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The step-type contrast structure for a second order semi-linear singularly perturbed differential-difference equation

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Abstract

The step-type contrast structure for a second order semi-linear singularly perturbed differential-difference equation is studied. Using the methods of boundary function and fractional steps, we construct the formula asymptotic expansion of the problem. At the same time, based on sewing techniques, the existence of the step-type contrast structure solution and the uniform validity of the asymptotic expansion are proved.

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Keywords: Singularly perturbed; Differential-difference equation; Contrast structure; Asymptotic expansion; Boundary function

1 Introduction

The boundary-value problems for singularly perturbed differential-difference equations arise in various practical problems in biomechanics and physics such as in variational problem in control theory and depolarization in Stein's model. Many scholars have done a lot of work on this field, especially for linear problems [1–6]. For nonlinear problems, some results [7–12] have also been obtained. However, most of these works are related to boundary layers, numerical solution, or the proof of the existence of the solution. Few of them concern the contrast structures and the uniform validity of the asymptotic expansions [7, 12]. Recently, the contrast structures have become the focus of attention in singular perturbation [13–16]. The fundamental characteristic of contrast structures is that there exists a t_* (or multiple t_*) within the domain of interest, which is called an internal transition point. The position of t_* is unknown in advance, and it needs to be determined thereafter. In the neighborhood of t_*, the solution $y(t,\mu)$ will have an abrupt structure change. In the different sides of t_*, if $y(t,\mu)$ approaches different reduced solutions, we call it step-type contrast structure. If $y(t,\mu)$ approaches the same reduced solution, we call it spike-type contrast structure. In [17], Wang, Xu, and Ni study the spike-type contrast structure for the following singularly perturbed differential-difference equation which only contains negative shift in it:

$$\mu^2 y''(t) = F\big(y(t), y(t-\sigma), t\big), \quad 0 \le t \le T; \tag{1.1}$$

$$y(t,\mu) = \alpha(t), \quad -\sigma \le t \le 0, \qquad y(T,\mu) = y^T. \tag{1.2}$$

In this paper, we study the step-type contrast structure for system (1.1), (1.2), where $0 < \mu \ll 1$ is a small parameter and σ is a delay argument. $\alpha(t)$ is a smooth function defined in $[-\sigma, 0]$. T is a positive constant that satisfies $\sigma \le T \le 2\sigma$. The restriction on T will not influence the essence of the problem and it is only convenient for our discussion.

2 Algorithm for the construction of asymptotics

Let $\mu y' = z$, then (1.1) can be rewritten as follows:

$$\mu y'(t) = z(t), \qquad \mu z'(t) = F\big(y(t), y(t-\sigma), t\big). \tag{2.1}$$

When necessary we impose several additional conditions on Eq. (2.1).

(H_1) Suppose that $F(y,u,t)$ is sufficiently smooth with respect to each argument and for $0 \le t \le T$, where $u = y(t-\sigma)$.

(H_2) Suppose that the reduced equation $F(\overline{y}(t), \overline{y}(t-\sigma), t) = 0$ has three disjoint real roots $\overline{y}(t) = \varphi_i(t)$ $(i = 1,2,3)$ in $[0,\sigma]$, but an isolate root $\overline{y}(t) = \psi_1(t)$ in $[\sigma, T]$.

(H_3) Suppose $F_y(\overline{y}(t), \overline{y}(t-\sigma), t) > 0$, when $\overline{y}(t) = \varphi_i(t)$ $(i = 1,3)$, or $\overline{y}(t) = \psi_1(t)$, while $F_y(\varphi_2(t), \varphi_2(t-\sigma), t) < 0$.

Let $t_* \in (0,\sigma)$ be the transfer point of the contrast structure, and it has the series form

$$t_* = t_0 + \mu t_1 + \cdots + \mu^k t_k + \cdots, \tag{2.2}$$

where t_k $(k = 0,1,\ldots)$ are unknown constants determined by the smooth connection at $t = t_*$.

Setting $x = (y,z)^T$ and using the method of boundary function, we construct a series formally satisfying (2.1), (1.2) in $[0,t_*]$, $[t_*,\sigma]$, $[\sigma, T]$, respectively.

$$x^{(1)}(t,\mu) = \sum_{k=0}^{\infty} \mu^k \big(\bar{x}_k^{(1)}(t,\mu) + \Pi x_k(\tau_0,\mu) + Q^{(-)} x_k(\tau_*,\mu)\big), \tag{2.3}$$

$$x^{(2)}(t,\mu) = \sum_{k=0}^{\infty} \mu^k \big(\bar{x}_k^{(2)}(t,\mu) + Q^{(+)} x_k(\tau_*,\mu) + \bar{Q}^{(-)} x_k(\tau_\sigma,\mu)\big), \tag{2.4}$$

$$x^{(3)}(t,\mu) = \sum_{k=0}^{\infty} \mu^k \big(\bar{x}_k^{(3)}(t,\mu) + \bar{Q}^{(+)} x_k(\tau_\sigma,\mu) + R x_k(\tau_T,\mu)\big), \tag{2.5}$$

where $\tau_0 = \frac{t}{\mu}$, $\tau_* = \frac{t-t_*}{\mu}$, $\tau_\sigma = \frac{t-\sigma}{\mu}$, $\tau_T = \frac{t-T}{\mu}$. $\Pi_k x(\tau_0)$, $Q_k^{(-)} x(\tau_*)$, $Q_k^{(+)} x(\tau_*)$, $\bar{Q}_k^{(-)} x(\tau_\sigma)$, $\bar{Q}_k^{(+)} x(\tau_\sigma)$, $R_k x(\tau_T)$ $(k \ge 0)$ are called boundary functions, and

$$\lim_{\tau_0 \to +\infty} \Pi_k x(\tau_0) = 0, \qquad \lim_{\tau_* \to -\infty} Q_k^{(-)} x(\tau_*) = 0, \qquad \lim_{\tau_\sigma \to +\infty} \bar{Q}_k^{(+)} x(\tau_\sigma) = 0,$$

$$\lim_{\tau_* \to -\infty} Q_k^{(-)} x(\tau_*) = 0, \qquad \lim_{\tau_\sigma \to +\infty} \bar{Q}_k^{(+)} x(\tau_\sigma) = 0, \qquad \lim_{\tau_T \to -\infty} R_k x(\tau_T) = 0$$

hold.

By the method of boundary function, we obtain

$$\bar{z}_0^{(1)}(t) = 0, \qquad F\big(\bar{y}_0^{(1)}(t), \alpha(t-\sigma), t\big) = 0; \tag{2.6}$$

$$\frac{d\bar{y}_{k-1}^{(1)}}{dt} = \bar{z}_k^{(1)}(t), \qquad \frac{d\bar{z}_{k-1}^{(1)}}{dt} = \bar{F}_y^{(1)}\bar{y}_k^{(1)}(t) + \bar{h}_k^{(1)}(t); \tag{2.7}$$

where $\bar{F}_y^{(1)}$ takes its values at $(\bar{y}_0^{(1)}(t), \alpha(t-\sigma), t)$ and $\bar{h}_k^{(1)}(t)$ are determined functions. Equation (2.6) coincides with the reduced equation of (2.1), so we have $\bar{y}_0^{(1)}(t) = \varphi_1(t)$, $\bar{z}_0^{(1)}(t) = 0$. Obviously, by (H_2) and (2.7), $\bar{x}_k^{(1)}(t)$ are completely determined. Similar to $\bar{x}_k^{(1)}(t)$, we obtain $\bar{y}_0^{(2)}(t) = \varphi_3(t)$, $\bar{z}_0^{(2)}(t) = 0$, $\bar{y}_0^{(3)}(t) = \psi_1(t)$, $\bar{z}_0^{(3)}(t) = 0$, and $\bar{x}_k^{(2)}(t)$, $\bar{x}_k^{(3)}(t)$ are completely determined.

For $\Pi_0 x(\tau_0)$, we have

$$\frac{d\Pi_0 y}{d\tau_0} = \Pi_0 z, \qquad \frac{d\Pi_0 z}{d\tau_0} = F\big(\varphi_1(0) + \Pi_0 y, \alpha(-\sigma), 0\big);$$
$$\Pi_0 y(0) = \alpha(0) - \varphi_1(0), \qquad \Pi_0 y(+\infty) = 0.$$

Let $\varphi_1(0) + \Pi_0 y = \tilde{y}^{(1)}$, $(\tilde{y}^{(1)})' = \tilde{z}^{(1)}$, and we get

$$\frac{d\tilde{y}^{(1)}}{d\tau_0} = \tilde{z}^{(1)}, \qquad \frac{d\tilde{z}^{(1)}}{d\tau_0} = F\big(\tilde{y}^{(1)}, \alpha(-\sigma), 0\big); \tag{2.8}$$

$$\tilde{y}^{(1)}(0) = \alpha(0), \qquad \tilde{y}^{(1)}(+\infty) = \varphi_1(0). \tag{2.9}$$

Integrating (2.8), we have

$$\tilde{z}^{(1)} = \pm\sqrt{2}\bigg(\int_{\varphi_1(0)}^{\tilde{y}^{(1)}} F\big(y, \alpha(-\sigma), 0\big)\, dy\bigg)^{\frac{1}{2}} \triangleq \pm\Phi_1\big(\tilde{y}^{(1)}\big). \tag{2.10}$$

By (H_2), the equilibrium $M_1(\varphi_1(0), 0)$ is a saddle point on the phase plane $(\tilde{y}^{(1)}, \tilde{z}^{(1)})$. Let the steady manifold be $\Sigma_1 : \tilde{z}^{(1)} = -\Phi_1(\tilde{y}^{(1)})$. Under the condition that the line $\tilde{y}^{(1)} = \alpha(0)$ intersects with Σ_1, the solution of problem (2.8), (2.9) exists.

For $\Pi_k x(\tau_0)$, we have the following system:

$$\frac{d\Pi_k y}{d\tau_0} = \Pi_k z, \qquad \frac{d\Pi_k z}{d\tau_0} = \tilde{F}_y \Pi_k y + G_k(\tau_0); \tag{2.11}$$

$$\Pi_k y(0) = -\bar{y}_k^{(1)}(0), \qquad \Pi_k y(+\infty) = 0, \tag{2.12}$$

where $\tilde{F}_y$ gets its value at the point $(\varphi_1(0) + \Pi_0 y, \alpha(-\sigma), 0)$. $G_k(\tau)$ are functions formed by $\bar{x}_i(t)$, $\Pi_i x(\tau_0)$ $(i = 0, 1, \ldots, k-1)$.

According to the Liouville formulas and the constant-change method, we infer that

$$\Pi_k y = \frac{\tilde{z}(\tau_0)}{\tilde{z}(0)}\big(-\bar{y}_k^{(1)}(0)\big) + \tilde{z}(\tau_0)\int_0^{\tau_0} \frac{1}{\tilde{z}^2(\eta)} \int_{+\infty}^{\eta} \tilde{z}(s) G_k(s)\, ds\, d\eta. \tag{2.13}$$

Thus, $\Pi_k x(\tau_0)$ are completely determined. The exponential decay of $\Pi_k x(\tau_0)$ can easily be obtained from (2.13).

For $Q_0^{(-)}x(\tau_*)$, we have

$$\frac{dQ_0^{(-)}y}{d\tau_*} = Q_0^{(-)}z, \qquad \frac{dQ_0^{(-)}z}{d\tau_*} = F\bigl(\varphi_1(t_0) + Q_0^{(-)}y, \alpha(t_0-\sigma), t_0\bigr); \tag{2.14}$$

$$Q_0^{(-)}y(0) = \varphi_2(t_0) - \varphi_1(t_0), \qquad Q_0^{(-)}y(-\infty) = 0. \tag{2.15}$$

Let $\varphi_1(t_0) + Q_0^{(-)}y = \tilde{y}^{(2)}$, $(\tilde{y}^{(2)})' = \tilde{z}^{(2)}$, and we get

$$\frac{d\tilde{y}^{(2)}}{d\tau_*} = \tilde{z}^{(2)}, \qquad \frac{d\tilde{z}^{(2)}}{d\tau_*} = F\bigl(\tilde{y}^{(2)}, \alpha(t_0-\sigma), t_0\bigr); \tag{2.16}$$

$$\tilde{y}^{(2)}(0) = \varphi_2(t_0), \qquad \tilde{y}^{(2)}(-\infty) = \varphi_1(t_0). \tag{2.17}$$

Integrating (2.16), we get

$$\tilde{z}^{(2)} = \pm\sqrt{2}\biggl(\int_{\varphi_1(t_0)}^{\tilde{y}^{(2)}} F\bigl(y, \alpha(t_0-\sigma), t_0\bigr)\,dy\biggr)^{\frac{1}{2}} \triangleq \pm\Phi_2\bigl(\tilde{y}^{(2)}\bigr). \tag{2.18}$$

By (H_2), the equilibrium $M_2(\varphi_1(t_0), 0)$ is a saddle point on the phase plane $(\tilde{y}^{(2)}, \tilde{z}^{(2)})$. Let the steady manifold be $\Sigma_2 : \tilde{z}^{(2)} = \Phi_2(\tilde{y}^{(2)})$. Under the condition that the line $\tilde{y}^{(2)} = \phi_2(t_0)$ intersects with Σ_2, the solution of problem (2.16), (2.17) exists.

For $Q_k^{(-)}x(\tau_*)$, we have the following system:

$$\frac{dQ_k^{(-)}y}{d\tau_*} = Q_k^{(-)}z, \qquad \frac{dQ_k^{(-)}z}{d\tau_*} = \tilde{F}_y^{(1)}Q_k^{(-)}y + G_k^{(1)}(\tau_*); \tag{2.19}$$

$$Q_k^{(-)}y(0) = \bigl(\phi_2'(t_0) - \phi_1'(t_0)\bigr)t_k + q_k^{(-)}(t_0, t_1, \ldots, t_{k-1}), \qquad Q_k^{(-)}y(+\infty) = 0, \tag{2.20}$$

where $\tilde{F}_y^{(1)}$ gets its value at the point $(\varphi_1(t_0) + Q_0^{(-)}y, \alpha(t_0-\sigma), t_0)$. $G_k^{(1)}(\tau_*)$ are functions compound formed by $\bar{x}_i(t)$, $Q_i^{(-)}x(\tau_0)$ $(i = 0, 1, \ldots, k-1)$.

Similar to (2.11), the solution of (2.19), (2.20) is

$$\begin{aligned} Q_k^{(-)}y = {}& \frac{\tilde{z}^{(2)}(\tau_*)}{\tilde{z}^{(2)}(0)}\bigl(\bigl(\phi_2'(t_0) - \phi_1'(t_0)\bigr)t_k + q_k^{(-)}\bigr) \\ & + \tilde{z}^{(2)}(\tau_*)\int_0^{\tau_*} \frac{1}{\tilde{z}^2(\eta)} \int_{-\infty}^{\eta} \tilde{z}^{(2)}(s)G_k^{(1)}(s)\,ds\,d\eta. \end{aligned} \tag{2.21}$$

Thus, $Q_k^{(-)}x(\tau_*)$ are completely determined.

For $Q_0^{(+)}x(\tau_*)$, we have

$$\frac{dQ_0^{(+)}y}{d\tau_*} = Q_0^{(+)}z, \qquad \frac{dQ_0^{(+)}z}{d\tau_*} = F\bigl(\varphi_3(t_0) + Q_0^{(+)}y, \alpha(t_0-\sigma), t_0\bigr); \tag{2.22}$$

$$Q_0^{(+)}y(0) = \varphi_2(t_0) - \varphi_3(t_0), \qquad Q_0^{(+)}y(+\infty) = 0. \tag{2.23}$$

Let $\varphi_3(t_0) + Q_0^{(+)}y = \tilde{y}^{(3)}$, $(\tilde{y}^{(3)})' = \tilde{z}^{(3)}$, and we get

$$\frac{d\tilde{y}^{(3)}}{d\tau_*} = \tilde{z}^{(3)}, \qquad \frac{d\tilde{z}^{(3)}}{d\tau_*} = F\bigl(\tilde{y}^{(3)}, \alpha(t_0-\sigma), 0\bigr); \tag{2.24}$$

$$\tilde{y}^{(3)}(0) = \varphi_2(t_0), \qquad \tilde{y}^{(3)}(+\infty) = \varphi_3(t_0). \tag{2.25}$$

Integrating (2.24), we have

$$\tilde{z}^{(3)} = \pm\sqrt{2}\biggl(\int_{\varphi_3(t_0)}^{\tilde{y}^{(3)}} F\bigl(y, \alpha(t_0-\sigma), 0\bigr)\,dy\biggr)^{\frac{1}{2}} \triangleq \pm\Phi_3\bigl(\tilde{y}^{(3)}\bigr). \tag{2.26}$$

By (H_2), the equilibrium $M_3(\varphi_3(t_0), 0)$ is a saddle point on the phase plane $(\tilde{y}^{(3)}, \tilde{z}^{(3)})$. Let the steady manifold be $\Sigma_3 : \tilde{z}^{(3)} = -\Phi_3(\tilde{y}^{(3)})$. Under the condition that the line $\tilde{y}^{(3)} = \varphi_2(t_0)$ intersects with Σ_3, the solution of problem (2.24), (2.25) exists.

For $Q_k^{(+)}x(\tau_*)$, we have the following system:

$$\frac{dQ_k^{(+)}y}{d\tau_*} = Q_k^{(+)}z, \qquad \frac{dQ_k^{(+)}z}{d\tau_*} = \tilde{F}_y^{(2)}Q_k^{(+)}y + G_k^{(2)}(\tau_*); \tag{2.27}$$

$$Q_k^{(+)}y(0) = \bigl(\phi_2'(t_0) - \phi_3'(t_0)\bigr)t_k + q_k^{(+)}(t_0, t_1, \ldots, t_{k-1}), \qquad Q_k^{(+)}y(+\infty) = 0, \tag{2.28}$$

where $\tilde{F}_y^{(2)}$ gets its value at the point $(\varphi_3(t_0) + Q_0^{(+)}y, \alpha(t_0-\sigma), t_0)$. $G_k^{(2)}(\tau_*)$ are functions compound formed by $\bar{x}_i^{(2)}(t)$, $Q_i^{(+)}x(\tau_*)$ $(i = 0, 1, \ldots, k-1)$.

Similarly, according to the Liouville formulas and the constant-change method, we get the solution of (2.27), (2.28):

$$\begin{aligned} Q_k^{(+)}y &= \frac{\tilde{z}^{(3)}(\tau_*)}{\tilde{z}^{(3)}(0)}\bigl(\bigl(\phi_2'(t_0) - \phi_3'(t_0)\bigr)t_k + q_k^{(+)}\bigr) \\ &\quad + \tilde{z}^{(3)}(\tau_*)\int_0^{\tau_*}\frac{1}{\tilde{z}^{(3)}(\eta)}\int_{+\infty}^{\eta}\tilde{z}^{(3)}(s)G_k^{(2)}(s)\,ds\,d\eta. \end{aligned} \tag{2.29}$$

Thus, $Q_k^{(+)}x(\tau_*)$ are completely determined. The exponential decay of $Q_k^{(+)}x(\tau_*)$ can easily be obtained from (2.29).

Specially, at the point $t = \sigma$, we set

$$y(\sigma, \mu) = p(\mu) = p_0 + \mu p_1 + \mu^2 p_2 + \cdots + \mu^k p_k + \cdots,$$

where p_k $(k = 0, 1, \ldots)$ are unknown constants determined by the smooth connection at $t = \sigma$.

$\bar{Q}_0^{(-)}x(\tau_\sigma)$ are determined by the following system:

$$\frac{d\bar{Q}_0^{(-)}y}{d\tau_\sigma} = \bar{Q}_0^{(-)}z, \qquad \frac{d\bar{Q}_0^{(-)}z}{d\tau_\sigma} = F\bigl(\varphi_3(\sigma) + \bar{Q}_0^{(-)}y, \alpha(0), \sigma\bigr); \tag{2.30}$$

$$\bar{Q}_0^{(-)}y(0) = p_0 - \varphi_3(\sigma), \qquad \bar{Q}_0^{(-)}y(-\infty) = 0. \tag{2.31}$$

Let $\varphi_3(\sigma) + \bar{Q}_0^{(-)}y(\tau_\sigma) = \tilde{y}^{(4)}$, $\bar{Q}_0^{(-)}z(\tau_\sigma) = \tilde{z}^{(4)}$, and we get

$$\frac{d\tilde{y}^{(4)}}{d\tau_\sigma} = \tilde{z}^{(4)}, \qquad \frac{d\tilde{z}^{(4)}}{d\tau_\sigma} = F\bigl(\tilde{y}^{(4)}, \alpha(0), \sigma\bigr); \tag{2.32}$$

$$\tilde{y}^{(4)}(0) = p_0, \qquad \tilde{y}^{(4)}(-\infty) = \varphi_3(\sigma). \tag{2.33}$$

Integrating (2.32), we get

$$\tilde{z}^{(4)} = \pm\sqrt{2}\left(\int_{\varphi_3(\sigma)}^{\tilde{y}^{(4)}} F\bigl(y,\alpha(0),\sigma\bigr)\,dy\right)^{\frac{1}{2}} \triangleq \pm\Phi_4\bigl(\tilde{y}^{(4)}\bigr).$$

By the virtue of condition (H_2), the equilibrium $(\varphi_3(\sigma),0)$ is a saddle point on the phase plane $(\tilde{y}^{(4)},\tilde{z}^{(4)})$. So passing through $(\varphi_3(\sigma),0)$ there exists a steady manifold $\Sigma_4 : \tilde{z}^{(4)} = \Phi_4(\tilde{y}^{(4)})$. Under the condition that the line $\tilde{y}^{(4)}(0) = p_0$ intersects with the manifold Σ_4, the solution of system (2.32), (2.33) exists.

$\bar{Q}_k^{(-)}x(\tau_\sigma)$ are determined by the following system:

$$\frac{d\bar{Q}_k^{(-)}y}{d\tau_\sigma} = \bar{Q}_k^{(-)}z, \qquad \frac{d\bar{Q}_k^{(-)}z}{d\tau_\sigma} = \tilde{F}_y^{(3)}\bar{Q}_k^{(-)}y + G_k^{(3)}(\tau_\sigma); \tag{2.34}$$

$$\bar{Q}_k^{(-)}y(0) = P_k - \bar{y}_k^{(2)}(\sigma), \qquad \bar{Q}_k^{(-)}y(-\infty) = 0, \tag{2.35}$$

where $\tilde{F}_z^{(3)}, \tilde{F}_y^{(3)}$ get their values at $(\varphi_3(\sigma)+\bar{Q}_0^{(3)}y,\alpha(0),\sigma)$. $G_k^{(3)}(\tau_\sigma)$ are determined functions.

In fact, the homogeneous system, corresponding to (2.34),

$$\frac{d\bar{Q}_k^{(-)}y}{d\tau_\sigma} = \bar{Q}_k^{(-)}z, \qquad \frac{d\bar{Q}_k^{(-)}z}{d\tau_\sigma} = \tilde{F}_y^{(3)}\bar{Q}_k^{(-)}y \tag{2.36}$$

is the variational equation of (2.30). Under the boundary condition $\bar{Q}_k^{(-)}y(0) = p_k - \bar{y}_k^{(2)}(\sigma)$, $\bar{Q}_k^{(-)}y(-\infty) = 0$, we get

$$\begin{cases} (\bar{Q}_k^{(-)}y(\tau_\sigma))^G = (p_k - \bar{y}_k(\sigma))\Psi_1(\tau_\sigma)\Psi_1^{-1}(0), \\ (\bar{Q}_k^{(-)}z(\tau_\sigma))^G = \frac{d\Phi_4(\tilde{y}^{(4)})}{d\tilde{y}^{(4)}}(p_k - \bar{y}_k(\sigma))\Psi_1(\tau_\sigma)\Psi_1^{-1}(0). \end{cases} \tag{2.37}$$

Next, let $\bar{Q}_k^{(-)}y^*$, $\bar{Q}_k^{(-)}z^*$ be the particular solution of (2.34). Introducing a new transformation

$$\bar{Q}_k^{(-)}y^* = \delta_1, \qquad \bar{Q}_k^{(-)}z^* = \frac{d\Phi_4(\tilde{y}^{(4)})}{d\tilde{y}^{(4)}}\bar{Q}_k^{(-)}y^* + \delta_2$$

and substituting it into (2.34), we get

$$\begin{cases} \frac{d\delta_1}{d\tau_\sigma} = \frac{d\Phi_4(\tilde{y}^{(4)})}{d\tilde{y}^{(4)}}\delta_1 + \delta_2, \\ \frac{d\delta_2}{d\tau_\sigma} = (\frac{d\Phi_4(\tilde{y}^{(4)})}{d\tilde{y}^{(4)}})\delta_2 + G_k^{(3)}(\tau_\sigma). \end{cases}$$

Let $\delta_2 = C\Psi_2(\tau_\sigma)$ be the general solution of $\frac{d\delta_2}{d\tau_\sigma} = (\frac{d\Phi_4(\tilde{y}^{(4)})}{d\tilde{y}^{(4)}})\delta_2$, then we get a particular solution

$$\delta_2 = \int_{-\infty}^{\tau_\sigma} \Psi_2(\tau_\sigma)\Psi_2^{-1}(s)G_k^{(3)}(s)\,ds,$$

of $\frac{d\delta_2}{d\tau_\sigma} = (\frac{d\Phi_4(\tilde{y}^{(4)})}{d\tilde{y}^{(4)}})\delta_2 + G_k^{(3)}(\tau_\sigma)$. Furthermore, we have

$$\delta_1 = \int_0^{\tau_\sigma} \Psi_1(\tau_\sigma)\Psi_1^{-1}(s)\left[\int_{-\infty}^{s} \Psi_2(s)\Psi_2^{-1}(p)G_k^{(3)}(p)\,dp\right]ds.$$

So, we have

$$\begin{cases} \bar{Q}_k^{(-)}y^*(\tau_\sigma) = \int_0^{\tau_\sigma} \Psi_1(\tau_\sigma)\Psi_1^{-1}(s)[\int_{-\infty}^{s} \Psi_2(s)\Psi_2^{-1}(p)G_k^{(3)}(p)\,dp]\,ds, \\ \bar{Q}_k^{(-)}z^*(\tau_\sigma) = \frac{d\Phi_4(\tilde{y}^4)}{d\tilde{y}^4} \cdot \bar{Q}_k^{(-)}y^*(\tau_\sigma) + \int_{-\infty}^{\tau_\sigma} \Psi_2(\tau_\sigma)\Psi_2^{-1}(s)G_k^{(3)}(s)\,ds. \end{cases}$$

Thus, we obtain

$$\begin{cases} \bar{Q}_k^{(-)}y(\tau_\sigma) = (p_k - \bar{y}_k(\sigma))\Psi_1(\tau_\sigma)\Psi_1^{-1}(0) + \bar{Q}_k^{(-)}y^*(\tau_\sigma), \\ \bar{Q}_k^{(-)}z(\tau_\sigma) = \frac{d\Phi_4(\tilde{y}^4)}{d\tilde{y}^4}(p_k - \bar{y}_k(\sigma))\Psi_1(\tau_\sigma)\Psi_1^{-1}(0) + \bar{Q}_k^{(-)}z^*(\tau_\sigma). \end{cases} \tag{2.38}$$

Now, $\bar{Q}_k^{(-)}x(\tau_\sigma)$ are all completely determined, but they contain the unknown numbers p_k. Obviously, the estimation about exponential decay of $\bar{Q}_k^{(-)}x(\tau_\sigma)$ can easily be obtained from (2.38).

Due to the deviation of arguments, the equations determining $\bar{Q}_k^{(+)}x(\tau_\sigma)$ will be relevant to $\Pi_j y(\tau_0)$, $0 \le j \le k$. Namely,

$$\begin{cases} \frac{d\bar{Q}_0^{(+)}y}{d\tau_\sigma} = \bar{Q}_0^{(+)}z, \\ \frac{d\bar{Q}_0^{(+)}z}{d\tau_\sigma} = F(\psi_1(\sigma) + \bar{Q}_0^{(+)}y, \varphi_1(0) + \Pi_0 y(\tau_\sigma), \sigma); \end{cases}$$
$$\bar{Q}_0^{(+)}y(0) = p_0 - \psi_1(\sigma), \qquad \bar{Q}_0^{(+)}y(+\infty) = 0.$$

Let $\psi_1(\sigma) + \bar{Q}_0^{(+)}y(\tau_\sigma) = \tilde{y}^{(5)}$, $\bar{Q}_0^{(+)}z(\tau_\sigma) = \tilde{z}^{(5)}$, we have

$$\frac{d\tilde{y}^{(5)}}{d\tau_\sigma} = \tilde{z}^{(5)}, \qquad \frac{d\tilde{z}^{(5)}}{d\tau_\sigma} = F\big(\tilde{y}^{(5)}, \varphi_1(0) + \Pi_0 y(\tau_\sigma), \sigma\big); \tag{2.39}$$

$$\tilde{y}^{(5)}(0) = p_0, \qquad \tilde{y}^{(5)}(+\infty) = \psi_1(\sigma). \tag{2.40}$$

Combining (2.8), (2.9) with (2.39), (2.40), we have a coupled system:

$$\begin{cases} \frac{d\tilde{y}^{(5)}}{d\tau_\sigma} = \tilde{z}^{(5)}, \qquad \frac{d\tilde{z}^{(5)}}{d\tau_\sigma} = F(\tilde{y}^{(5)}, \tilde{y}^{(1)}, \sigma); \\ \frac{d\tilde{y}^{(1)}}{d\tau_0} = \tilde{z}^{(1)}, \qquad \frac{d\tilde{z}^{(1)}}{d\tau_0} = F(\tilde{y}^{(1)}, \alpha(-\sigma), 0); \end{cases}$$
$$\tilde{y}^{(1)}(0) = \alpha(0), \qquad \tilde{y}^{(1)}(+\infty) = \varphi_1(0), \qquad \tilde{y}^{(5)}(0) = p_0, \qquad \tilde{y}^{(5)}(+\infty) = \psi_1(\sigma).$$

Here, the phase space $(\tilde{y}^{(5)}, \tilde{z}^{(5)}, \tilde{y}^{(1)}, \tilde{z}^{(1)})$ is the direct sum of $(\tilde{y}^{(5)}, \tilde{z}^{(5)})$ and $(\tilde{y}^{(1)}, \tilde{z}^{(1)})$. The equilibrium $\mathrm{M}(\psi_1(\sigma), 0, \varphi_1(0), 0)$ is a hyperbolic saddle point because the characteristic equation at $\mathrm{M}(\psi_1(\sigma), 0, \varphi_1(0), 0)$ is

$$\big[\lambda^2 - F_{\tilde{y}^{(5)}}\big]\big[\lambda^2 - F_{\tilde{y}^{(1)}}\big] = 0$$

and its eigenvalues satisfy

$$\lambda_1\lambda_2 = -F_{\tilde{y}^{(5)}} < 0, \qquad \lambda_3\lambda_4 = -F_{\tilde{y}^{(1)}} < 0.$$

Thus, going through equilibrium M there exist a two-dimensional stable submanifold $W^s(M)$ and a two-dimensional unstable submanifold $W^u(M)$. Set

$$W^s(M) : Z = \bar{\Phi}(Y),$$

where $Y = (\tilde{y}^{(5)}, \tilde{y}^{(1)})^T$, $Z = (\tilde{z}^{(5)}, \tilde{z}^{(1)})^T$, $\bar{\Phi} = (\Phi_1, \Phi_5)^T$. Obviously, the projection of $W^s(M)$ on the phase plane $(\tilde{y}^{(1)}, \tilde{z}^{(1)})$ is Σ_1. Namely, $(W^s(M))^{\perp}_{(\tilde{y}^{(1)}, \tilde{z}^{(1)})} = \Sigma_1$. Set

$$\big(W^s(M)\big)^{\perp}_{(\tilde{y}^{(5)}, \tilde{z}^{(5)})} = \Sigma_5,$$

then

$$\tilde{z}^{(5)} = \Phi_5\big(\tilde{y}^{(5)}, \tilde{y}^{(1)}\big).$$

Under the condition that the plane $\tilde{y}^{(5)}(0) = p_0$ intersects with the steady submanifold Σ_5 in the phase space, the solution of (2.39), (2.40) exists.

$\bar{Q}_k^{(+)} x(\tau_\sigma)$ are determined by the following system:

$$\begin{aligned} &\frac{d\bar{Q}_k^{(+)} y}{d\tau_\sigma} = \bar{Q}_k^{(+)} z, \qquad \frac{d\bar{Q}_k^{(+)} z}{d\tau_\sigma} = \tilde{F}_y^{(4)} \bar{Q}_k^{(+)} y + G_k^{(4)}(\tau_\sigma); \\ &\bar{Q}_k^{(+)} y(0) = P_k - \bar{y}_k^{(3)}(\sigma), \qquad \bar{Q}_k^{(+)} y(+\infty) = 0, \end{aligned}$$

where $\tilde{F}_z^{(+)}$, $\tilde{F}_y^{(+)}$ take their values at $(\psi_1(\sigma) + \bar{Q}_0^{(+)} y, \varphi_1(0) + \Pi_0 y, \sigma)$. $H_k^{(+)}(\tau_\sigma)$ are determined functions.

In a similar manner, for solving $\bar{Q}_k^{(-)} x(\tau_\sigma)$, we can obtain

$$\begin{cases} \bar{Q}_k^{(+)} y(\tau_\sigma) = (p_k - \bar{y}_k^{(3)}(\sigma))\Psi_3(\tau_\sigma)\Psi_3^{-1}(0) + \bar{Q}_k^{(+)} y^*(\tau_\sigma), \\ \bar{Q}_k^{(+)} z(\tau_\sigma) = \frac{d\Phi_5(\tilde{y}^5)}{d\tilde{y}^5}(p_k - \bar{y}_k^{(3)}(\sigma))\Psi_3(\tau_\sigma)\Psi_3^{-1}(0) + \bar{Q}_k^{(+)} z^*(\tau_\sigma), \end{cases}$$

where

$$\bar{Q}_k^{(+)} y^*(\tau_\sigma) = \int_0^{\tau_\sigma} \Psi_3(\tau_\sigma)\Psi_3^{-1}(s)\bigg[\int_{+\infty}^{s} \Psi_4(s)\Psi_4^{-1}(p) G_k^{(4)}(p)\,dp\bigg]\,ds$$

and

$$\bar{Q}_k^{(+)} z^*(\tau_\sigma) = \frac{d\Phi_5(\tilde{y}^5)}{d\tilde{y}^5} \cdot \bar{Q}_k^{(+)} y^*(\tau_\sigma) + \int_{+\infty}^{\tau_\sigma} \Psi_4(\tau_\sigma)\Psi_4^{-1}(s) G_k^{(4)}(s)\,ds.$$

Thus, $\bar{Q}_k^{(+)} x(\tau_\sigma)$ are completely determined.

Lemma *Under conditions* (H_1), (H_2), (H_3),

$$\Pi_k x(\tau_0), \qquad Q_k^{(\pm)} x(\tau_*), \qquad \bar{Q}_k^{(\pm)} x(\tau_\sigma) \quad (k \geq 0)$$

all satisfy exponential decay.

For boundary functions $R_k x(\tau_T)$ $(k \geq 0)$, they have no essential influence on the interior layer and the solving method of them completely coincides with $\Pi_k x(\tau_0)$ $(k \geq 0)$. We will not discuss them in detail.

3 The smooth connection of the asymptotic solution

In order to get a smooth solution in $[0,T]$, $y^{(1)}(t,\mu)$ and $y^{(2)}(t,\mu)$ must join smoothly at $t=t_*$, at the same time, $y^{(2)}(t,\mu)$ and $y^{(3)}(t,\mu)$ join smoothly at $t=\sigma$. Namely,

$$\frac{d}{dt}y^{(1)}(t_*,\mu)=\frac{d}{dt}y^{(2)}(t_*,\mu), \tag{3.1}$$

$$\frac{d}{dt}y^{(2)}(\sigma,\mu)=\frac{d}{dt}y^{(3)}(\sigma,\mu). \tag{3.2}$$

Substituting (2.3), (2.4) and (2.5) into (3.1), (3.2) respectively, we get a series of equations:

$$\frac{d}{d\tau}Q_0^{(-)}y(0)=\frac{d}{d\tau}Q_0^{(+)}y(0), \tag{3.3}$$

$$\varphi_1'(t_0)+\frac{d}{d\tau}Q_1^{(-)}y(0)=\varphi_3'(t_0)+\frac{d}{d\tau}Q_1^{(+)}y(0), \tag{3.4}$$

$$\vdots$$

$$\left(\bar{y}_{k-1}^{(1)}\right)'(t_0)+\frac{d}{d\tau}Q_k^{(-)}y(0)=\left(\bar{y}_{k-1}^{(2)}\right)'(t_0)+\frac{d}{d\tau}Q_k^{(+)}y(0) \tag{3.5}$$

and

$$\frac{d}{d\tau}\bar{Q}_0^{(-)}y(0)=\frac{d}{d\tau}\bar{Q}_0^{(+)}y(0), \tag{3.6}$$

$$\varphi_3'(\sigma)+\frac{d}{d\tau}\bar{Q}_1^{(-)}y(0)=\psi_1'(\sigma)+\frac{d}{d\tau}\bar{Q}_1^{(+)}y(0), \tag{3.7}$$

$$\vdots$$

$$\left(\bar{y}_{k-1}^{(2)}\right)'(\sigma)+\frac{d}{d\tau}\bar{Q}_k^{(-)}y(0)=\left(\bar{y}_{k-1}^{(3)}\right)'(\sigma)+\frac{d}{d\tau}\bar{Q}_k^{(+)}y(0). \tag{3.8}$$

Substituting (2.8), (2.26) into (3.3), we have

$$\int_{\varphi_1(t_0)}^{\varphi_2(t_0)}F\bigl(y,\alpha(t_0-\sigma),t_0\bigr)\,dy=\int_{\varphi_3(t_0)}^{\varphi_2(t_0)}F\bigl(y,\alpha(t_0-\sigma),t_0\bigr)\,dy,$$

that is,

$$H(t_0)\equiv\int_{\varphi_1(t_0)}^{\varphi_3(t_0)}F\bigl(y,\alpha(t_0-\sigma),t_0\bigr)\,dy=0, \tag{3.9}$$

which is the equation for finding t_0.

(H_4) Suppose that (3.9) is solvable for t_0 $(0<t_0<\sigma)$, and $\frac{d}{dt}H(t_0)\neq 0$.

Therefore, $Q_0^{(\mp)}y(\tau)$ exists. By (3.5), the equation to determine t_k is

$$H'(t_0)t_k=P_k, \tag{3.10}$$

where P_k are known constants. Thus $Q_k^{\pm}x(\tau_*)$ are all completely determined.

In the following, we will seek the value of p_k. Let

$$G(p_0)=\tilde{z}^{(4)}(0,p_0)-\tilde{z}^{(5)}(0,p_0)=\Phi_4(p_0)-\Phi_5\bigl(p_0,\alpha(0)\bigr). \tag{3.11}$$

(H_5) Suppose that there exists a solution $p_0 = \bar{p}_0$ for (3.11) that satisfies $\frac{dG}{dp_0}|_{p_0=\bar{p}_0} < 0$.
For p_k, by virtue of (2.38) and (3.8), we have

$$\begin{aligned}&\left(\frac{d\Phi_4(p_0)}{dp_0} - \frac{d\Phi_5(p_0)}{dp_0}\right)p_k \\ &\quad = \left(\left(\bar{y}^{(3)}_{k-1}(\sigma)\right)' - \left(\bar{y}^{(2)}_{k-1}(\sigma)\right)'\right) - \frac{d\Phi_5(p_0)}{dp_0}\left(\bar{y}^{(3)}_{k-1}(\sigma)\right) + \frac{d\Phi_4(p_0)}{dp_0}\left(\bar{y}^{(2)}_{k-1}(\sigma)\right) \\ &\qquad - \int_{-\infty}^{0} \Psi_2(0)\Psi_2^{-1}(s)G_k^{(3)}(s)\,ds + \int_{+\infty}^{0} \Psi_4(0)\Psi_4^{-1}(s)G_k^{(4)}(s)\,ds.\end{aligned}$$

By (H_5), the coefficient of p_k is not equal to zero , so p_k is determined. Thus $\bar{Q}_k^{(\pm)}x(\tau_\sigma)$ are all completely determined.

4 The existence of the complex solution

In this section, using the method of sewing connection, we prove the existence of the solution about problem (2.1), (1.2) and give out the estimates of the remainder. The solution of (2.1), (1.2) may be considered as a solution which is smoothly connected by the solutions of the following auxiliary problems.

The left problem ($0 \le t \le t_*$):

$$\mu^2\left(y^{(1)}\right)'' = F\left(y^{(1)}(t), \alpha(t-\sigma), t\right), \tag{4.1}$$

$$y^{(1)}(0,\mu) = \alpha(0), \qquad y^{(1)}(t_*,\mu) = \phi_2(t_*). \tag{4.2}$$

The middle problem ($t_* \le t \le \sigma$):

$$\mu^2\left(y^{(2)}\right)'' = F\left(y^{(2)}(t), \alpha(t-\sigma), t\right), \tag{4.3}$$

$$y^{(2)}(t_*,\mu) = \phi_2(t_*), \qquad y^{(2)}(\sigma,\mu) = \bar{p}(\mu), \tag{4.4}$$

where $\bar{p}(\mu) = p_0 + \mu(p_1 + \delta)$. Here, we do not expand the parameter t_*. δ is a parameter.

The right problem ($\sigma \le t \le T$):

$$\mu^2\left(y^{(3)}\right)'' = F\left(y^{(3)}(t), y^{(1)}(t-\sigma,\mu), t\right), \tag{4.5}$$

$$y^{(3)}(\sigma,\mu) = \bar{p}(\mu), \qquad y^{(3)}(T,\mu) = y^T, \tag{4.6}$$

where $\bar{p}(\mu) = p_0 + \mu(p_1 + \delta)$.

The problems (4.1), (4.2), (4.3), (4.4), and (4.5), (4.6) are all boundary layer problems, so their solutions exist and have the following form:

$$y^{(1)}(t,\mu) = \varphi_1(t) + \Pi_0 y(\tau_0) + Q_0^{(-)}y(\tau_*) + O(\mu), \tag{4.7}$$

$$y^{(2)}(t,\mu) = \varphi_3(t) + Q_0^{(+)}y(\tau_*) + \bar{Q}_0^{(-)}y(\tau_\sigma) + O(\mu), \tag{4.8}$$

$$y^{(3)}(t,\mu) = \psi_1(t) + \bar{Q}_0^{(+)}y(\tau_\sigma) + R_0 y(\tau_T) + O(\mu). \tag{4.9}$$

Considering (4.2) and (4.4), we see

$$y^{(1)}(t_*,\mu) = y^{(2)}(t_*,\mu), \quad t_* \in (0,1), \tag{4.10}$$

which implies that $y(t,\mu)$ is continuous at $t = t_*$. Therefore, t_* can be determined by the following formula:

$$z^{(1)}(t_*,\mu) = z^{(2)}(t_*,\mu). \tag{4.11}$$

It yields

$$\Delta(t_*) = H(t_*) + O(\mu) = H(t_0) + \frac{d}{dt}H(t_0)(t_* - t_0) + O\big((t_* - t_0)^2\big) + O(\mu), \tag{4.12}$$

where t_0 is known by (3.9). Let $t_* = t_0 \pm k\mu$ and put it into (4.12), we obtain

$$\Delta(t_0 \pm k\mu) = \pm k\mu \frac{d}{dt}H(t_0) + O(\mu). \tag{4.13}$$

Let k in (4.13) be sufficiently large and select μ sufficiently small, then the symbols of the right-hand side of (4.13) are different. By virtue of the intermediate value theorem, there exists $\bar{t}_* \in (t_0 - k\mu, t_0 + k\mu)$ such that $\Delta(\bar{t}_*) = 0$. So (4.11) holds, and $\bar{t}_* = t_0 + O(\mu)$.

From (4.4), (4.6) we know that $y^{(2)}(t,\mu)$, $y^{(3)}(t,\mu)$ are continuous at $t = \sigma$. For their smooth connection, $\frac{d}{dt}y^{(2)}(\sigma,\mu) = \frac{d}{dt}y^{(3)}(\sigma,\mu)$ is necessary.

Let $W(\bar{p},\mu) = \frac{d}{dt}y^{(2)}(\sigma,\mu) - \frac{d}{dt}y^{(3)}(\sigma,\mu)$. Considering the smooth condition (3.6), (3.7), we have

$$\begin{aligned} W(\bar{p},\mu) &= \mu\left[\big(\bar{y}_1^{(2)}(\sigma)\big)' - \big(\bar{y}_1^{(3)}(\sigma)\big)' + \frac{d}{d\tau}\bar{Q}_1^{(-)}y(0) - \frac{d}{d\tau}\bar{Q}_1^{(+)}y(0)\right] + o(\mu) \\ &= \mu\delta \frac{dG}{dp_0}\bigg|_{p_0=\bar{p}_0} + o(\mu). \end{aligned}$$

When μ is sufficiently small and δ has different sign, $G(\bar{p},\mu)$ also has different sign. By virtue of the intermediate value theorem, there exists $\bar{p}^* \in [p_1 - \delta, p_1 + \delta]$ such that $G(\bar{p}^*,\mu) = 0$.

We write the results in the following theorem.

Theory 1 *Under conditions* (H_4)–(H_5), *the smooth solution* $y(t,\mu)$ *of* (1.1), (1.2) *exists in the interval* $[0,T]$. *The zeroth asymptotic expansion of* (1.1), (1.2) *is*

$$y(t,\mu) = \begin{cases} \varphi_1(t) + \Pi_0 y(\tau_0) + Q_0^{(-)}y(\tau_*) + O(\mu), & 0 \le t \le t_*, \\ \varphi_3(t) + Q_0^{(+)}y(\tau_*) + \bar{Q}_0^{(-)}y(\tau_\sigma) + O(\mu), & t_* \le t \le \sigma, \\ \psi_1(t) + \bar{Q}_0^{(+)}y(\tau_\sigma) + R_0 y(\tau_T) + O(\mu), & \sigma \le t \le T. \end{cases}$$

Similarly, we can obtain the higher order asymptotic expansion.

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