \eta_{\acute{\imath}})\big\},$$

then $(\mathcal{M}^2, \sigma^2)$ is also a C-distance space.

Now define a mapping $G : \mathcal{M}^2 \to \mathcal{M}$ such that

$$G(\xi_1, \xi_2) = \frac{\xi_1}{3} \quad \text{for all } (\xi_1, \xi_2) \in \mathcal{M}^2,$$

and a mapping $\Gamma : \mathcal{M} \to \mathcal{M}^2$ such that

$$\Gamma(\xi) = \big(\Gamma_1(\xi), \Gamma_2(\xi)\big),$$

where $\Gamma_{\acute{\imath}} : \{1,2\} \to \{1,2\}$ are defined as

$$\begin{pmatrix} \Gamma_1(1) & \Gamma_1(2) \\ \Gamma_2(1) & \Gamma_2(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The mapping $\Gamma G : \mathcal{M}^2 \to \mathcal{M}^2$, which is the composition of G and Γ, is defined as follows:

$$\Gamma G(\xi_1, \xi_2) = \bigl(G(\xi_{\Gamma_1(1)}, \xi_{\Gamma_2(2)}), G(\xi_{\Gamma_2(1)}, \xi_{\Gamma_2(2)})\bigr) = \left(\frac{\xi_1}{3}, \frac{\xi_2}{3}\right).$$

Consider

$$\sigma\bigl(G(\xi_1, \xi_2), G(\eta_1, \eta_2)\bigr) = \sigma\left(\frac{\xi_1}{3}, \frac{\eta_1}{3}\right).$$

We need to show that

$$\sigma\bigl(G(\xi_1, \xi_2), G(\eta_1, \eta_2)\bigr) \le \delta\Bigl[\sup_{\acute{\imath}\le 2}\bigl\{\sigma\bigl(\xi_{\acute{\imath}}, G(\xi_1, \xi_2)\bigr)\bigr\} + \sup_{\acute{\imath}\le 2}\bigl\{\sigma\bigl(\eta_{\acute{\imath}}, G(\eta_1, \eta_2)\bigr)\bigr\}\Bigr], \tag{A}$$

where $\delta \in [0, \frac{1}{2})$.

If $\xi = (0, \frac{1}{\vartheta_1})$ and $\eta = (\frac{1}{\vartheta_2}, 0)$, then

$$\sigma\left(G\left(0, \frac{1}{\vartheta_1}\right), G\left(\frac{1}{\vartheta_2}, 0\right)\right) = \sigma\left(0, \frac{1}{3\vartheta_2}\right) = \frac{1}{9\vartheta_2}. \tag{1}$$

Now consider

$$\begin{aligned}
&\sup_{\acute{\imath}\le 2}\bigl\{\sigma\bigl(\xi_{\acute{\imath}}, G(\xi_1, \xi_2)\bigr)\bigr\} + \sup_{\acute{\imath}\le 2}\bigl\{\sigma\bigl(\eta_{\acute{\imath}}, G(\eta_1, \eta_2)\bigr)\bigr\} \\
&\quad = \sup_{\acute{\imath}\le 2}\left\{\sigma\left(\xi_{\acute{\imath}}, \frac{\xi_1}{3}\right)\right\} + \sup_{\acute{\imath}\le 2}\left\{\sigma\left(\eta_{\acute{\imath}}, \frac{\eta_1}{3}\right)\right\} \\
&\quad = \sup\left\{\sigma\left(\xi_1, \frac{\xi_1}{3}\right), \sigma\left(\xi_2, \frac{\xi_1}{3}\right)\right\} + \sup\left\{\sigma\left(\eta_1, \frac{\eta_1}{3}\right), \sigma\left(\eta_2, \frac{\eta_1}{3}\right)\right\} \\
&\quad = \sup\left\{\sigma(0,0), \sigma\left(\frac{1}{\vartheta_1}, 0\right)\right\} + \sup\left\{\sigma\left(\frac{1}{\vartheta_2}, \frac{1}{3\vartheta_2}\right), \sigma\left(0, \frac{1}{3\vartheta_2}\right)\right\} \\
&\quad = \sup\left\{0, \frac{1}{4\vartheta_1}\right\} + \sup\left\{\frac{2}{3\vartheta_2}, \frac{1}{9\vartheta_2}\right\} \\
&\quad = \frac{1}{4\vartheta_1} + \frac{2}{3\vartheta_2}.
\end{aligned} \tag{2}$$

From (1) and (2) we get

$$\frac{1}{9\vartheta_2} < \frac{1}{3}\left(\frac{1}{4\vartheta_1} + \frac{2}{3\vartheta_2}\right),$$

that is,

$$\sigma\left(G\left(0, \frac{1}{\vartheta_1}\right), G\left(\frac{1}{\vartheta_2}, 0\right)\right) < \frac{1}{3}\left[\sup_{\acute{\imath}\le 2}\left\{\sigma\left(\xi_{\acute{\imath}}, G\left(0, \frac{1}{\vartheta_1}\right)\right)\right\} + \sup_{\acute{\imath}\le 2}\left\{\sigma\left(\eta_{\acute{\imath}}, G\left(\frac{1}{\vartheta_2}, 0\right)\right)\right\}\right].$$

Similarly, for other values of ξ and η, condition (A) is easily verified, so G is a Γ-Kannan type contraction. Then the mapping ΓG is Kannan contraction and it has a fixed point.

Now $\xi^{\vartheta} = \Gamma G(\xi^{\vartheta-1})$ for $\xi \in \mathcal{M}^2$. Choose $\xi = (\xi_1^0, \xi_2^0)$

$$\begin{aligned}
(\xi_1^1, \xi_2^1) &= \Gamma G(\xi_1^0, \xi_2^0) = \left(\frac{\xi_1^0}{3}, \frac{\xi_2^0}{3}\right), \\
(\xi_1^2, \xi_2^2) &= \Gamma G(\xi_1^1, \xi_2^1) = \left(\frac{\xi_1^0}{3^2}, \frac{\xi_2^0}{3^2}\right), \\
&\vdots \\
(\xi_1^{\vartheta}, \xi_2^{\vartheta}) &= \Gamma G(\xi_1^{\vartheta-1}, \xi_2^{\vartheta-1}) = \left(\frac{\xi_1^0}{3^{\vartheta}}, \frac{\xi_2^0}{3^{\vartheta}}\right).
\end{aligned}$$

Applying limit $\vartheta \to \infty$, we get

$$\lim_{\vartheta \to \infty} (\xi_1^{\vartheta}, \xi_2^{\vartheta}) = (0,0) = (O_1, O_2),$$

which is a unique fixed point for ΓG and a unique multidimensional fixed point for G, *i.e.*,

$$O_{\acute{\imath}} = G(O_{\Gamma_{\acute{\imath}}(1)}, O_{\Gamma_{\acute{\imath}}(2)}).$$

Theorem 3.2 *The mapping ΓG on a complete symmetric C-distance space $(\mathcal{M}, \sigma)$ satisfies generalized contraction if*

$$\begin{aligned}
&\sigma^r(\Gamma G(\xi), \Gamma G(\eta)) \le q \max\{\sigma^r(\xi, \eta), \sigma^r(\xi, \Gamma G(\xi)), \sigma^r(\eta, \Gamma G(\eta))\} \\
&\quad \text{for all } \xi, \eta \in \mathcal{M}^r, 0 < q < 1,
\end{aligned}$$

then any Picard sequence of a self-mapping ΓG on $\mathcal{M}^r$ is Cauchy and G possesses a unique multiple fixed point.

Proof Let $\varkappa \in \mathcal{M}^r$, $\varkappa(1) = \Gamma G(\varkappa)$, $\varkappa(\vartheta + 1) = \Gamma G(\varkappa(\vartheta))$. Consider

$$\begin{aligned}
\sigma^r(\varkappa(\vartheta), \varkappa(\vartheta+1)) &= \sigma^r(\Gamma G(\varkappa(\vartheta-1)), \Gamma G(\varkappa(\vartheta))) \\
&\le q \max\{\sigma^r(\varkappa(\vartheta-1)), \varkappa(\vartheta)), \sigma^r(\varkappa(\vartheta-1), \Gamma G(\varkappa(\vartheta-1)), \\
&\quad \sigma^r(\varkappa(\vartheta), \Gamma G(\varkappa(\vartheta))\}, \quad \text{where } q \in (0,1) \\
&\le q \max\{\sigma^r(\varkappa(\vartheta-1), \varkappa(\vartheta)), \sigma^r(\varkappa(\vartheta), \varkappa(\vartheta+1))\}.
\end{aligned}$$

If

$$\max\{\sigma^r(\varkappa(\vartheta-1), \varkappa(\vartheta)), \sigma^r(\varkappa(\vartheta), \varkappa(\vartheta+1))\} = \sigma^r(\varkappa(\vartheta), \varkappa(\vartheta+1)),$$

then

$$\sigma^r(\varkappa(\vartheta), \varkappa(\vartheta+1)) \le q\sigma^r(\varkappa(\vartheta), \varkappa(\vartheta+1)) < \sigma^r(\varkappa(\vartheta), \varkappa(\vartheta+1)),$$

which is not possible, so

$$\max\{\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)),\sigma^r(\varkappa(\vartheta),\varkappa(\vartheta+1))\}=\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)).$$

Thus

$$\begin{aligned}\sigma^r(\varkappa(\vartheta),\varkappa(\vartheta+1)) &\le q\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)).\\ &\le q^2\sigma^r(\varkappa(\vartheta-2),\varkappa(\vartheta-1)).\\ &\;\;\vdots\\ &\le q^{\vartheta-1}\sigma^r(\varkappa(1),\varkappa(2)).\end{aligned}$$

Hence

$$\lim_{\vartheta\to\infty}\sigma^r(\varkappa(\vartheta),\varkappa(\vartheta+1))=0. \tag{3}$$

Similarly, we can prove that

$$\lim_{\vartheta\to\infty}\sigma^r(\varkappa(\vartheta+1),\varkappa(\vartheta))=0. \tag{4}$$

Now, to show $(\varkappa(\vartheta))_{\vartheta\in\mathbb{N}}$ is a Cauchy sequence, for $l>\vartheta$, consider

$$\begin{aligned}&\sigma^r(\varkappa(\vartheta),\varkappa(l))\\ &\quad=\sigma^r(\Gamma G(\varkappa(\vartheta-1)),\Gamma G(\varkappa(l-1)))\\ &\quad\le q\max\{\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)),\sigma^r(\varkappa(\vartheta-1),\Gamma G(\varkappa(\vartheta-1))),\\ &\qquad\sigma^r(\varkappa(l-1),\Gamma G(\varkappa(l-1)))\},\quad\text{where } q\in(0,1)\\ &\quad\le q\max\{\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)),\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)),\sigma^r(\varkappa(l-1),\varkappa(l))\}.\end{aligned} \tag{5}$$

Now we consider the following cases.

Case 1: If

$$\begin{aligned}&\max\{\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)),\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)),\sigma^r(\varkappa(l-1),\varkappa(l))\}\\ &\quad=\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)),\end{aligned}$$

then inequality (5) gives

$$\sigma^r(\varkappa(\vartheta),\varkappa(l))\le q\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)).$$

Apply limit $l,\vartheta\to\infty$ over the above expression and using (3), we get

$$\lim_{l,\vartheta\to\infty}\sigma^r(\varkappa(\vartheta),\varkappa(l))=0.$$

Case 2: If

$$\max\{\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)),\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)),\sigma^r(\varkappa(l-1),\varkappa(l))\}$$
$$=\sigma^r(\varkappa(l-1),\varkappa(l)),$$

then inequality (5) implies

$$\sigma^r(\varkappa(\vartheta),\varkappa(l))\leq q\sigma^r(\varkappa(l-1),\varkappa(l)).$$

Again applying limit $l,\vartheta\to\infty$ over the above expression and using (3), we get

$$\lim_{l,\vartheta\to\infty}\sigma^r(\varkappa(\vartheta),\varkappa(l))=0.$$

Case 3: If

$$\max\{\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)),\sigma^r(\varkappa(\vartheta-1),\varkappa(\vartheta)),\sigma^r(\varkappa(l-1),\varkappa(l))\}$$
$$=\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)),$$

then

$$\sigma^r(\varkappa(\vartheta),\varkappa(l))\leq q\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)). \tag{6}$$

Consider

$$\begin{aligned}&\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1))\\&\quad=\sigma^r(\Gamma G(\varkappa(\vartheta-2)),\Gamma G(\varkappa(l-2)))\\&\quad\leq q\max\{\sigma^r(\varkappa(\vartheta-2),\varkappa(l-2)),\sigma^r(\varkappa(\vartheta-2),\Gamma G(\varkappa(\vartheta-2))),\\&\qquad\sigma^r(\varkappa(l-2),\Gamma G(\varkappa(l-2)))\},\quad\text{where } q\in(0,1)\\&\quad\leq q\max\{\sigma^r(\varkappa(\vartheta-2),\varkappa(l-2)),\sigma^r(\varkappa(\vartheta-2),\varkappa(\vartheta-1)),\sigma^r(\varkappa(l-2),\varkappa(l-1))\}.\end{aligned}\tag{7}$$

Now, if

$$\max\{\sigma^r(\varkappa(\vartheta-2),\varkappa(l-2)),\sigma^r(\varkappa(\vartheta-2),\varkappa(\vartheta-1)),\sigma^r(\varkappa(l-2),\varkappa(l-1))\}$$
$$=\sigma^r(\varkappa(\vartheta-2),\varkappa(\vartheta-1)),$$

then inequality (7) implies

$$\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1))\leq q\sigma^r(\varkappa(\vartheta-2),\varkappa(\vartheta-1)).$$

Applying lim $l,\vartheta\to\infty$ over the above expression and using (5), we get

$$\lim_{l,\vartheta\to\infty}\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1))=0.$$

If

$$\max\{\sigma^r(\varkappa(\vartheta-2),\varkappa(l-2)),\sigma^r(\varkappa(\vartheta-2),\varkappa(\vartheta-1)),\sigma^r(\varkappa(l-2),\varkappa(l-1))\}$$
$$= q\sigma^r(\varkappa(l-2),\varkappa(l-1)),$$

then (7) implies

$$\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)) \le q\sigma^r(\varkappa(l-2),\varkappa(l-1)).$$

Applying limit $l,\vartheta \to \infty$ over the above expression and using (3), we get

$$\lim_{l,\vartheta\to\infty}\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)) = 0.$$

If

$$\max\{\sigma^r(\varkappa(\vartheta-2),\varkappa(l-2)),\sigma^r(\varkappa(\vartheta-2),\varkappa(\vartheta-1)),\sigma^r(\varkappa(l-2),\varkappa(l-1))\}$$
$$= \sigma^r(\varkappa(\vartheta-2),\varkappa(l-2)),$$

then condition (7) gives

$$\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1)) \le q\sigma^r(\varkappa(\vartheta-2),\varkappa(l-2)). \tag{8}$$

Using (8) in (6) we get

$$\begin{aligned}\sigma^r(\varkappa(\vartheta),\varkappa(l)) &\le q\sigma^r(\varkappa(\vartheta-1),\varkappa(l-1))\\ &\le q^2\sigma^r(\varkappa(\vartheta-2),\varkappa(l-2))\\ &\le q^3\sigma^r(\varkappa(\vartheta-3),\varkappa(l-3))\\ &\quad\vdots\\ &\le q^{\vartheta-1}\sigma^r(\varkappa(1),\varkappa(l-\vartheta-2)).\end{aligned}$$

Applying limit $l,\vartheta \to \infty$, it follows

$$\lim_{l,\vartheta\to\infty}\sigma^r(\varkappa(\vartheta),\varkappa(l)) = 0.$$

Similarly, we can show that

$$\lim_{l,\vartheta\to\infty}\sigma^r(\varkappa(l),\varkappa(\vartheta)) = 0,$$

which implies $(\varkappa(\vartheta))_{\vartheta\in\mathbb{N}}$ is Cauchy. Because of the completeness of space, a Cauchy sequence will converge, and from the proposition, the limit of a convergent Cauchy sequence in a C-distance space is the multiple fixed point of the operator G. Since the space is C-distance space, *i.e.*, the limit of a convergent Cauchy sequence is unique, so a fixed point of the operator G will be unique. □

Assuming Γ_i as identity maps in the above result, we define the following corollary.

Corollary 1 *If a mapping G on a complete C- distance space* $(\mathcal{M},\sigma)$ *satisfies*

$$\sigma\big(G(\xi),G(\eta)\big)\le q\sigma^r(\xi,\eta)\quad \textit{for all } \xi,\eta\in\mathcal{M}^r, 0<q<1,$$

then G possesses a unique multiple fixed point.

Proof The proof of this result can be easily deduced by using the above theorem. □

Theorem 3.3 *If a self-mapping* ΓG *on a complete C-distance space* $(\mathcal{M},\sigma)$ *satisfies*

$$\psi\big(\sigma^r\big(\Gamma G(\xi),\Gamma G(\eta)\big)\big)\le\theta(M_{\xi,\eta})-\varphi(M_{\xi,\eta})\quad \textit{for all } \xi,\eta\in\mathcal{M}^r,$$

where

$$M_{\xi,\eta}=\max\big\{\sigma^r(\xi,\eta),\sigma^r\big(\xi,\Gamma G(\xi)\big),\sigma^r\big(\eta,\Gamma G(\eta)\big)\big\},$$

ψ *is defined in Definition 2.4,* $\theta:[0,\infty)\to[0,\infty)$ *is upper semicontinuous and* $\varphi:[0,\infty)\to[0,\infty)$ *is a lower semicontinuous function with* $\theta(0)=\varphi(0)=0$ *and*

$$\textit{for } \xi>0,\quad \psi(\xi)>\theta(\xi)-\varphi(\xi), \tag{9}$$

then a Picard sequence is Cauchy and G possesses a unique multiple fixed point.

Proof If $\varkappa(\vartheta)=\varkappa(\vartheta+1)$, then G has a fixed point. Suppose $\varkappa(\vartheta)\ne\varkappa(\vartheta+1)$. Let $\varkappa\in\mathcal{M}^r$, $\varkappa(1)=\Gamma G(\varkappa)$, $\varkappa(\vartheta+1)=\Gamma G(\varkappa(\vartheta))$.

To show $\sigma^r(\varkappa(\vartheta),\varkappa(\vartheta+1))$ is a decreasing sequence, consider

$$\begin{aligned}\psi\big(\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big)\big) &= \psi\big(\sigma^r\big(\Gamma G\big(\varkappa(\vartheta-1)\big),\Gamma G\big(\varkappa(\vartheta)\big)\big)\big)\\ &\le\theta(M_{\varkappa(\vartheta-1),\varkappa(\vartheta)})-\varphi(M_{\varkappa(\vartheta-1),\varkappa(\vartheta)}),\end{aligned}\tag{10}$$

where

$$\begin{aligned}M_{\varkappa(\vartheta-1),\varkappa(\vartheta)} &= \max\big\{\sigma^r\big(\varkappa(\vartheta-1),\varkappa(\vartheta)\big),\sigma^r\big(\varkappa(\vartheta-1),\Gamma G\big(\varkappa(\vartheta-1)\big)\big),\\ &\qquad \sigma^r\big(\varkappa(\vartheta),\Gamma G\big(\varkappa(\vartheta)\big)\big)\big\}\\ &= \max\big\{\sigma^r\big(\varkappa(\vartheta-1),\varkappa(\vartheta)\big),\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big)\big\}.\end{aligned}$$

If

$$M_{\varkappa(\vartheta-1),\varkappa(\vartheta)}=\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big),$$

then condition (10) implies

$$\begin{aligned}\psi\big(\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big)\big) &\le\theta\big(\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big)\big)-\varphi\big(\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big)\big)\\ &<\psi\big(\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big)\big),\end{aligned}$$

which is a contradiction. Thus we get

$$M_{\varkappa(\vartheta-1),\varkappa(\vartheta)} = \sigma^r\big(\varkappa(\vartheta-1),\varkappa(\vartheta)\big),$$

then condition (10) implies

$$\begin{aligned}\psi\big(\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big)\big) &\leq \theta\big(\sigma^r\big(\varkappa(\vartheta-1),\varkappa(\vartheta)\big)\big) - \varphi\big(\sigma^r\big(\varkappa(\vartheta-1),\varkappa(\vartheta)\big)\big) \\ &< \psi\big(\sigma^r\big(\varkappa(\vartheta-1),\varkappa(\vartheta)\big)\big). \end{aligned} \tag{11}$$

Since ψ is increasing

$$\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big) < \sigma^r\big(\varkappa(\vartheta-1),\varkappa(\vartheta)\big).$$

Hence $\sigma^r(\varkappa(\vartheta),\varkappa(\vartheta+1))$ is a decreasing sequence, so there exists $r \geq 0$ such that

$$\lim_{\vartheta\to\infty} \sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big) = r.$$

If $r > 0$ then apply limit $\vartheta \to \infty$ in inequality (11)

$$\psi(r) < \psi(r),$$

a contradiction, so $r = 0$. Hence

$$\lim_{\vartheta\to\infty} \sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big) = 0.$$

Similar steps can be followed to prove

$$\lim_{\vartheta\to\infty} \sigma^r\big(\varkappa(\vartheta+1),\varkappa(\vartheta)\big) = 0.$$

Hence

$$\lim_{\vartheta\to\infty} \big[\sigma^r\big(\varkappa(\vartheta),\varkappa(\vartheta+1)\big) + \sigma^r\big(\varkappa(\vartheta+1),\varkappa(\vartheta)\big)\big] = 0.$$

Now, to show

$$\lim_{l,\vartheta\to\infty} \big[\sigma^r\big(\varkappa(\vartheta),\varkappa(l)\big) + \sigma^r\big(\varkappa(l),\varkappa(\vartheta)\big)\big] = 0,$$

suppose that $(\varkappa(\vartheta))_{\vartheta\in N}$ is not Cauchy. Choose $\epsilon > 0$ and $\delta \in (0,\epsilon)$, there exist subsequences ϑ_p, l_p such that $l_p > \vartheta_p > p$ and

$$\sigma^r\big(\varkappa(\vartheta_p),\varkappa(l_p)\big) \geq \delta. \tag{12}$$

Let l_p be the smallest integer satisfying the above condition, then

$$\sigma^r\big(\varkappa(\vartheta_p),\varkappa(l_p-1)\big) < \delta. \tag{13}$$

Consider

$$\psi\big(\sigma^r\big(\varkappa(\vartheta_p),\varkappa(l_p)\big)\big) = \psi\big(\sigma^r\big(\Gamma G\big(\varkappa(\vartheta_p-1)\big),\Gamma G\big(\varkappa(l_p-1)\big)\big)\big) \\ \leq \theta(M_{\varkappa(\vartheta_p-1),\varkappa(l_p-1)}) - \varphi(M_{\varkappa(\vartheta_p-1),\varkappa(l_p-1)}),$$

where

$$M_{\varkappa(\vartheta_p-1),\varkappa(l_p-1)} = \max\big\{\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(l_p-1)\big),\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(\vartheta_p)\big), \\ \sigma^r\big(\varkappa(l_p-1),\varkappa(l_p)\big)\big\}.$$

If

$$M_{\varkappa(\vartheta_p-1),\varkappa(l_p-1)} = \sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(\vartheta_p)\big),$$

then

$$\psi\big(\sigma^r\big(\varkappa(\vartheta_p),\varkappa(l_p)\big)\big) \leq \theta\big(\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(\vartheta_p)\big)\big) - \varphi\big(\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(\vartheta_p)\big)\big) \\ < \psi\big(\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(\vartheta_p)\big)\big).$$

Since ψ is nondecreasing, so

$$\sigma^r\big(\varkappa(\vartheta_p),\varkappa(l_p)\big) < \sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(\vartheta_p)\big).$$

Apply limit $l,\vartheta\to\infty$, it follows

$$\lim_{l,\vartheta\to\infty}\sigma^r\big(\varkappa(\vartheta_p),\varkappa(l_p)\big) < \lim_{\vartheta\to\infty}\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(\vartheta_p)\big) = 0,$$

a contradiction. Now, if

$$M_{\varkappa(\vartheta_p-1),\varkappa(l_p-1)} = \sigma^r\big(\varkappa(l_p-1),\varkappa(l_p)\big),$$

then a similar argument as above leads to contradiction. Thus

$$M_{\varkappa(\vartheta_p-1),\varkappa(l_p-1)} = \sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(l_p-1)\big),$$

then

$$\psi\big(\sigma^r\big(\varkappa(\vartheta_p),\varkappa(l_p)\big)\big) \\ \leq \theta\big(\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(l_p-1)\big)\big) - \varphi\big(\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(l_p-1)\big)\big) \\ < \psi\big(\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(l_p-1)\big)\big).$$

Consider

$$\psi\big(\sigma^r\big(\varkappa(\vartheta_p-1),\varkappa(l_p-1)\big)\big) \\ = \psi\big(\sigma^r\big(\Gamma G\big(\varkappa(\vartheta_p-2)\big),\Gamma G\big(\varkappa(l_p-2)\big)\big)\big) \\ \leq \theta(M_{\varkappa(\vartheta_p-2),\varkappa(l_p-2)}) - \varphi(M_{\varkappa(\vartheta_p-2),\varkappa(l_p-2)}),$$

where

$$M_{\varkappa(\vartheta_p-2),\varkappa(l_p-2)} = \max\bigl\{\sigma^r\bigl(\varkappa(\vartheta_p-2),\varkappa(l_p-2)\bigr),\sigma^r\bigl(\varkappa(\vartheta_p-2),\varkappa(\vartheta_p-1)\bigr),\\ \sigma^r\bigl(\varkappa(l_p-2),\varkappa(l_p-1)\bigr)\bigr\}.$$

If

$$M_{\varkappa(\vartheta_p-2),\varkappa(l_p-2)} = \sigma^r\bigl(\varkappa(\vartheta_p-2),\varkappa(\vartheta_p-1)\bigr)$$

or

$$M_{\varkappa(\vartheta_p-2),\varkappa(l_p-2)} = \sigma^r\bigl(\varkappa(l_p-2),\varkappa(l_p-1)\bigr),$$

both lead to contradiction. The only option we are left with is

$$M_{\varkappa(\vartheta_p-2),\varkappa(l_p-2)} = \sigma^r\bigl(\varkappa(\vartheta_p-2),\varkappa(l_p-2)\bigr).$$

It follows that

$$\begin{aligned}&\psi\bigl(\sigma^r\bigl(\varkappa(\vartheta_p-1),\varkappa(l_p-1)\bigr)\bigr)\\ &\quad\le\theta\bigl(\sigma^r\bigl(\varkappa(\vartheta_p-2),\varkappa(l_p-2)\bigr)\bigr)-\varphi\bigl(\sigma^r\bigl(\varkappa(\vartheta_p-2),\varkappa(l_p-2)\bigr)\bigr)\\ &\quad<\psi\bigl(\sigma^r\bigl(\varkappa(\vartheta_p-2),\varkappa(l_p-2)\bigr)\bigr).\end{aligned}$$

Continuing this process, it follows that

$$\psi\bigl(\sigma^r\bigl(\varkappa(\vartheta_p),\varkappa(l_p)\bigr)\bigr)<\psi\bigl(\sigma^r\bigl(\varkappa(1),\varkappa\bigl(l_p-(\vartheta_p-1)\bigr)\bigr)\bigr).$$

Since ψ is nondecreasing, so

$$\sigma^r\bigl(\varkappa(\vartheta_p),\varkappa(l_p)\bigr)<\sigma^r\bigl(\varkappa(1),\varkappa\bigl(l_p-(\vartheta_p-1)\bigr)\bigr).$$

Applying limit $p\to\infty$ and using (12) and (13) in the above inequality results in

$$\delta\le\lim_{p\to\infty}\sigma^r\bigl(\varkappa(\vartheta_p),\varkappa(l_p)\bigr)<\lim_{p\to\infty}\sigma^r\bigl(\varkappa(1),\varkappa\bigl(l_p-(\vartheta_p-1)\bigr)\bigr)<\delta,$$

which is a contradiction. So

$$\lim_{p\to\infty}\sigma^r\bigl(\varkappa(\vartheta_p),\varkappa(l_p)\bigr)=0.$$

Similarly,

$$\lim_{p\to\infty}\sigma^r\bigl(\varkappa(l_p),\varkappa(\vartheta_p)\bigr)=0,$$

then we have

$$\lim_{p\to\infty}\bigl[\sigma^r\bigl(\varkappa(\vartheta_p),\varkappa(l_p)\bigr)+\sigma^r\bigl(\varkappa(l_p),\varkappa(\vartheta_p)\bigr)\bigr]=0.$$

Hence $(\varkappa(\vartheta))_{\vartheta\in\mathbb{N}}$ is Cauchy, and since the space is complete, there will be some $\nu \in \mathcal{M}^r$ such that

$$\lim_{\vartheta\to\infty} \sigma^r\bigl(\varkappa(\vartheta), \nu\bigr) = 0 \quad \text{and} \quad \lim_{\vartheta\to\infty} \sigma^r\bigl(\nu, \varkappa(\vartheta)\bigr) = 0.$$

In a C-distance space, the limit of a convergent Cauchy sequence is a multidimensional fixed point of G, and the uniqueness of limit ensures that G possesses a unique multiple fixed point. □

Corollary 2 *If a mapping* ΓG *on a complete* C*-distance space* $(\mathcal{M}, \sigma)$ *satisfies*

$$\psi\bigl(\sigma^r\bigl(\Gamma G(\xi), \Gamma G(\eta)\bigr)\bigr) \le \psi(M_{\xi,\eta}) - \varphi(M_{\xi,\eta}) \quad \textit{for all } \xi, \eta \in \mathcal{M}^r,$$

where

$$M_{\xi,\eta} = \max\bigl\{\sigma^r(\xi, \eta), \sigma^r\bigl(\xi, \Gamma G(\xi)\bigr), \sigma^r\bigl(\eta, \Gamma G(\eta)\bigr)\bigr\},$$

ψ *defined in Definition 2.4,* $\varphi : [0,\infty) \to [0,\infty)$ *a lower semicontinuous function with* $\varphi(0) = 0$ *and*

$$\textit{for } \xi > 0, \quad \varphi(\xi) > 0,$$

then a Picard sequence of a self-mapping ΓG *on* $\mathcal{M}^r$ *is Cauchy and* G *possesses a multiple fixed point, which will be unique.*

4 An application

Theory of differential and integral equations is arising with fundamental tools such as fixed point theory, functional analysis, and topology. Most of the problems of applied mathematics are reduced to finding fixed points of certain mappings. For solving various problems of integral calculus, researchers have tried to generalize contractive conditions, mappings, and metric spaces, see [6–11, 17].

Let $a, b \in \mathbb{R}$ with $a < b$, and let $I = [a, b]$. Consider $\mathcal{M} = C(I)$, the space of all continuous real-valued functions defined on I, with the distance

$$\sigma(\alpha, \beta) = \max_{t\in I} \bigl|\alpha(t) - \beta(t)\bigr|^p \quad \text{for all } \alpha, \beta \in \mathcal{M}, p > 1,$$

then $(\mathcal{M}, \sigma)$ is a complete C-distance space.

Consider the following system of equations:

$$\begin{aligned} \eta_1(t) &= \omega + \int_a^t J\bigl(\eta_1(u), \eta_2(u), \ldots, \eta_r(u)\bigr)\,du, \\ \eta_i(t) &= \omega + \int_a^t J\bigl(\eta_i(u), \eta_{i+1}(u), \ldots, \eta_r(u), \eta_1(u), \ldots, \eta_{i-1}(u)\bigr)\,du \end{aligned} \tag{14}$$

for $i = 1, 2, \ldots, r$, where η_is are elements of $\mathcal{M}$, $u \in I$ and $J : \mathbb{R}^r \longrightarrow \mathbb{R}$ is a mapping verifying:

(i) J is continuous;

(ii) for all $(x_1, x_2, \dots, x_r), (y_1, y_2, \dots, y_r) \in \mathbb{R}^r$,

$$\left|J(x_1, x_2, \dots, x_r) - J(y_1, y_2, \dots, y_r)\right| \le k\Big(\max_{1\le i\le r} \lambda_i |x_i - y_i|^p\Big)^{\frac{1}{p}},$$

where $k, \lambda_1, \lambda_2, \dots, \lambda_r$ are real numbers.

Define a mapping $G : \mathcal{M}^r \to \mathcal{M}$ for all $\eta = (\eta_1, \eta_2, \dots, \eta_r)$ in $\mathcal{M}^r$ and $\omega \in \mathbb{R}$ such that

$$G(\eta_1, \eta_2, \dots, \eta_r)(t) = \omega + \int_a^t J\big(\eta_1(u), \eta_2(u), \dots, \eta_r(u)\big)\, du.$$

Clearly, $G \in C(I)$. Now, to prove the existence of solution of system (14), we need to prove that G has a multiple fixed point.

For $(\eta_1, \eta_2, \dots, \eta_r), (\xi_1, \xi_2, \dots, \xi_r) \in \mathcal{M}^r$, consider

$$\begin{aligned}
&\sigma\big(G(\eta_1, \eta_2, \dots, \eta_r), G(\xi_1, \xi_2, \dots, \xi_r)\big) \\
&\quad = \max_{t\in I}\left|G(\eta_1, \eta_2, \dots, \eta_r)(t) - G(\xi_1, \xi_2, \dots, \xi_r)(t)\right|^p \\
&\quad = \max_{t\in I}\left|\bigg(\omega + \int_a^t J\big(\eta_1(u), \eta_2(u), \dots, \eta_r(u)\big)\, du\bigg)\right. \\
&\qquad \left. - \bigg(\omega + \int_a^t J\big(\xi_1(u), \xi_2(u), \dots, \xi_r(u)\big)\, du\bigg)\right|^p \\
&\quad = \max_{t\in I}\left|\int_a^t \Big(J\big(\eta_1(u), \eta_2(u), \dots, \eta_r(u)\big) - J\big(\xi_1(u), \xi_2(u), \dots, \xi_r(u)\big)\Big)\, du\right|^p \\
&\quad \le \max_{t\in I}\bigg(\int_a^t \left|J\big(\eta_1(u), \eta_2(u), \dots, \eta_r(u)\big) - J\big(\xi_1(u), \xi_2(u), \dots, \xi_r(u)\big)\right| du\bigg)^p \\
&\quad \le \max_{t\in I}\bigg(\int_a^t \Big(k \max_{1\le i\le r} \lambda_i\Big(\max_{u\in I}\left|\eta_i(u) - \xi_i(u)\right|^p\Big)\Big)^{\frac{1}{p}}\, du\bigg)^p \\
&\quad = k\lambda \max_{t\in I}\bigg(\int_a^t \big(\sigma(\eta_i, \xi_i)\big)^{\frac{1}{p}}\, du\bigg)^p \\
&\quad \le k\lambda\sigma(\eta_i, \xi_i)(b-a)^p \\
&\quad \le k\lambda \sup_{1\le i\le r} \sigma(\eta_i, \xi_i)(b-a)^p \\
&\quad \le k\lambda\sigma^r\big((\eta_1, \eta_2, \dots, \eta_r), (\xi_1, \xi_2, \dots, \xi_r)\big)(b-a)^p
\end{aligned}$$

with $\lambda = \max_{1\le i\le r} \lambda_i$. If $k\lambda(b-a)^p < 1$, then by Corollary 1 G has a unique multiple fixed point, which is a solution of system (14).

Acknowledgements
Not applicable.

Funding
None.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 April 2020 Accepted: 1 October 2020 Published online: 12 October 2020

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