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The Nehari manifold method for discrete fractional p-Laplacian equations

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Abstract

The aim of this paper is to investigate the multiplicity of homoclinic solutions for a discrete fractional difference equation. First, we give a variational framework to a discrete fractional p-Laplacian equation. Then two nontrivial and nonnegative homoclinic solutions are obtained by using the Nehari manifold method.

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Keywords: Discrete fractional p-Laplacian; Homoclinic solutions; Nehari manifold

1 Introduction and main result

Denote by $\mathbb{Z}$ the set of whole integers and let T be a positive real number. Set

$$-\Delta_T u(j) = \frac{1}{T^2}\big[u\big((j+1)T\big) - 2u(jT) + u\big((j-1)T\big)\big]$$

for $u : \mathbb{Z} \to \mathbb{R}$. The well-known second order difference equation

$$-\Delta_T u(j) + V(j)u(j) = f\big(j, u(j)\big) \quad \text{in } \mathbb{Z} \tag{1.1}$$

can be regarded as the discrete version of the Schrödinger type equation, which can be used to describe a planetary system or an electron in an electromagnetic field. Here potential function $V : \mathbb{Z} \to [0, \infty)$ and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$. Particularly, homoclinic orbits play a very important role in studying the dynamics of discrete Schrödinger equations. In recent tears, second order difference equations and homoclinic orbits have been the research focus. The literature on such a field is very rich, we collect some papers; see, for example, [2, 7, 18–20, 22, 23, 30]. Especially, Agarwal, Perera and O'Regan in [2] first considered the existence of solutions for second order difference equations like (1.1) by using variational methods.

Recently, Ciaurri *et al.* in [10] considered the following discrete fractional Laplace equation:

$$(-\Delta_T)^s u = f, \tag{1.2}$$

where $(-\Delta_T)^s$ is the so-called discrete fractional Laplacian given by

$$(-\Delta_T)^s u(j) = \frac{1}{\Gamma(-s)} \int_0^\infty \bigl(e^{t\Delta T} u(j) - u(j)\bigr) \frac{dt}{t^{1+2s}}.$$

Here $s \in (0,1)$, Γ is the Gamma function and $v(t,j) = e^{t\Delta_T} u(j)$ is the solution of the following problem:

$$\begin{cases} \partial_t v(t,j) = \Delta_T v(t,j), & \text{in } \mathbb{Z}_T \times (0,\infty), \\ v(0,j) = u(j), & \text{on } \mathbb{Z}_T, \end{cases}$$

where $\mathbb{Z}_T = \{Tj : j \in \mathbb{Z}\}$.

Set

$$\mathcal{L}_s = \left\{ u : \mathbb{Z}_T \to \mathbb{R} \,\Big|\, \sum_{k\in\mathbb{Z}} \frac{|u(k)|}{(1+|k|)^{1+2s}} < \infty \right\}$$

and

$$\mathcal{K}_s^T(k) = \frac{4^s \Gamma(1/2+s)}{\sqrt{\pi}\,|\Gamma(-s)|} \cdot \frac{\Gamma(|k|-s)}{T^{2s}\Gamma(|k|+1+s)}$$

for any $k \in \mathbb{Z} \setminus \{0\}$ and $\mathcal{K}_s^T(0) = 0$. Then by [10, Theorem 1.1]

$$(-\Delta_T)^s u(j) = \sum_{k\in\mathbb{Z}, k\neq j} \bigl(u(j) - u(k)\bigr) \mathcal{K}_s^T(j-k),$$

provided $u \in \mathcal{L}_s$. As showed in [10, Theorem 1.1], there exist positive constants $c_s \leq C_s$ such that

$$\frac{c_s}{T^{2s}|j|^{1+2s}} \leq \mathcal{K}_s^T(j) \leq \frac{C_s}{T^{2s}|j|^{1+2s}},$$

for any $j \in \mathbb{Z} \setminus \{0\}$. An interesting result is that $\lim_{s\to 1^-} (-\Delta_T)^s u(j) = -\Delta_T u(j)$ if u is bounded. In particular, [10] stated that the solutions of (1.2) converge to the solutions of following fractional Laplacian problem:

$$(-\Delta)^s u = f \quad \text{in } \mathbb{R}.$$

Here $(-\Delta)^s$ is the fractional Laplacian defined for any $x \in \mathbb{R}$ as

$$(-\Delta)^s v(x) = \mathcal{C}(s) \lim_{R\to 0^+} \int_{\mathbb{R}\setminus B_R(x)} \frac{v(x) - v(y)}{|x-y|^{1+2s}}\, dy$$

along any $v \in C_0^\infty(\mathbb{R})$, where $B_R(x) = (x-R, x+R)$ and $\mathcal{C}(s) > 0$ is a constant. For further details about the fractional Laplacian and fractional Sobolev spaces, we refer to [12]. The numerical analysis of fractional difference equations is difficulty, since the discrete fractional Laplace operator is nonlocal and singular; see for example [1, 17] and the references cited therein. In [35], Xiang and Zhang first studied the following discrete fractional Laplacian equation:

$$\begin{cases} (-\Delta_T)^s u(k) + V(k)u(k) = \lambda f(k,u(k)) & \text{in } \mathbb{Z}, \\ u(k) \to 0 & \text{as } |k| \to \infty, \end{cases} \tag{1.3}$$

where $s \in (0,1)$, $V : \mathbb{Z} \to (0,\infty)$, $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to the second variable and satisfies asymptotically linear growth at infinity. Under some suitable hypotheses, two solutions were obtained by using the mountain pass theorem and Ekeland's variational principle.

Recently, the study of fractional Laplacian and related problems has been received an increasing amount of attention. The fractional Laplacian appears in many fields, such as anomalous diffusion, quantum mechanics, finance, optimization and game theory; see [4, 8, 21, 31] and the references therein. For the applications of fractional operators, we refer to [3, 5, 9, 11, 14–16, 24–29, 33, 34, 36, 37] and the references therein.

Motivated by above papers, we study the following nonlinear discrete fractional p-Laplacian equation:

$$\begin{cases} (-\Delta_d)^s_p u(k) + V(k)|u(k)|^{p-2}u(k) \\ \quad = \lambda a(k)|u(k)|^{q-2}u(k) + b(k)|u(k)|^{r-2}u(k) & \text{for } k \in \mathbb{Z}, \\ u(k) \to 0 & \text{as } |k| \to \infty, \end{cases} \tag{1.4}$$

where $s \in (0,1)$, $V : \mathbb{Z} \to (0,\infty)$, $1 < q < p$, $a \in \ell^{\frac{p}{p-q}}$, $p < r < \infty$, $b \in \ell^\infty$ and $(-\Delta_d)^s_p$ is defined as follows: for each $j \in \mathbb{Z}$,

$$(-\Delta_d)^s_p u(j) = 2 \sum_{m \in \mathbb{Z}, m \neq j} \bigl|u(j) - u(k)\bigr|^{p-2}\bigl(u(j) - u(m)\bigr)K_{s,p}(j-m).$$

Here the discrete kernel $K_{s,p}$ satisfies the requirement that there exist constants $0 < c_{s,p} \leq C_{s,p} < \infty$ such that

$$\begin{cases} \frac{c_{s,p}}{|j|^{1+ps}} \leq K_{s,p}(j) \leq \frac{C_{s,p}}{|j|^{1+ps}} & \text{for any } j \in \mathbb{Z} \setminus \{0\}; \\ K_{s,p}(0) = 0. \end{cases} \tag{1.5}$$

Note that when $p = 2$ the discrete fractional p-Laplacian $(-\Delta_d)^s_p$ reduces to $(-\Delta_T)^s$ with $T = 1$. As usual, we say that a function $u : \mathbb{Z} \to \mathbb{R}$ is a homoclinic solution of Eq. (1.4) if $u(k) \to 0$ as $|k| \to \infty$.

In this paper we always assume that the function V satisfies

(V) there exists $V_0 > 0$ such that $V(k) \geq V_0 > 0$ for all $k \in \mathbb{Z}$, and $V(k) \to \infty$ as $|k| \to \infty$.

Set $M_b = \sup_{k \in \mathbb{Z}} b(k)$ and

$$\Lambda_0 = \|a\|^{-1}_{\frac{p}{p-q}} V_0^{\frac{q}{p}} \left(\frac{(p-q)}{(r-q)M_b V_0^{-\frac{r}{p}}} \right)^{\frac{p-q}{r-p}} \frac{r-p}{r-q}.$$

Theorem 1.1 *Assume that V satisfies (V), $1 < q < p < r < \infty$, $a \in \ell^{\frac{p}{p-q}}$ and $0 \leq b \in \ell^{\infty}$. Then for all $0 < \lambda < \Lambda_0$ Eq. (1.4) admits at least two nontrivial and nonnegative homoclinic solutions.*

To the best of our knowledge, our paper is the first time of use of the Nehari manifold method to study the multiplicity of solutions for discrete fractional p-Laplacian equations. It is worth mentioning that the weight function a may change sign in this paper. But for the case that both a and b are sign changing functions, the existence of two solutions is still an open problem. The authors will consider the case in the further.

The paper is organized as follows. In Sect. 2, we present a variational framework to Eq. (1.4) and show some basic results. In Sect. 3, we give the definitions of Nehari manifold and fibering map. Moreover, some properties of the fibering map are given. In Sect. 4, using the Nehari manifold method, we obtain two distinct nontrivial and nonnegative homoclinic solutions of Eq. (1.4).

2 Variational setting and preliminaries

In this section, we first recall some basic definitions, which can be found in [13, 19, 35]. Then we introduce a variational framework to Eq. (1.4) and discuss its properties. For any $1 \leq \nu < \infty$, we define ℓ^{ν} as

$$\ell^{\nu} := \left\{ u : \mathbb{Z} \to \mathbb{R} \;\middle|\; \sum_{j\in\mathbb{Z}} \left|u(j)\right|^{\nu} < \infty \right\},$$

with the norm

$$\|u\|_{\nu} = \left(\sum_{j\in\mathbb{Z}} \left|u(j)\right|^{\nu} \right)^{1/\nu}.$$

Set

$$\|u\|_{\infty} := \sup_{j\in\mathbb{Z}} \left|u(j)\right| < \infty.$$

Define

$$\ell^{\infty} = \left\{ u : \mathbb{Z} \to \mathbb{R} | \|u\|_{\infty} < \infty \right\}.$$

Then $(\ell^{\nu}, \|\cdot\|_{\nu})$ and $(\ell^{\infty}, \|\cdot\|_{\infty})$ are Banach spaces; see [13]. Clearly, $\ell^{\nu_1} \subset \ell^{\nu_2}$ if $1 \leq \nu_1 \leq \nu_2 \leq \infty$. From now on, we shortly denote by $\|\cdot\|_{\nu}$ the norm of ℓ^{ν} for all $\nu \in [1, \infty]$.

For interval $I \subset \mathbb{R}$, we define

$$\ell_I^{\nu} := \left\{ u : I \to \mathbb{R} \;\middle|\; \sum_{j\in I} \left|u(j)\right|^{\nu} < \infty \right\}.$$

Define W as

$$W = \left\{ u : \mathbb{Z} \to \mathbb{R} \;\middle|\; \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|u(j) - u(k)\right|^{p} K_{s,p}(j-k) + \sum_{j\in\mathbb{Z}} V(j) \left|u(j)\right|^{p} < \infty \right\}.$$

Equip W with the norm

$$\|u\|_W = \left([u]_{s,p}^p + \sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^p \right)^{1/p},$$

where

$$[u]_{s,p} := \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|u(j) - u(k)\right|^p K_{s,p}(j-m) \right)^{1/p}.$$

Lemma 2.1 *If $u \in \ell^p$, then $[u]_{s,p} < \infty$. Moreover, there exists $\mathcal{C} > 0$ only depending on s and p such that $[u]_{s,p} \leq \mathcal{C}\|u\|_p$ for all $u \in \ell^p$.*

Proof The proof is similar to [32]. Let $u \in \ell^p$. Then

$$\begin{aligned}
[u]_{s,p}^p &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|u(j) - u(k)\right|^p K_{s,p}(j-m) \\
&\leq 2^{p-1} C_{s,p} \sum_{j\in\mathbb{Z}} \sum_{k\neq j} \frac{|u(j)|^p + |u(k)|^p}{|j-k|^{1+ps}} \\
&= 2^{p-1} C_{s,p} \sum_{k\neq 0} \frac{|u(0)|^p + |u(k)|^p}{|k|^{1+ps}} + 2^{p-1} C_{s,p} \left(\sum_{j\neq 0} \sum_{k\neq 0} \frac{|u(j)|^p}{|k|^{1+ps}} \right) \\
&\quad + 2^{p-1} C_{s,p} \left(\sum_{j\neq 0} \sum_{k\neq 0} \frac{|u(k+j)|^p}{|k|^{1+ps}} \right) \\
&= 2^{p-1} C_{s,p} \sum_{k\neq 0} \frac{|u(0)|^p + |u(k)|^p}{|k|^{1+ps}} + 2^{p-1} C_{s,p} \left(\sum_{k\neq 0} \sum_{j\neq 0} \frac{|u(j)|^p}{|k|^{1+ps}} \right) \\
&\quad + 2^{p-1} C_{s,p} \left(\sum_{k\neq 0} \sum_{j\neq 0} \frac{|u(k+j)|^p}{|k|^{1+ps}} \right) \\
&\leq 32^{p-1} C_{s,p} \sum_{k\neq 0} \frac{1}{|k|^{1+ps}} \sum_{j\in\mathbb{Z}} \left|u(j)\right|^p \\
&= \mathcal{C}^p \sum_{j\in\mathbb{Z}} \left|u(j)\right|^p,
\end{aligned}$$

where $0 < \mathcal{C} = (32^{p-1} C_{s,p} \sum_{k\neq 0} \frac{1}{|k|^{1+2s}})^{1/p} < \infty$. Therefore, the proof is complete. □

Lemma 2.2 *The norm*

$$\|u\| := \left(\sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^p \right)^{1/p}$$

is an equivalent norm of W. Moreover, $(W, \|\cdot\|_W)$ is a Banach space.

Proof The proof is similar to [32], for completeness, we give its details. Using assumption (V) and Lemma 2.1, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^p \le \|u\|_W^p &\le C\sum_{j\in\mathbb{Z}}\left|u(j)\right|^p + \sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^p \\ &\le C\frac{1}{V_0}\sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^p + \sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^p \\ &= C\sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^p,\end{aligned}$$

which leads to $\|u\| = (\sum_{j\in\mathbb{Z}} V(j)|u(j)|^p)^{1/p}$ being an equivalent norm of W.

Finally we show that $(W, \|\cdot\|_W)$ is complete. Let $\{v_n\}_n$ be a Cauchy sequence in W. Observe that

$$\|u\|_p \le V_0^{-\frac{1}{p}}\|u\|$$

for all $u \in W$. Then $\{v_n\}_n$ is also a Cauchy sequence in ℓ^p. By the completeness of ℓ^p, there exists $u \in \ell^p$ such that $v_n \to u$ in ℓ^p. Furthermore, Lemma 2.1 and assumption (V) show that $v_n \to u$ strongly in W as $n \to \infty$.

In conclusion, the proof is complete. □

Moreover, we have the following compactness result.

Lemma 2.3 *Assume (V). Then the embedding $W \hookrightarrow \ell^\nu$ is compact for any $p \le \nu < \infty$.*

Proof The proof is similar to that in [19] and [35]. We first show that the result holds for the case $\nu = p$. It follows from assumption (V) that

$$\|u\|_p \le V_0^{-\frac{1}{p}}\|u\| \quad \text{for all } u \in W,$$

which shows that the embedding $W \hookrightarrow \ell^p$ is continuous.

Next we prove that $W \hookrightarrow \ell^p$ is compact. Let $\{v_n\}_n \subset W$ and assume that there exists $D > 0$ such that $\|v_n\|_W^p \le D$ for all $n \in \mathbb{N}$. Now we show that $\{v_n\}_n$ strongly converges to some function in ℓ^p. Using the reflexivity of W, there exist a subsequence of $\{v_n\}_n$ still denoted by $\{v_n\}_n$ and function $u \in W$ such that $v_n \rightharpoonup u$ in W. By assumption (V), for any $\delta > 0$ there exists $j_0 \in \mathbb{N}$ such that for all $|j| > j_0$

$$V(j) > \frac{1+D}{\delta}.$$

Set $I = [-j_0, j_0]$ and define

$$W_I := \left\{u : I \to \mathbb{R} \,\middle|\, \sum_{j\in I}\sum_{j\neq k\in I}\left|u(j)-u(k)\right|^p K_{s,p}(j-k) + \sum_{j\in I} V(j)\left|u(j)\right|^p < \infty\right\}.$$

Observe that the dimension of W_I is finite. Then $\{v_n\}_n$ is a bounded sequence in W_I, due to which yields $\{v_n\}_n$ is bounded in ℓ_I^p. Thus, up to a subsequence we may assume that

$v_n \to u$ on I. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\sum_{j\in I} |v_n(j) - u(j)|^p \leq \frac{\delta}{1+D}.$$

Then, for all $n > n_0$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |v_n(j) - u(j)|^p &< \frac{\delta}{1+D} + \frac{\delta}{1+D} \sum_{|j|>j_0} V(j)|v_n(j) - u(j)|^p \\ &\leq \frac{\delta}{1+D}\left(1 + \|v_n\|_W^p\right) \leq \delta.\end{aligned}$$

Thus, we deduce that $v_n \to u$ in ℓ^p.

Now we consider the case $\nu > p$. Note that

$$\|u(j)\|_\infty \leq \left(\sum_{j\in\mathbb{Z}} |u(j)|^p\right)^{1/p}$$

for all $u \in \ell^p$. Then

$$\begin{aligned}\left(\sum_{j\in\mathbb{Z}} |u(j)|^\nu\right)^{1/\nu} &= \|u\|_\infty \left(\sum_{j\in\mathbb{Z}} \left(\frac{|u(j)|}{\|u\|_\infty}\right)^r\right)^{1/\nu} \\ &\leq \|u\|_\infty \left(\sum_{j\in\mathbb{Z}} \left(\frac{|u(j)|}{\|u\|_\infty}\right)^p\right)^{1/\nu} \\ &= \|u\|_\infty^{1-\frac{p}{\nu}} \left(\sum_{j\in\mathbb{Z}} |u(j)|^p\right)^{1/\nu} \\ &\leq \|u\|_p^{1-\frac{p}{r}} \|u\|_p^{\frac{p}{\nu}} \\ &= \|u\|_p\end{aligned}$$

for all $u \in \ell^p \setminus \{0\}$. Thus,

$$\|u\|_\nu \leq \|u\|_p$$

for all $u \in \ell^p$. This inequality together with the result of the case $\nu = p$ leads to the proof. □

To obtain some properties of energy functional associated with Eq. (1.4), we need the following result.

Lemma 2.4 *Assume that U is a compact subset of W. Then for any $\delta > 0$ there is a $j_0 \in \mathbb{N}$ such that*

$$\left[\sum_{|j|>j_0} V(j)|u(j)|^p\right]^{1/p} < \delta \quad \textit{for any } u \in U.$$

Proof The proof can be found in [32]. □

For each $u \in W$, we define the associated energy functional with Eq. (1.4) as

$$I_\lambda(u) = \Psi(u) - F(u),$$

where

$$\Psi(u) = \frac{1}{p}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|u(j)-u(m)\right|^p K_{s,p}(j-m) + \frac{1}{p}\sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^p$$

and

$$F(u) = \sum_{j\in\mathbb{Z}}\left(\lambda\frac{a(j)}{q}\left|u(j)\right|^q + \frac{b(j)}{r}\left|u(j)\right|^r\right).$$

Lemma 2.5 *If V satisfies (V), then Ψ is well-defined, of class $C^1(W,\mathbb{R})$ and*

$$\begin{aligned}\left\langle\Psi'(u),v\right\rangle &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|u(j)-u(m)\right|^{p-2}\big(u(j)-u(m)\big)\big(v(j)-v(m)\big)K_{s,p}(j-m)\\ &\quad+\sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^{p-2}u(j)v(j),\end{aligned}$$

for all $u, v \in W$.

Proof By Lemma 2.1, we know that Ψ is well-defined on W. Fix $u, v \in W$. We first prove that

$$\begin{aligned}&\lim_{t\to 0^+}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p}{p}K_{s,p}(j-m)\\ &\quad=\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|u(j)-u(m)\right|^{p-2}\big(u(j)-u(m)\big)\big(v(j)-v(m)\big)K_{s,p}(j-m).\end{aligned} \tag{2.1}$$

Choose $C > 0$ such that $\|u\|_W, \|v\|_W \le C$. For any $\varepsilon > 0$ there exists $h_1 \in \mathbb{N}$ such that

$$\left(\sum_{|j|>h}\sum_{|m|>h}\left|u(j)-u(m)\right|^p K_{s,p}(j-m)\right)^{\frac{1}{p}} < \varepsilon \tag{2.2}$$

for all $h > h_1$. Indeed, for any $h \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{|j|>h}\sum_{|m|>h}\left|u(j)-u(m)\right|^p K_{s,p}(j-m) &\le C_{s,p}2^{p-1}\sum_{|j|>h}\sum_{|m|>h,m\neq j}\frac{(|u(j)|^p+|u(m)|^p)}{|j-m|^{1+ps}}\\ &\le 2^p C_{s,p}\sum_{|j|>h}\sum_{|m|>h,m\neq j}\frac{|u(j)|^p}{|j-m|^{1+ps}}\\ &\le 2^p C_{s,p}\left(\sum_{k\neq 0}\frac{1}{|k|^{1+ps}}\right)\sum_{|j|>h}\left|u(j)\right|^p.\end{aligned}$$

It follows from $u \in W$ that (2.2) holds. For $h \in \mathbb{N}$, if $|j| \leq h$ and $|m| > 2h$, then $|j-m| \geq |m|-|j| \geq |m|-h > \frac{|m|}{2}$. Thus,

$$\begin{aligned}
&\sum_{|j|\leq h}\sum_{|m|>2h,m\neq j}\frac{|u(j)|^p}{|j-m|^{1+ps}}\\
&\quad\leq \sum_{|j|\leq h}\sum_{|m|>2h,m\neq j}\frac{2^{1+ps}|u(j)|^p}{|m|^{1+ps}}\\
&\quad\leq 2^{1+ps}\Bigl(\sum_{|j|\leq h}\bigl|u(j)\bigr|^p\Bigr)\sum_{|m|>2h}\frac{1}{|m|^{1+ps}}.
\end{aligned}$$

Then there exists $h_2 \in \mathbb{N}$ such that

$$\Bigl(\sum_{|j|\leq h}\sum_{|m|>2h}\bigl|u(j)-u(m)\bigr|^p K_{s,p}\Bigr)^{\frac{1}{p}} < \varepsilon \tag{2.3}$$

for all $h > h_2$. Fix $h > \max\{h_1.h_2\}$. Clearly, there exists $t_0 \in (0,1)$ such that for all $0 < t < t_0$

$$\begin{aligned}
&\sum_{|j|\leq 2h}\sum_{|m|\leq 2h}\biggl|\frac{|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p}{p}\\
&\qquad-\bigl|u(j)-u(m)\bigr|^{p-2}\bigl(u(j)-u(m)\bigr)\bigl(v(j)-v(m)\bigr)\biggr|K_{s,p}(j-m)\\
&\quad<\varepsilon.
\end{aligned}$$

Fix $0 < t < t_0$. For $j,m \in \mathbb{Z}$, by the mean value theorem, we can choose $0 < t_{j,m} < t$ such that

$$\begin{aligned}
&\frac{(|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p)}{tp}K_{s,p}(j-m)\\
&\quad=\bigl|y(j)-y(m)\bigr|^{p-2}\bigl(y(j)-y(m)\bigr)\bigl(v(j)-v(m)\bigr)K_{s,p}(j-m)
\end{aligned}\tag{2.4}$$

where $y(j) = u(j) + t_{j,m}v(j)$. Clearly, $y \in W$ and $\|y\|_W \leq 2C$. Observe that

$$\begin{aligned}
&\biggl|\sum_{|j|\leq h}\sum_{|m|>2h}\bigl|u(j)-u(m)\bigr|^{p-2}\bigl(u(j)-u(m)\bigr)\bigl(v(j)-v(m)\bigr)K_{s,p}\biggr|\\
&\quad\leq \sum_{|j|\leq h}\sum_{|m|>wh}\bigl|u(j)-u(m)\bigr|^{p-1}\bigl|v(j)-v(m)\bigr|K_{s,p}\\
&\quad\leq \Bigl(\sum_{|j|\leq h}\sum_{|m|>2h}\bigl|u(j)-u(m)\bigr|^p K_{s,p}\Bigr)^{\frac{p-1}{p}}\Bigl(\sum_{|j|\leq h}\sum_{|m|>2h}\bigl|v(j)-v(m)\bigr|^p K_{s,p}\Bigr)^{\frac{1}{p}} \leq C\varepsilon.
\end{aligned}\tag{2.5}$$

By Hölder's inequality and (2.2)–(2.5),

$$\begin{aligned}
&\biggl|\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p}{p}K_{s,p}(j-m)\\
&\qquad-\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\bigl|u(j)-u(m)\bigr|^{p-2}\bigl(u(j)-u(m)\bigr)\bigl(v(j)-v(m)\bigr)K_{s,p}(j-m)\biggr|
\end{aligned}$$

$$
\begin{aligned}
&\leq \varepsilon + \sum_{|j|\leq h}\sum_{|m|>h} + \sum_{|j|>h}\sum_{|m|\leq h} \\
&\quad + \sum_{|j|>h}\sum_{|m|>h} \bigl|\bigl(\phi_p\bigl(y(j)-y(m)\bigr) - \phi_p\bigl(u(j)-u(m)\bigr)\bigr)\bigl(v(j)-v(m)\bigr)\bigr| K_{s,p}(j-m) \\
&\leq C\varepsilon + \sum_{|j|\leq h}\sum_{|m|>2h} + \sum_{|j|>2h}\sum_{|m|\leq h} \\
&\quad + \sum_{|j|>h}\sum_{|m|>h} \bigl|\bigl(\phi_p\bigl(y(j)-y(m)\bigr) - \phi_p\bigl(u(j)-u(m)\bigr)\bigr)\bigl(v(j)-v(m)\bigr)\bigr| K_{s,p}(j-m) \\
&\leq C\varepsilon,
\end{aligned}
$$

where $\phi_p(\tau) := |\tau|^{p-2}\tau$ for all $\tau \in \mathbb{R}$. Thus, (2.1) holds true. An analogous argument gives

$$
\lim_{t\to 0^+} \frac{\|u+tv\|^p - \|u\|^p}{pt} = \sum_{j\in\mathbb{Z}} V(j)\bigl|u(j)\bigr|^{p-2} u(j)v(j).
$$

Thus, we get

$$
\begin{aligned}
\bigl\langle \Psi'(u), v\bigr\rangle = {}& \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \bigl|u(j)-u(m)\bigr|^{p-2}\bigl(u(j)-u(m)\bigr)\bigl(v(j)-v(m)\bigr)K_{s,p}(j-m) \\
&+ \sum_{j\in\mathbb{Z}} V(j)\bigl|u(j)\bigr|^{p-2} u(j)v(j).
\end{aligned}
$$

Thus, Ψ is Gâteaux differentiable in W. Finally, we prove that $\Psi' : W \to W^*$ is continuous. To this aim, we assume that $\{u_n\}_n$ is a sequence in W such that $u_n \to u$ in W as $n \to \infty$. By Lemma 2.4, for any $\varepsilon > 0$ there exists $h \in \mathbb{N}$ such that

$$
\Bigl(\sum_{|j|>h}\sum_{|m|>h} \bigl|u_n(j)-u_n(m)\bigr|^p K_{s,p}(j-m)\Bigr)^{1/p} < \varepsilon \quad \text{for all } n \in \mathbb{N}
$$

and

$$
\Bigl(\sum_{|j|>h}\sum_{|m|>h} \bigl|u(j)-u(m)\bigr|^p K_{s,p}(j-m)\Bigr)^{1/p} < \varepsilon.
$$

In addition, there exists $n_0 \in \mathbb{N}$ such that

$$
\Bigl(\sum_{|j|\leq 2h}\sum_{|m|\leq 2h} \bigl|\bigl[\phi\bigl(u_n(j)-u_n(m)\bigr) - \phi\bigl(u(j)-u(m)\bigr)\bigr]K_{s,p}^{\frac{1}{p'}}(j-m)\bigr|^{p'}\Bigr)^{1/p'} < \varepsilon
$$

for all $n \geq n_0$, where $p' = \frac{p}{p-1}$. For any $v \in W$ with $\|v\|_W \leq 1$, and for any $n \geq n_0$, by the Hölder inequality and a similar discussion to above, we deduce

$$
\begin{aligned}
&\Bigl|\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \bigl[\phi\bigl(u_n(j)-u_n(m)\bigr) - \phi\bigl(u(j)-u(m)\bigr)\bigr]\bigl(v(j)-v(m)\bigr)K_{s,p}(j-m)\Bigr| \\
&\quad \leq \Bigl(\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \bigl|\bigl[\phi\bigl(u_n(j)-u_n(m)\bigr) - \phi\bigl(u(j)-u(m)\bigr)\bigr]K_{s,p}^{\frac{1}{p'}}(j-m)\bigr|^{p'}\Bigr)^{1/p'}
\end{aligned}
$$

$$\times \left(\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left| v(j) - v(m) \right|^p K_{s,p}(j-m) \right)^{1/p}$$
$$\le C\varepsilon \|v\|_W.$$

Similarly, one can show that

$$\left| \sum_{k\in\mathbb{Z}} V(k)\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)v \right| \le C\varepsilon \|v\|_W$$

as $n \to \infty$. Thus,

$$\left\| \Psi'(u_n) - \Psi'(u) \right\| = \sup_{\|v\|\le 1} \left| \left\langle \Psi'(u_n) - \Psi'(u), v \right\rangle \right| \to 0.$$

This means that Ψ' is continuous.

Consequently, we prove that $\Psi \in C^1(W, \mathbb{R})$. □

Lemma 2.6 *Assume that V satisfies (V), $1 < q < p < r < \infty$, $a \in \ell^{\frac{p}{p-q}}$ and $0 \le b \in \ell^\infty$. Then $F \in C^1(W, \mathbb{R})$ with*

$$\left\langle F'(u), v \right\rangle = \sum_{j\in\mathbb{Z}} \left(\lambda a(j)\left|u(j)\right|^{q-2} u(j)v(j) + b(j)\left|u(j)\right|^{r-2} u(j)v(j) \right)$$

for all $u, v \in W$.

Proof Using the same discussion as [19] and [35], one can prove the lemma. □

Gathering Lemma 2.5 and Lemma 2.6, we know that $I_\lambda \in C^1(W, \mathbb{R})$.

Lemma 2.7 *Assume that V satisfies (V), $1 < q < p < r < \infty$, $a \in \ell^{\frac{p}{p-q}}$ and $0 \le b \in \ell^\infty$. Then a critical point of I_λ is a homoclinic solution of Eq. (1.4) for all $\lambda > 0$.*

Proof Let $u \in W$ be a critical point of I_λ, that is, $I'_\lambda(u) = 0$. Then

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left| u(j) - u(m) \right|^{p-2} \left(u(j) - u(m) \right)\left(v(j) - v(m) \right) K_{s,p}(j-m) \\ &\quad + \sum_{j\in\mathbb{Z}} V(j)\left|u(j)\right|^{p-2} u(j)v(j) \\ &= \lambda \sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^{q-2} u(j)v(j) + \sum_{j\in\mathbb{Z}} b(j)\left|u(j)\right|^{r-2} u(j)v(j) \end{aligned} \tag{2.6}$$

for all $v \in W$. For each $k \in \mathbb{Z}$, we define γ_k as

$$\gamma_k(j) := \begin{cases} 1, & j = k; \\ 0, & j \ne k. \end{cases}$$

Clearly, $\gamma_k \in W$. Choosing $v = \gamma_k$ in (2.6), we obtain

$$\begin{aligned} &2\sum_{j\neq k}\left|u(k)-u(j)\right|^{p-2}\left(u(k)-u(j)\right)K_{s,p}(k-j)+V(k)\left|u(k)\right|^{p-2}u(k) \\ &\quad=\lambda a(k)\left|u(k)\right|^{q-2}u(k)+b(k)\left|u(k)\right|^{r-2}u(k), \end{aligned}$$

which means that u is a solution of (1.4). Obviously, $u(k)\to 0$ as $|k|\to\infty$, this means that u is a homoclinic solution of (1.4). □

3 Nehari manifold and Fibering map analysis

In this section, we give some definitions and properties of Nehari manifold. Some ideas are inspired from [6] and [32]. In present section, we always assume V satisfies (V), $a \in \ell^{\frac{p}{p-q}}$ and $0 \leq b \in \ell^{\infty}$.

Define the Nehari manifold as follows:

$$\mathcal{N}_\lambda = \left\{u \in W\backslash\{0\} | \left\langle I'_\lambda(u), u\right\rangle = 0\right\}.$$

Obviously,

$$\begin{aligned} &u \in \mathcal{N}_\lambda \quad \text{if and only if} \quad u \in W\setminus\{0\} \quad \text{and} \\ &\|u\|_W^p = \lambda\sum_{j\in\mathbb{Z}}a(j)\left|u(j)\right|^q+\sum_{j\in\mathbb{Z}}b(j)\left|u(j)\right|^r. \end{aligned}$$

For each $u \in W$, we define the fibering map $\Phi_{\lambda,u} : (0,\infty) \to \mathbb{R}$ as

$$\begin{aligned} \Phi_{\lambda,u}(t) &= I_\lambda(tu) \\ &= \frac{t^p}{p}\|u\|_W^p - \frac{t^q}{q}\lambda\sum_{j\in\mathbb{Z}}a(j)\left|u(j)\right|^q \\ &\quad - \frac{t^r}{r}\sum_{j\in\mathbb{Z}}b(j)\left|u(j)\right|^r, \end{aligned}$$

for all $t > 0$. Then a simple calculation yields

$$\Phi'_{\lambda,u}(t) = t^{p-1}\|u\|_W^p - \lambda t^{q-1}\sum_{j\in\mathbb{Z}}a(j)\left|u(j)\right|^q - t^{r-1}\sum_{j\in\mathbb{Z}}b(j)\left|u(j)\right|^r$$

and

$$\begin{aligned} \Phi''_{\lambda,u}(t) &= (p-1)t^{p-2}\|u\|_W^p - (q-1)t^{q-2}\lambda\sum_{j\in\mathbb{Z}}a(j)\left|u(j)\right|^q \\ &\quad - (r-1)t^{r-2}\sum_{j\in\mathbb{Z}}b(j)\left|u(j)\right|^r. \end{aligned}$$

In particular, if $u \in \mathcal{N}_\lambda$, then

$$\Phi'_{\lambda,u}(1) = 0 \quad \text{and} \quad \Phi''_{\lambda,u}(1) = (p-q)\lambda\sum_{j\in\mathbb{Z}}a(j)\left|u(j)\right|^q + (p-r)\sum_{j\in\mathbb{Z}}b(j)\left|u(j)\right|^r.$$

Since 1 may be a minimum point, maximum point, or saddle point of $\Phi_{\lambda,u}$, we divide $\mathcal{N}_\lambda$ into three subsets $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$, which are defined respectively as

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \Phi''_{\lambda,u}(1) > 0\},\\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \Phi''_{\lambda,u}(1) = 0\},\\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \Phi''_{\lambda,u}(1) < 0\}.
\end{aligned}$$

Lemma 3.1 *Suppose that $u \in W \setminus \{0\}$ and $t > 0$. Then $tu \in \mathcal{N}_\lambda$ if and only if $\Phi'_{\lambda,u}(t) = 0$.*

Proof Let $tu \in \mathcal{N}_\lambda$. Then

$$t^p \|u\|_W^p - \lambda t^q \sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q - t^r \sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^r = 0,$$

which leads to $t\Phi'_{\lambda,u}(t) = 0$. Therefore, we can prove that $tu \in \mathcal{N}_\lambda$ if and only if $\Phi'_{\lambda,u}(t) = 0$. □

Lemma 3.2 *If u is a local minimizer of I_λ on $\mathcal{N}_\lambda$ and $u \notin \mathcal{N}_\lambda^0$, then $I'_\lambda(u) = 0$.*

Proof The proof is similar to that in [6]; see also [32]. For completeness, we give its proof. Assume that u is a local minimizer of I_λ on $\mathcal{N}_\lambda$. By Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that

$$I'_\lambda(u) = \mu J'(u),$$

where $J(u)$ is given by

$$J(u) = \|u\|_W^p - \lambda \sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q - \sum_{j\in\mathbb{Z}} b(j)\left|u(j)\right|^r.$$

Since $u \in \mathcal{N}_\lambda$, we deduce $\langle I'_\lambda(u), u\rangle = 0$. Hence, $\mu\langle J'(u), u\rangle = 0$. It follows from $u \notin \mathcal{N}_\lambda^0$ that

$$\left\langle J'(u), u\right\rangle = \Phi''_{\lambda,u}(1) \neq 0.$$

Consequently, $\mu = 0$. Furthermore, we obtain $I'_\lambda(u) = 0$. □

Lemma 3.3 *The functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof Let $u \in \mathcal{N}_\lambda$. Then

$$I_\lambda(u) = \left(\frac{1}{p} - \frac{1}{r}\right)\|u\|_W^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q.$$

It follows from $q < p < r$ and the Hölder inequality that

$$\begin{aligned} I_\lambda(u) &\geq \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|_W^p - \lambda\left(\frac{1}{q}-\frac{1}{r}\right)\left(\sum_{j\in\mathbb{Z}}\left|a(j)\right|^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\left(\sum_{j\in\mathbb{Z}}\left|u(j)\right|^p\right)^{\frac{q}{p}} \\ &\geq \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|_W^p - \lambda\left(\frac{1}{q}-\frac{1}{r}\right)\left(\sum_{j\in\mathbb{Z}}\left|a(j)\right|^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} V_0^{-\frac{q}{p}}\left(\sum_{j\in\mathbb{Z}}V(k)\left|u(j)\right|^p\right)^{\frac{q}{p}} \\ &\geq \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|_W^p - \lambda\left(\frac{1}{q}-\frac{1}{r}\right)\left(\sum_{j\in\mathbb{Z}}\left|a(j)\right|^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} V_0^{-\frac{q}{p}}\|u\|_W^q, \end{aligned}$$

which shows that I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. □

Lemma 3.4 *For any $u \in W \setminus \{0\}$, we have the following results.*

(1) *If $\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q > 0$, then there exist $0 < t_1 < t_{\max} < t_2 < \infty$ such that $\Phi'_{\lambda,u}(t_1) = \Phi'_{\lambda,u}(t_2) = 0$ and $t_1 u \in \mathcal{N}^+_{\lambda,u}$ and $t_2 u \in \mathcal{N}^-_{\lambda,u}$.*

(2) *If $\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q < 0$, there exists a unique $t_u > 0$ such that $\Phi'_{\lambda,u}(t_u) = 0$ and $t_u u \in \mathcal{N}^-_\lambda$.*

Proof For all $t > 0$, we define

$$g(t) := t^{p-q}\|u\|_W^p - t^{r-q}\sum_{j\in\mathbb{Z}} b(j)\left|u(j)\right|^r.$$

By direct calculation one obtains that

$$g'(t) = t^{p-q-1}\left((p-q)\|u\|_W^p - t^{r-p}(r-q)\sum_{j\in\mathbb{Z}} b(j)\left|u(j)\right|^r\right).$$

Note that

$$\Phi'_{\lambda,u}(t) = t^{p-1}\|u\|_W^p - \lambda t^{q-1}\sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q - t^{r-1}\sum_{j\in\mathbb{Z}} b(j)\left|u(j)\right|^r.$$

Then

$$\Phi'_{\lambda,u}(t) = t^{q-1}\left(g(t) - \lambda\sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q\right).$$

By $\sum_{j\in\mathbb{Z}} b(j)|u(j)|^r > 0$, one can verify that $g \in C^1(0,\infty)$, $\lim_{t\to 0^+} g(t) = 0$ and $\lim_{t\to\infty} g(t) = -\infty$. Thus g has a unique maximum point $t_{\max} > 0$ and

$$t_{\max} = \left(\frac{(p-q)\|u\|_W^p}{(r-q)\sum_{j\in\mathbb{Z}} b(j)|u(j)|^r}\right)^{\frac{1}{r-p}}.$$

Moreover, we know that g is decreasing on $(0, t_{\max})$ and increasing on $(t_{\max}, \infty)$. Then

$$\begin{aligned} g(t_{\max}) &= t_{\max}^{p-q}\frac{r-p}{r-q}\|u\|_W^p \\ &\geq \left(\frac{(p-q)\|u\|_W^p}{(r-q)M_b V_0^{-\frac{r}{p}}\|u\|_W^r}\right)^{\frac{p-q}{r-p}}\frac{r-p}{r-q}\|u\|_W^p \\ &= \left(\frac{(p-q)}{(r-q)M_b V_0^{-\frac{r}{p}}}\right)^{\frac{p-q}{r-p}}\frac{r-p}{r-q}\|u\|_W^q. \end{aligned}$$

If $\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q > 0$, then we deduce from the Hölder inequality that

$$\begin{aligned} \lambda\sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q &\leq \|a\|_{\frac{p}{p-q}}\|u\|_p^q \\ &\leq \lambda\|a\|_{\frac{p}{p-q}}V_0^{-\frac{q}{p}}\|u\|_W^q \\ &\leq \lambda\|a\|_{\frac{p}{p-q}}V_0^{-\frac{q}{p}}\left(\frac{(p-q)}{(r-q)M_b V_0^{-\frac{r}{p}}}\right)^{-\frac{p-q}{r-p}}\frac{r-q}{r-p}g(t_{\max}). \end{aligned}$$

Since

$$\lambda < \|a\|_{\frac{p}{p-q}}^{-1}V_0^{\frac{q}{p}}\left(\frac{(p-q)}{(r-q)M_b V_0^{-\frac{r}{p}}}\right)^{\frac{p-q}{r-p}}\frac{r-p}{r-q},$$

we have $\lambda\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q < g(t_{\max})$. Thus there exist t_1, t_2 satisfying $0 < t_1 < t_{\max} < t_2 < \infty$ such that

$$g(t_1) = g(t_2) = \lambda\sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q,$$

which means that $\Phi'_{\lambda,u}(t_1) = \Phi'_{\lambda,u}(t_2) = 0$. Moreover, since g is increasing on $(0, t_{\max})$ and decreasing on $(t_{\max}, \infty)$, we have $t_1 u \in \mathcal{N}_\lambda^+$ and $t_2 u \in \mathcal{N}_\lambda^-$.

If $\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q < 0$, there exists a unique $t_u > t_{\max}$ such that $g(t_u) = \lambda\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q$ and $t_u u \in \mathcal{N}_\lambda^-$. □

Remark 3.1 Lemma 3.4 implies that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are non-empty sets.

Lemma 3.5 *If $\lambda \in (0, \lambda_0)$, then $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof Let $u \in \mathcal{N}_\lambda^0$. Then

$$\|u\|_W^p - \lambda\sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q - \sum_{j\in\mathbb{Z}} b(j)\left|u(j)\right|^r = 0 \tag{3.1}$$

and

$$(p-1)\|u\|_W^p - (q-1)\lambda\sum_{j\in\mathbb{Z}} a(j)\left|u(j)\right|^q - (r-1)\sum_{j\in\mathbb{Z}} b(j)\left|u(j)\right|^r = 0. \tag{3.2}$$

Multiplying (3.1) by $(r-1)$ and inserting it in (3.2), we have

$$\begin{aligned}(r-p)\|u\|_W^p &= \lambda(r-q)\sum_{j\in\mathbb{Z}} a(j)\big|u(j)\big|^q \\ &\le \lambda(r-q)\|a\|_{\frac{p}{p-q}}\|u\|_p^q \\ &\le \lambda(r-q)\|a\|_{\frac{p}{p-q}} V_0^{-\frac{q}{p}}\|u\|_W^q.\end{aligned}$$

Thus,

$$\|u\|_W \le \lambda\left(\left(\frac{r-q}{r-p}\right)\|a\|_{\frac{p}{p-q}} V_0^{-\frac{q}{p}}\right)^{\frac{1}{p-q}}. \tag{3.3}$$

On the other hand, multiplying (3.1) by $(q-1)$ and inserting it in (3.2), we get

$$(p-q)\|u\|_W^p = (r-q)\sum_{j\in\mathbb{Z}} b(j)\big|u(j)\big|^r \le (r-q)M_b\|u\|_p^r \le (r-q)M_b V_0^{-\frac{r}{p}}\|u\|_W^r.$$

Then

$$\|u\|_W \ge \left(\frac{(p-q)V_0^{\frac{r}{p}}}{(r-q)M_b}\right)^{\frac{1}{r-p}},$$

this together with (3.3) yields

$$\lambda \ge \left(\frac{(p-q)V_0^{\frac{r}{p}}}{(r-q)M_b}\right)^{\frac{1}{r-p}}\left(\left(\frac{r-q}{r-p}\right)\|a\|_{\frac{p}{p-q}} V_0^{-\frac{q}{p}}\right)^{-\frac{1}{p-q}}.$$

This is a contradiction. Thus, $\mathcal{N}_\lambda^0 = \emptyset$. □

Set

$$A_0 = \left(\frac{(p-q)V_0^{\frac{r}{p}}}{(r-q)M_b}\right)^{\frac{1}{r-p}}$$

and

$$A_\lambda := \lambda\left(\left(\frac{r-q}{r-p}\right)\|a\|_{\frac{p}{p-q}} V_0^{-\frac{q}{p}}\right)^{\frac{1}{p-q}}.$$

Lemma 3.6 *If* $\lambda \in (0, \Lambda_0)$, *then*

$$\|u\|_W \ge A_0 \quad \textit{for all } u \in \mathcal{N}_\lambda^-$$

and

$$\|v\|_W \le A_\lambda \quad \textit{for all } v \in \mathcal{N}_\lambda^+.$$

Proof Let $u \in \mathcal{N}_\lambda^-$. Then

$$\|u\|_W^p - \lambda \sum_{j\in\mathbb{Z}} a(j)|u(j)|^q - \sum_{j\in\mathbb{Z}} b(j)|u(j)|^r = 0$$

and

$$(p-1)\|u\|_W^p - (q-1)\lambda \sum_{j\in\mathbb{Z}} a(j)|u(j)|^q - (r-1)\sum_{j\in\mathbb{Z}} b(j)|u(j)|^r < 0.$$

A similar discussion to Lemma 3.4 leads to

$$\|u\|_W \geq \left(\frac{(p-q)V_0^{\frac{r}{p}}}{(r-q)M_b}\right)^{\frac{1}{r-p}},$$

Let $v \in \mathcal{N}_\lambda^+$. It follows that

$$(r-p)\|u\|_W^p \leq \lambda(r-q)\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q.$$

Using a similar argument to Lemma 3.4, one has

$$\|u\|_W \leq \lambda\left(\left(\frac{r-q}{r-p}\right)\|a\|_{\frac{p}{p-q}} V_0^{-\frac{q}{p}}\right)^{\frac{1}{p-q}}.$$

The proof is complete. □

4 Proof of Theorem 1.1

In this section, we prove the main result.

Theorem 4.1 *For all $\lambda \in (0,\Lambda_0)$, the functional I_λ has a nontrivial and nonnegative minimizer on $\mathcal{N}_\lambda^+$.*

Proof Define

$$c_\lambda^+ := \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u).$$

It follows from Lemma 3.3 that $c_\lambda^+ \in (-\infty,\infty)$. More precisely, $-\infty < c_\lambda^+ < 0$. Indeed, for $u \in \mathcal{N}_\lambda^+$ we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\|u\|_W^p - \frac{1}{q}\lambda\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q - \frac{1}{r}\sum_{j\in\mathbb{Z}} b(j)|u(j)|^r \\ &= \left(\frac{1}{p}-\frac{1}{q}\right)\lambda\sum_{j\in\mathbb{Z}} a(j)|u(j)|^q + \left(\frac{1}{p}-\frac{1}{r}\right)\sum_{j\in\mathbb{Z}} b(j)|u(j)|^r \\ &\leq -\frac{(r-p)(r-q)}{pqr}\sum_{j\in\mathbb{Z}} b(j)|u(j)|^r < 0. \end{aligned}$$

This means that $c_\lambda^+ < 0$. Motivated by [32], we divide the rest proof into the following three steps.

Step 1. The strong convergence of minimizing sequence.

Suppose that $\{u_n\}_n \subset \mathcal{N}_\lambda^+$ is a minimizing sequence. Then

$$\|u_n\|_W^p = \lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_n(j)\right|^q + \sum_{j\in\mathbb{Z}} b(j)\left|u_n(j)\right|^r$$

and

$$(p-q)\lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_n(j)\right|^q + (p-r)\sum_{j\in\mathbb{Z}} b(j)\left|u_n(j)\right|^r > 0. \tag{4.1}$$

By Lemma 3.3, I_λ is coercive on $\mathcal{N}_\lambda$. Thus, $\{u_n\}_n$ is bounded in $\mathcal{N}_\lambda$.

By Lemma 2.3, there exist a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and u_0 such that

$$\begin{aligned} &u_n \rightharpoonup u_0 \quad \text{in } W, \\ &u_n \to u_0 \quad \text{in } L^\nu(\Omega)(p \le \nu < \infty). \end{aligned}$$

Since $a \in \ell^{\frac{p}{p-q}}$, it follows that

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} a(j)\left|u_n(j)-u_0(j)\right|^q \\ &\quad \le \|a\|_{\frac{p}{p-q}} \|u_n - u_0\|_p^q \to 0 \end{aligned}$$

as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \sum_{j\in\mathbb{Z}} a(j)\left|u_n(j)-u_0(j)\right|^q = 0.$$

We also have

$$\lim_{n\to\infty} \sum_{j\in\mathbb{Z}} b(j)\left|u_n(j)-u_0(j)\right|^r = 0.$$

If $u_n \nrightarrow u_0$ in W, then

$$\|u_0\|_W^p < \liminf_{n\to\infty} \|u_n\|_W^p.$$

Then

$$\begin{aligned} &\|u_0\|_W^p - \lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_0(j)\right|^q - \sum_{j\in\mathbb{Z}} b(j)\left|u_0(j)\right|^r \\ &\quad < \liminf_{n\to\infty}\left[\|u_n\|_W^p - \lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_n(j)\right|^q - \sum_{j\in\mathbb{Z}} b(j)\left|u_n(j)\right|^r\right] = 0. \end{aligned} \tag{4.2}$$

Using Lemma 3.4, for u_0 there exists $0 < t_{u_0} \neq 1$ such that $t_{u_0}u_0 \in \mathcal{N}_\lambda^+$.

Set $z(t) = I_\lambda(tu_0)$ for all $t > 0$. Obviously, $t_{u_0}u_0$ is a minimizer of $z(t)$. Thus,

$$I_\lambda(t_{u_0}u_0) < I_\lambda(u_0) \le \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u\in\mathcal{N}^+} I_\lambda(u),$$

which is impossible. Thus, we get $u_n \to u_0$ in W.

Step 2. $u_0 \in \mathcal{N}_\lambda^+$.

By the definition of $\mathcal{N}_\lambda^+$, it suffices to show that

$$(p-q)\lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_0(j)\right|^q + (p-r)\sum_{j\in\mathbb{Z}} b(j)\left|u_0(j)\right|^r > 0. \tag{4.3}$$

Arguing by contradiction, we assume that

$$(p-q)\lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_0(j)\right|^q + (p-r)\sum_{j\in\mathbb{Z}} b(j)\left|u_0(j)\right|^r = 0.$$

Clearly, $u_0 \neq 0$, since $I_\lambda(u_0) < 0$. Then $u_0 \in \mathcal{N}_\lambda^0$, which contradicts Lemma 3.5. Thus, we prove that $u_0 \in \mathcal{N}_\lambda^+$.

Step 3. Existence of nonnegative minimizers.

It follows from $u_0 \in \mathcal{N}_\lambda^+$ that

$$(p-q)\lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_0(j)\right|^q + (p-r)\sum_{j\in\mathbb{Z}} b(j)\left|u_0(j)\right|^r > 0. \tag{4.4}$$

Then, by $I_\lambda(u_0) = \inf_{u\in\mathcal{N}^+} I_\lambda(u) < 0$, we prove that u_0 is a minimizer of I_λ on $\mathcal{N}_\lambda^+$. Furthermore, we can show that $|u_0|$ is a minimizer of I_λ on $\mathcal{N}_\lambda^+$. Since $I_\lambda(|u_0|) \le I_\lambda(u_0)$ and

$$(p-q)\lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_0(j)\right|^q + (p-r)\sum_{j\in\mathbb{Z}} b(j)\left|u_0(j)\right|^r > 0,$$

it suffices to show that $|||u_0|||_W^p - \lambda\sum_{j\in\mathbb{Z}} a(j)|u_0(j)|^q - \sum_{j\in\mathbb{Z}} b(j)|u_0(j)|^r = 0$. By $|||u_0|||_W^p \le \|u_0\|_W^p$, we obtain $|||u_0|||_W^p - \lambda\sum_{j\in\mathbb{Z}} a(j)|u_0(j)|^q - \sum_{j\in\mathbb{Z}} b(j)|u_0(j)|^r \le 0$. If

$$|||u_0|||_W^p - \lambda \sum_{j\in\mathbb{Z}} a(j)\left|u_0(j)\right|^q - \sum_{j\in\mathbb{Z}} b(j)\left|u_0(j)\right|^r < 0,$$

then $\Phi'_{\lambda,|u_0|}(1) < 0$. By Lemma 3.4, there is a $t_{|u_0|} > 0$ such that $t_{|u_0|}|u_0| \in \mathcal{N}_\lambda^+$ and $\Phi'_{\lambda,|u_0|}(t_{|u_0|}) = 0$. Thus, $t_{|u_0|} \neq 1$. Observe that $t_{|u_0|}$ is a minimizer of $\widetilde{g}(t) := I_\lambda(t|u_0|)$. Thus,

$$I_\lambda\left(t_{|u_0|}|u_0|\right) < I_\lambda\left(|u_0|\right) \le I_\lambda(u_0) = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u),$$

which is impossible. Thus, $|u_0| \in \mathcal{N}_\lambda^+$ and $I_\lambda(|u_0|) = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u_0)$. Consequently, we obtain a nonnegative minimizer of I_λ on $\mathcal{N}_\lambda^+$.

Therefore, we complete the proof. □

Theorem 4.2 *For all $\lambda \in (0, \Lambda_0)$, I_λ has a nontrivial and nonnegative minimizer on $\mathcal{N}_\lambda^-$.*

Proof Using a similar discussion to Theorem 4.1, one can show that I_λ possesses a minimizer u_1 on $\mathcal{N}_\lambda^-$. Moreover, Lemma 3.6 shows that u_1 is nontrivial. Furthermore, one can use a similar discussion to Theorem 4.1 to prove that $|u_1|$ is a minimizer of I_λ on $\mathcal{N}_\lambda^-$. Therefore, the proof is complete. □

Proof of Theorem 1.1 Gathering Theorem 4.1 with Theorem 4.2, we see that I_λ has two nonnegative and nonnegative local minimizers. Then it follows from Lemma 3.2 that I_λ has two critical points on W, which are two nontrivial and nonnegative local least energy solutions of problem (1.4). □

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