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On weak and strong convergence results for generalized equilibrium variational inclusion problems in Hilbert spaces

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Abstract

We introduce a new iterative method for finding a common element of the set of fixed points of pseudo-contractive mapping, the set of solutions to a variational inclusion and the set of solutions to a generalized equilibrium problem in a real Hilbert space. We provide some results about strongly and weakly convergent of the iterative scheme sequence to a point $p \in \Omega$ which is the unique solution of a variational inequality, where Ω is an intersection of set as given by $\Omega = F(S) \cap (A+B)^{-1}(0) \cap N^{-1}(0) \cap \mathrm{GEP}(F,M) \neq \emptyset$. This gives us a common solution. Also, We show that our results extend some published recent results in this field. Finally, we provide an example to illustrate our main result.

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Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively, C a nonempty, closed and convex subset of H. Recall that a mapping $S: C \to C$ is said to be pseudo-contractive if and only if $\|Su - Sv\|^2 \leq \|u-v\|^2 + \|(I-S)u-(I-S)v\|^2$ for all $u,v \in C$. Equivalently, $\langle u-v, Su-Sv\rangle \leq \|u-v\|^2$ for all $u,v\in C$. A mapping $S: C\to C$ is said to be k-strictly pseudo-contractive if and only if there exists $0 \leq k < 1$ such that $\|Su-Sv\|^2 \leq \|u-v\|^2 + k\|(I-S)u-(I-S)v\|^2$ for all $u,v \in C$. Equivalently, $\langle u-v, Su-Sv\rangle \leq \|u-v\|^2 - k\|(I-S)u-(I-S)v\|^2$ for all $u,v\in C$. A mapping L-Lipschitz if there exists $L \geq 0$ such that $\|Su - Sv\| \leq L\|u-v\|$ for all $u,v \in C$. The mapping S is called nonexpansive if $L = 1$ and is called contractive if $L < 1$. A mapping S is called firmly nonexpansive if $\|Su-Sv\|^2 \leq \|u-v\|^2 - \|(I-S)u-(I-S)v\|^2$ for all $u,v\in C$. Every nonexpansive mapping is a k-strictly pseudo-contractive mapping and every k-strictly pseudo-contractive mapping is pseudo-contractive. Assume that $S: C\to C$ be a strictly pseudo-contractive. We denote by $F(S)$ the fixed point set of S, that is, $F(S) = \{x \in C : S(x) = x\}$. There is a lot of work associated with the fixed point algorithms (see for example, [1–6]). Also, there are many papers and books about iterative schemes for numerical estimations in different area of this field (see for example [7–12]).

Let $A : C \to H$ be a nonlinear mapping and F be a bi-function from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The generalized equilibrium problem is to find $x^* \in C$ such that $F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0$, for all $y \in C$. The set of solutions of x^* is denoted by $\mathrm{GEP}(F, A)$ ([13]). If $A = 0$, then $\mathrm{GEP}(F, A)$ is denoted by $\mathrm{EP}(F)$. If $F(x, y) = 0$ for all $x, y \in C$, then $\mathrm{GEP}(F, A)$ is denoted by $\mathrm{VI}(C, A) = \{x^* \in C : \langle Ax^*, y - x^*\rangle \geq 0, y \in C\}$. This is the set of solutions of the variational inequality for A ([14–16]). If $C = H$, then $\mathrm{VI}(H, A) = A^{-1}(0)$ where $A^{-1}(0) = \{x \in H : Ax = 0\}$. Recall that a mapping $A : C \to H$ is said to be monotone whenever $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in C$. A mapping A is said to be α-strongly monotone whenever there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2$ for all $u, v \in C$. A mapping A is said to be α-inverse strongly monotone whenever there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$ for all $u, v \in C$. For such a case, A is said to be α-inverse strongly monotone. Note that any α-inverse strongly monotone mapping A is Lipschitz and $\|Au - Av\| \leq \frac{1}{\alpha}\|u - v\|$. Let $A : H \to H$ be a single-valued nonlinear mapping, $B : H \to 2^H$ a set-valued mapping. The variational inclusion is to find $p \in H$ such that

$$\theta \in A(p) + B(p), \tag{1}$$

where θ is a zero vector in H. When $A = 0$, then (1) becomes the inclusion problem introduced by Rockafellar ([17]). Let $B : H \to 2^H$ be a mapping. The effective domain of B is denoted by $D(B)$, namely, $D(B) = \{x \in H : Bx \neq \emptyset\}$. The graph of B is $G(B) = \{(u, v) \in H \times H : v \in Bu\}$. A set-valued mapping B is said to be monotone whenever $\langle x - y, f - h\rangle \geq 0$ for all $x, y \in D(B)$, $f \in Bx$ and $h \in By$. A monotone operator B is maximal if the graph $G(B)$ of B is not properly contained in the graph of any other monotone mapping. Also, a monotone mapping B is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - h\rangle \geq 0$ for every $(y, h) \in G(B)$ implies $f \in Bx$. For a maximal monotone operator B on H and $r > 0$, we define a single-valued operator $J_r^B x = (I + rB)^{-1} : H \to D(B)$, which is called the resolvent of B for r. It is well known that $J_r^B x$ is firmly nonexpansive, that is, $\langle x - y, J_r^B x - J_r^B y\rangle \geq \|J_r^B x - J_r^B y\|^2$ for all $x, y \in H$, and that a solution of (1) is a fixed point of $J_r^B(I - rA)$ for all $r > 0$ (see[18]).

A basic problem for maximal monotone operator B is to find

$$x \in H \quad \text{such that } 0 \in Bx. \tag{2}$$

A well-known method for solving problem (2) is the proximal point algorithm: $x_1 = x \in H$, and

$$x_{n+1} = J_{r_n}^B x_n, \quad n = 1, 2, 3, \ldots,$$

where $J_{r_n}^B = (I + r_n B)^{-1}$ and $\{r_n\} \subset (0, \infty)$. For any initial guess $x^* \in H$, the proximal point algorithm generates an iterative sequence as $x_{n+1} = J_{r_n}^B(x_n + e_n)$, where e_n is the error sequence, then Rockafellar ([17, 19]) proved that the sequence $\{x_n\}$ converges weakly to an element of $B^{-1}(0)$. To ensure convergence, it is assumed that $\|e_{n+1}\| \leq \varepsilon_n\|x_{n+1} - x_n\|$ with $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ ([17]). This criterion was then improved by Han and He as $\|e_{n+1}\| \leq \varepsilon_n\|x_{n+1} - x_n\|$ with $\sum_{n=0}^{\infty} \varepsilon_n^2 < \infty$ ([20]). Then Kamimura and Takahashi introduced the following iterative method:

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)J_{r_n}^B(x_n + e_n),$$

where $u \in H$ is fixed and (λ_n) is a real sequence ([3]). They proved that the sequence $\{x_n\}$ converges strongly to $x^* = P_{(B)^{-1}(0)}(u)$. Then Ceng, Wu and Yao obtained the norm convergence under the following conditions:

(i) $\lim_{n\to\infty} r_n = \infty$,

(ii) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$,

(iii) $\|e_{n+1}\| \leq \varepsilon_n \|x_{n+1} - x_n\|$ with $\sum_{n=0}^{\infty} \varepsilon_n^2 < \infty$ ([21]).

In 2013, Tian and Wang show that, if $\{r_n\}$ be bounded below away from zero, then the norm convergence is still guaranteed for bounded (r_n), especially for constant sequence ([22]). In the literature, there are a large number references associated with the proximal point algorithm (see for example, [20–31]).

In 2008, Takahashi and Takahashi introduced an iterative method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions to a generalized equilibrium problem in a real Hilbert space ([13]). In 2019, Qin, Cho and Yao introduced the following iterative scheme in Banach space E:

$$\begin{cases} x_0 \in C \cap D, \\ y_n = \beta_n T x_n + (1 - \beta_n) x_n, \\ x_{n+1} = P_{C\cap D}^E (\alpha_n f(x_n) + \delta_n R_{r_n}^M (x_n - r_n N x_n + e_n) + \gamma_n y_n), \quad n \geq 0, \end{cases} \tag{3}$$

where $\{e_n\}$ is a sequence in E such that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, C and D is two nonempty closed and convex subsets of E, $P_{C\cap D}^E$ is a sunny nonexpansive retraction from E onto $C \cap D$, $M : D \to 2^E$ is an m-accretive operator, $N : C \to E$ is an α-inverse strongly accretive operator, R_r^M the resolvent of N for each $r > 0$, $f : C \to E$ is a k-contraction, $T : C \to E$ is a k-strict pseudo-contraction with a nonempty fixed point set ([32]). They proved that the sequence $\{x_n\}$ generated by (3) converges strongly to $x^* = P_{F(T)\cap(N+M)^{-1}(0)}^{C\cap D} f(x^*)$, where x^* is the unique solution of the variational inequality $\langle f(x^*) - x^*, J_q(y - x^*)\rangle \leq 0$, $y \in F(T) \cap (N + M)^{-1}(0)$ ([32]).

The purpose of this paper is to prove the strong and weak convergence of new algorithms under different criteria of the errors $\{e_n\}$. We use a new technique of argument for dealing with strong and weak convergence, also, suggest and propose the new accuracy criteria for modified approximate proximal point algorithms. Applications of the main results are also provided. In this paper, motivated by the mentioned above results, we present an iterative method which converges strong and weak to a common element of the fixed point set of pseudo-contractive mapping and the zero set of the sums of maximal monotone operators and the set of solutions to a generalized equilibrium problem in a real Hilbert space.

1 Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. It is well known that, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| = \inf_{y\in C} \|x - y\| =: d(x, C)$. It is well known that P_C is nonexpansive and monotone mapping of H onto C and satisfies the following:

(1) $\langle x - P_C x, z - P_C x\rangle \leq 0$ for all $x \in H$, $z \in C$.

(2) $\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2$ for all $x \in H$, $z \in C$.

(3) The relation $\langle P_C x - P_C z, x - z\rangle \geq \|P_C x - P_C z\|^2$ holds for all $z, x \in H$.

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from (2) that

$$p \in \mathrm{VI}(C,A) \quad \Leftrightarrow \quad p = P_C(p - \lambda Ap) \quad \text{for some } \lambda > 0.$$

For solving the equilibrium problem for a bi-function $F : C \times C \to \mathbb{R}$, we assume that F satisfy the following conditions:

(A_1) $F(x,x) = 0$ for all $x \in C$,

(A_2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$,

(A_3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$,

(A_4) for each $x \in C$, the function $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Putting $F(x,y) = \langle Ax, y - x\rangle$ for every $x, y \in C$, we see that the equilibrium problem is reduced to the variational inequality.

Lemma 1.1 ([33]) *Assume that B is a maximal monotone operator. The followings hold.*

(a) $D(J_r^B) = H$,

(b) *J_r^B is single-valued and firmly nonexpansive*

(c) $F(J_r^B) = \Gamma = \{x \in D(B) : 0 \in B(x)\}$,

(d) *its graph $G(B)$ is weak-to-strong closed in $H \times H$.*

Lemma 1.2 ([34, 35]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A_1)–(A_4) and C is a nonempty, closed and convex subset of H. For $r > 0$ and $x \in H$, consider the map $T_r : H \to C$ defined by*

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \textit{ for all } y \in C \right\}.$$

For each $c \in H$, we have $T_r(x) \neq \emptyset$, T_r is single-valued, $\mathrm{EP}(F)$ is closed and convex, $F(T_r) = \mathrm{EP}(F)$ and T_r is firmly nonexpansive, that is, $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle$ for all $x, y \in H$.

Lemma 1.3 ([36]) *Assume that C is a nonempty, closed and convex subset of H, F is a bi-function from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), A_F is the multivalued mapping from H into itself defined by $A_F x = \{z \in C : F(z,y) \leq \langle y - x, z\rangle$ for all $y \in C\}$ whenever $x \in C$ and $A_F x = \emptyset$ otherwise. In this case, A_F is a maximal monotone operator with the domain $T_r(x) = (I + rA_F)^{-1}x$, for all $x \in H$ and $r > 0$.*

Lemma 1.4 ([5]) *Assume that H is a real Hilbert space, C is a closed convex subset of H and $T : C \to C$ is a continuous pseudo-contractive mapping. In this case, $F(T)$ is a closed convex subset of C and $(I - T)$ is demiclosed at zero, that is, $x = T(x)$ whenever $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$.*

Lemma 1.5 ([37]) *If $\{x_n\}, \{a_n\} \subset \mathbb{R}^+$, $\{\lambda_n\} \subset (0,1)$ and $\{\gamma_n\} \subset \mathbb{R}$ are some sequences such that $x_{n+1} \leq (1 - \lambda_n)x_n + \lambda_n\gamma_n + a_n$ for all $n \geq 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\limsup_{n\to\infty} \gamma_n \leq 0$ and $\sum_{n=0}^{\infty} a_n < \infty$, then $\lim_{n\to\infty} x_n = 0$.*

Lemma 1.6 ([38]) *Assume that H is a real Hilbert space. For each $x_j \in H$ and $a_j \in [0,1]$ for $j = 1,2,3$ with $a_1 + a_2 + a_3 = 1$ the following equality holds*:

$$\|a_1x_1 + a_2x_2z + a_3x_3\|^2 = a_1\|x_1\|^2 + a_2\|x_2\|^2 + a_3\|x_3\|^2 - \sum_{1\le i,j\le 3} a_ia_j\|x_i - x_j\|^2.$$

Lemma 1.7 ([36]) *Suppose that B is a maximal monotone operator on H. In this case, we have*

$$\frac{\lambda - r}{r}\langle J_\lambda^B x - J_r^B x, J_\lambda^B x - x\rangle \ge \|J_\lambda^B x - J_r^B x\|^2 \quad \forall \lambda, r > 0 \text{ and } x \in H.$$

Lemma 1.8 ([5]) *Suppose that H is a real Hilbert space. For every $x, y \in H$, we have $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$.*

Lemma 1.9 ([39]) *Assume that $\{x_n\}$ is sequences of real numbers and there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \le x_n$ for all $k \in \mathbb{N}$. There exists a nondecreasing sequence $\{t_i\} \subset \mathbb{N}$ such that $x_{t_i} \le x_{t_i+1}$ and $x_i \le x_{t_i+1}$ for all $i \ge 1$. In fact, $t_i = \max\{k \le i : x_k \le x_{k+1}\}$.*

2 Weak and strong convergence theorems

Now, we are ready to state and prove our main results.

Theorem 2.1 *Suppose that C is a nonempty, closed and convex subset of H, F is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), M is an α-inverse strongly monotone mapping from C into H, A is a β-inverse strongly monotone map from C into H, B and N are two maximal monotone operators on H such that their domains contained in C, $f : C \to C$ is a ρ-contractive map with $\rho \in (0, \frac{1}{2})$ and $S : C \to C$ is a Lipschitz pseudo-contractive mapping with Lipschitz constants K such that $\Omega = F(S) \cap (A + B)^{-1}(0) \cap N^{-1}(0) \cap \mathrm{GEP}(F, M) \neq \emptyset$. Assume that $\{b_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are some sequences in $(0,1)$ and $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$ are the sequences generated by*

$$\begin{cases} x_0 \in C, \\ F(y_n, y) + \langle Mx_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ u_n = J_{\lambda_n}^B(y_n - \lambda_n Ay_n), \\ z_n = b_n f(x_n) + (1 - b_n)J_{s_n}^N(u_n + e_n), \\ x_{n+1} = (1 - \beta_n)z_n + \beta_n S(\delta_n z_n + (1 - \delta_n)Sz_n) \quad \forall n \ge 0. \end{cases} \tag{4}$$

If

- (d_1) $0 < c \le \lambda_n \le d < 2\beta$, $0 < a \le r_n \le b < 2\alpha$,
- (d_2) $0 < c < \beta_n \le \delta_n < d < \frac{1}{\sqrt{1+K^2}+1}$, $s_n > s > 0$,
- (d_3) $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$,
- (d_4) $\|e_n\| \le \frac{\varepsilon_n}{2}\max\{\|u_n - J_{s_n}^N(u_n + e_n)\|, \|J_{s_n}^N(u_n + e_n) - p\|\}$ *with* $\sum_{n=0}^{\infty} \varepsilon_n < \infty$,

then $\{x_n\}$ converges strongly to a point $p \in \Omega$ which is the unique solution of the variational inequality $\langle (I - f)p, x - p\rangle \ge 0$ for all $x \in \Omega$.

Proof We first show that $I-\lambda_n A$ is nonexpansive. For all $u,v\in C$ and $0<\lambda_n<2\beta$, we obtain

$$\begin{aligned}\bigl\|(I-\lambda_n A)u-(I-\lambda_n A)v\bigr\|^2 &= \bigl\|(u-v)-\lambda_n(Au-Av)\bigr\|^2\\ &\le \|u-v\|^2-2\lambda_n\langle u-v,Au-Av\rangle+\lambda_n^2\|Au-Av\|^2\\ &\le \|u-v\|^2-\lambda_n\beta\|Au-Av\|^2+\lambda_n^2\|Au-Av\|^2\\ &= \|u-v\|^2+\lambda_n(\lambda_n-2\beta)\|Au-Av\|^2\\ &\le \|u-v\|^2. \end{aligned}\tag{5}$$

This proves that $I-\lambda_n A$ is nonexpansive. Let $p\in\Omega$. Observe that y_n can be re-written as $y_n=T_{r_n}(x_n-r_nMx_n)$, $n\ge 0$. From (d_2) and Lemma 1.2, we have

$$\begin{aligned}\|y_n-p\|^2 &= \bigl\|T_{r_n}(x_n-r_nMx_n)-p\bigr\|^2\\ &= \bigl\|T_{r_n}(x_n-r_nMx_n)-T_{r_n}(p-r_nMp)\bigr\|^2\\ &\le \bigl\|(x_n-r_nMx_n)-(p-r_nMp)\bigr\|^2\\ &= \|x_n-p\|^2+r_n(r_n-2\alpha)\|Mx_n-Mp\|^2\\ &\le \|x_n-p\|^2. \end{aligned}\tag{6}$$

From (4), (5) and using the fact that $J^B_{\lambda_n}$ is nonexpansive, we have

$$\begin{aligned}\|u_n-p\|^2 &= \bigl\|J^B_{\lambda_n}(y_n-\lambda_n Ay_n)-p\bigr\|^2\\ &\le \bigl\|J^B_{\lambda_n}(y_n-\lambda_n Ay_n)-J^B_{\lambda_n}(p-\lambda_n Ap)\bigr\|^2\\ &\le \bigl\|(y_n-\lambda_n Ay_n)-(p-\lambda_n Ap)\bigr\|^2\\ &\le \|y_n-p\|^2+\lambda_n(\lambda_n-2\beta)\|Ay_n-Ap\|^2\\ &\le \|x_n-p\|^2+\lambda_n(\lambda_n-2\beta)\|Ay_n-Ap\|^2\\ &\le \|x_n-p\|^2. \end{aligned}\tag{7}$$

Set $t_n=(1-\delta_n)z_n+\delta_n Sz_n$ for all $n\ge 1$. By using Lemma 1.6, we have

$$\begin{aligned}\|t_n-p\|^2 &= \bigl\|(1-\delta_n)z_n+\delta_n Sz_n-p\bigr\|^2\\ &\le (1-\delta_n)\|z_n-p\|^2+\delta_n\|Sz_n-p\|^2-(1-\delta_n)\delta_n\|z_n-Sz_n\|^2\\ &\le (1-\delta_n)\|z_n-p\|^2+\delta_n\bigl(\|z_n-p\|^2+\|z_n-Sz_n\|^2\bigr)-(1-\delta_n)\delta_n\|z_n-Sz_n\|^2\\ &\le \|z_n-p\|^2+\delta_n^2\|z_n-Sz_n\|^2. \end{aligned}\tag{8}$$

From (4) and (8), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \bigl\|(1-\beta_n)z_n+\beta_n S\bigl((1-\delta_n)z_n+\delta_n Sz_n\bigr)-p\bigr\|^2\\ &= \bigl\|(1-\beta_n)z_n+\beta_n St_n-p\bigr\|^2\\ &\le (1-\beta_n)\|z_n-p\|^2+\beta_n\|St_n-p\|^2-(1-\beta_n)\beta_n\|z_n-St_n\|^2 \end{aligned}$$

$$\begin{aligned}
&\le (1-\beta_n)\|z_n-p\|^2+\beta_n\bigl(\|t_n-p\|^2+\|t_n-St_n\|^2\bigr)-(1-\beta_n)\beta_n\|z_n-St_n\|^2\\
&\le (1-\beta_n)\|z_n-p\|^2+\beta_n\|t_n-p\|^2+\beta_n\|t_n-St_n\|^2-(1-\beta_n)\beta_n\|z_n-St_n\|^2\\
&\le \|z_n-p\|^2+\delta_n^2\beta_n\|z_n-Sz_n\|^2+\beta_n\|t_n-St_n\|^2\\
&\quad -(1-\beta_n)\beta_n\|z_n-St_n\|^2. \qquad (9)
\end{aligned}$$

Thus,

$$\begin{aligned}
\|t_n-St_n\|^2 &= \bigl\|(1-\delta_n)z_n+\delta_n Sz_n-St_n\bigr\|^2\\
&\le (1-\delta_n)\|z_n-St_n\|^2+\delta_n\|Sz_n-St_n\|^2\\
&\quad -(1-\delta_n)\delta_n\|z_n-Sz_n\|^2. \qquad (10)
\end{aligned}$$

Since S is K-Lipschitz and $z_n-t_n=\delta_n(z_n-Sz_n)$, by using (10) we get

$$\begin{aligned}
&\|t_n-St_n\|^2\\
&\quad\le (1-\delta_n)\|z_n-St_n\|^2+\delta_n K^2\|z_n-t_n\|^2-(1-\delta_n)\delta_n\|z_n-Sz_n\|^2\\
&\quad\le (1-\delta_n)\|z_n-St_n\|^2+\delta_n^3K^2\|z_n-Sz_n\|^2-(1-\delta_n)\delta_n\|z_n-Sz_n\|^2\\
&\quad= (1-\delta_n)\|z_n-St_n\|^2-\delta_n\bigl(1-\delta_n-\delta_n^2K^2\bigr)\|z_n-Sz_n\|^2.
\end{aligned}$$

This together with (9) implies that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|z_n-p\|^2+\beta_n\bigl((1-\delta_n)\|z_n-St_n\|^2\\
&\quad -\delta_n\bigl(1-\delta_n-\delta_n^2K^2\bigr)\|z_n-Sz_n\|^2\bigr)\\
&\quad +\delta_n^2\beta_n\|z_n-Sz_n\|^2-(1-\beta_n)\beta_n\|z_n-St_n\|^2\\
&\le \|z_n-p\|^2-\delta_n\bigl(1-2\delta_n-\delta_n^2K^2\bigr)\|z_n-Sz_n\|^2\\
&\quad +\beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2. \qquad (11)
\end{aligned}$$

Since $0<c<\beta_n\le\delta_n<d<\frac{1}{\sqrt{1+K^2}+1}$ for all $n\ge 1$, we conclude that

$$\|x_{n+1}-p\|^2\le\|z_n-p\|^2. \qquad (12)$$

Put $v_n=J^N_{\lambda_n}(u_n+e_n)$ for all $n\ge 0$. By using Lemma 1.1, we obtain

$$\begin{aligned}
\|v_n-p\|^2 &\le \|u_n+e_n-p\|^2-\|u_n+e_n-v_n\|^2\\
&= \|u_n-p\|^2-\|u_n-v_n\|^2+2\langle e_n,v_n-p\rangle\\
&\le \|u_n-p\|^2-\|u_n-v_n\|^2+2\|e_n\|\|v_n-p\|.
\end{aligned}$$

Since

$$\begin{aligned}
\|e_n\| &\le \frac{\varepsilon_n}{2}\max\bigl\{\|u_n-v_n\|,\|v_n-p\|\bigr\}\\
&\le \frac{\varepsilon_n}{2}\bigl(\|v_n-p\|+\|u_n-v_n\|\bigr),
\end{aligned}$$

this implies that

$$\begin{aligned}\|v_n - p\|^2 &\leq \|u_n - p\|^2 - \|u_n - v_n\|^2 + 2\frac{\varepsilon_n}{2}\big(\|v_n - q\| + \|u_n - v_n\|\big)\|v_n - p\|.\\ &\leq \|u_n - p\|^2 - \left(1 - \frac{\varepsilon_n}{2}\right)\|u_n - v_n\|^2 + 2\varepsilon_n\|v_n - q\|^2.\end{aligned}$$

Since $\varepsilon_n \to 0$, for all $n \geq m_0$, we see that there exists an integer $m_0 \geq 0$ such that $1 - 2\varepsilon_n > 0$. It follows from (7) that

$$\begin{aligned}\|v_n - p\|^2 &\leq \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|u_n - p\|^2 - \frac{1 - \frac{\varepsilon_n}{2}}{1 - 2\varepsilon_n}\|u_n - v_n\|^2\\ &\leq \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|u_n - p\|^2 - \|u_n - v_n\|^2\\ &\leq \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|^2 - \|u_n - v_n\|^2.\end{aligned} \tag{13}$$

It follows that

$$\|v_n - p\| \leq \left(1 + \frac{\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|. \tag{14}$$

It follows from (4) and the last inequality that

$$\begin{aligned}\|z_n - p\| &= \big\|b_n f(x_n) + (1 - b_n)v_n - p\big\|\\ &\leq b_n\big\|f(x_n) - p\big\| + (1 - b_n)\|v_n - p\|\\ &\leq b_n\big(\rho\|x_n - p\| + \big\|f(p) - p\big\|\big) + (1 - b_n)\left(1 + \frac{\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|.\\ &\leq \left(1 + \frac{\varepsilon_n}{1 - 2\varepsilon_n}\right)\big(1 - b_n(1 - \rho)\big)\|x_n - p\| + b_n\big\|f(p) - p\big\|.\end{aligned}$$

Now, by induction we have

$$\|x_{n+1} - p\| \leq \prod_{i=0}^{n}\left(1 + \frac{\varepsilon_i}{1 - 2\varepsilon_i}\right)\max\left\{\frac{1}{1 - \rho}\big\|f(p) - p\big\|, \|x_0 - p\|\right\}, \quad \forall n \geq 0. \tag{15}$$

Indeed when $n = 0$, from (12) we have

$$\begin{aligned}\|x_1 - p\| &\leq \left(1 + \frac{\varepsilon_0}{1 - 2\varepsilon_0}\right)\big(1 - b_0(1 - \rho)\big)\|x_0 - p\| + b_0\big\|f(p) - p\big\|\\ &\leq \left(1 + \frac{\varepsilon_0}{1 - 2\varepsilon_0}\right)\big[\big(1 - b_0(1 - \rho)\big)\|x_0 - p\| + b_0\big\|f(p) - p\big\|\big]\\ &\leq \left(1 + \frac{\varepsilon_0}{1 - 2\varepsilon_0}\right)\max\left\{\frac{1}{1 - \rho}\big\|f(p) - p\big\|, \|x_0 - p\|\right\},\end{aligned}$$

which implies that (15) holds for $n = 0$. Assume that (15) holds for $n \geq 1$. Then it follows that $\|x_n - p\| \leq \prod_{i=0}^{n-1}(1 + \frac{\varepsilon_i}{1-2\varepsilon_i})\max\{\frac{1}{1-\rho}\|f(p) - p\|, \|x_0 - p\|\}$. Hence, from (12) we have

$$\|x_{n+1} - p\| \leq \left(1 + \frac{\varepsilon_n}{1 - 2\varepsilon_n}\right)\big(1 - b_n(1 - \rho)\big)\|x_n - p\| + b_n\big\|f(p) - p\big\|$$

$$\begin{aligned}
&\le \left(1+\frac{\varepsilon_n}{1-2\varepsilon_n}\right)\big[\big(1-b_n(1-\rho)\big)\|x_n-p\|+b_n\big\|f(p)-p\big\|\big] \\
&\le \left(1+\frac{\varepsilon_n}{1-2\varepsilon_n}\right)\prod_{i=0}^{n-1}\left(1+\frac{\varepsilon_i}{1-2\varepsilon_i}\right)\max\left\{\frac{1}{1-\rho}\big\|f(p)-p\big\|,\|x_0-p\|\right\} \\
&= \prod_{i=0}^{n}\left(1+\frac{\varepsilon_i}{1-2\varepsilon_i}\right)\max\left\{\frac{1}{1-\rho}\big\|f(p)-p\big\|,\|x_0-p\|\right\}.
\end{aligned}$$

This indicates that (15) holds for $n+1$. Therefore, (15) holds for $n \ge 0$. We have

$$\begin{aligned}
\|x_{n+1}-p\| &\le \prod_{i=0}^{n}\left(1+\frac{\varepsilon_i}{1-2\varepsilon_i}\right)\max\left\{\frac{1}{1-\rho}\big\|f(p)-p\big\|,\|x_0-p\|\right\} \\
&\le \prod_{i=0}^{\infty}\left(1+\frac{\varepsilon_i}{1-2\varepsilon_i}\right)\max\left\{\frac{1}{1-\rho}\big\|f(p)-p\big\|,\|x_0-p\|\right\}.
\end{aligned}$$

Since $\sum_{n=0}^{+\infty}\varepsilon_n < +\infty$, it follows that $\prod_{n=m_0}^{+\infty}(1+\frac{\varepsilon_n}{1-2\varepsilon_n}) < +\infty$. Thus, $\{\|x_n-p\|\}$ is bounded. So, $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$. Let $p = P_\Omega f(p)$. We have from (6), (7), (11), (14) and Lemma 1.8

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|z_n-p\|^2-\delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2 \\
&\quad +\beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2 \\
&= \big\|b_nf(x_n)+(1-b_n)v_n-p\big\|^2-\delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2 \\
&\quad +\beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2 \\
&\le \big\|b_n\big(f(x_n)-p\big)+(1-b_n)(v_n-p)\big\|^2-\delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2 \\
&\quad +\beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2 \\
&\le (1-b_n)\|v_n-p\|^2+2b_n\big\langle f(x_n)-p,x_{n+1}-p\big\rangle \\
&\quad +\beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2-\delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2 \\
&\le (1-b_n)\left[\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2-\|u_n-v_n\|^2\right] \\
&\quad +2b_n\big\langle f(x_n)-p,x_{n+1}-p\big\rangle \\
&\quad +(1-b_n)\big[\lambda_n(\lambda_n-2\beta)\|Ay_n-Ap\|^2+r_n(r_n-2\alpha)\|Mx_n-Mp\|^2\big] \\
&\quad +\beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2-\delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2 \\
&\le (1-b_n)\left[\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2-\|u_n-v_n\|^2\right] \\
&\quad +2b_n\big[\big\langle f(x_n)-p,x_n-p\big\rangle+\big\langle f(x_n)-p,x_{n+1}-x_n\big\rangle\big] \\
&\quad +(1-b_n)\big[\lambda_n(\lambda_n-2\beta)\|Ay_n-Ap\|^2+r_n(r_n-2\alpha)\|Mx_n-Mp\|^2\big] \\
&\quad +\beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2-\delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2 \\
&\le (1-b_n)\left[\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2-\|u_n-v_n\|^2\right] \\
&\quad +2b_n\big[\big\langle f(x_n)-f(p),x_n-p\big\rangle+\big\langle f(p)-p,x_n-p\big\rangle+\|x_{n+1}-x_n\|\big\|f(x_n)-p\big\|\big]
\end{aligned}$$

$$
\begin{aligned}
&+ (1-b_n)\big[\lambda_n(\lambda_n-2\beta)\|Ay_n-Ap\|^2 + r_n(r_n-2\alpha)\|Mx_n-Mp\|^2\big] \\
&+ \beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2 - \delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2 \\
\le{}& (1-b_n)\left[\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2 - \|u_n-v_n\|^2\right] \\
&+ 2b_n\big[\rho\|x_n-p\|^2 + \langle f(p)-p, x_n-p\rangle + \|x_{n+1}-x_n\|\,\|f(x_n)-p\|\big] \\
&+ (1-b_n)\big[\lambda_n(\lambda_n-2\beta)\|Ay_n-Ap\|^2 + r_n(r_n-2\alpha)\|Mx_n-Mp\|^2\big] \\
&+ \beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2 - \delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2 \\
\le{}& \big(1-b_n(1-2\rho)\big)\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2 - (1-b_n)\|u_n-v_n\|^2 \\
&+ 2b_n\big[\langle f(p)-p, x_n-p\rangle + \|x_{n+1}-x_n\|\,\|f(x_n)-p\|\big] \\
&+ (1-b_n)\big[\lambda_n(\lambda_n-2\beta)\|Ay_n-Ap\|^2 + r_n(r_n-2\alpha)\|Mx_n-Mp\|^2\big] \\
&+ \beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2 - \delta_n\big(1-2\delta_n-\delta_n^2K^2\big)\|z_n-Sz_n\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-p\| \le{}& \big(1-b_n(1-2\rho)\big)\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2 \\
&+ 2b_n\big[\langle f(p)-p, x_n-p\rangle + \|x_{n+1}-x_n\|\,\|f(x_n)-p\|\big].
\end{aligned}
\tag{16}
$$

Next, we split the proof into two cases.

Case 1: Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|\}$ is decreasing for all $n \ge n_0$. Therefore, we obtain $\lim_{n\to\infty}\|x_n-p\| = d$. Consequently, we obtain

$$
\begin{aligned}
&(1-b_n)\big[\|u_n-v_n\|^2 + \lambda_n(2\beta-\lambda_n)\|Ay_n-Ap\|^2 + r_n(2\alpha-r_n)\|Mx_n-Mp\|^2\big] \\
&\quad + \beta_n(\delta_n-\beta_n)\|z_n-St_n\|^2 + \delta_n\big(2\delta_n+\delta_n^2K^2-1\big)\|z_n-Sz_n\|^2 \\
&\le \big(1-b_n(1-2\rho)\big)\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\
&\quad + 2b_n\big[\langle f(p)-p, x_n-p\rangle + \|x_{n+1}-x_n\|\,\|f(x_n)-p\|\big].
\end{aligned}
$$

We find from the restrictions (d_1)–(d_4) that

$$
\begin{aligned}
&\lim_{n\to\infty}\|u_n-v_n\| = \lim_{n\to\infty}\|Ay_n-Ap\|, \\
&\lim_{n\to\infty}\|Mx_n-Mp\| = \lim_{n\to\infty}\|z_n-St_n\| = \lim_{n\to\infty}\|z_n-Sz_n\| = 0.
\end{aligned}
\tag{17}
$$

From $\|x_{n+1}-u_n\| \le \|x_{n+1}-z_n\| + \|z_n-u_n\|$, $\|z_n-u_n\| \le b_n\|f(x_n)-u_n\| + (1-b_n)\|v_n-u_n\|$, $\|x_{n+1}-z_n\| \le \|z_n-St_n\|$ and the restrictions (d_3) we get

$$
\lim_{n\to\infty}\|x_{n+1}-z_n\| = \lim_{n\to\infty}\|z_n-u_n\| = \lim_{n\to\infty}\|x_{n+1}-u_n\| = 0. \tag{18}
$$

Observe that

$$\begin{aligned}
\|u_n-p\|^2 &= \big\|J^B_{\lambda_n}(y_n-\lambda_n Ay_n)-J^B_{\lambda_n}(p-\lambda_n Ap)\big\|^2\\
&\le \big\langle (y_n-\lambda_n Ay_n)-(P-\lambda_n Ap),u_n-p\big\rangle\\
&= \frac{1}{2}\big\|(y_n-\lambda_n Ay_n)-(p-\lambda_n Ap)\big\|^2+\frac{1}{2}\|u_n-p\|^2\\
&\quad -\frac{1}{2}\big\|(y_n-\lambda_n Ay_n)-(p-\lambda_n Ap)-(u_n-p)\big\|^2\\
&\le \frac{1}{2}\big[\|y_n-p\|^2+\|u_n-p\|^2-\big\|(y_n-u_n)-\lambda_n(Ay_n-Ap)\big\|^2\big]\\
&= \frac{1}{2}\big[\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2+2\lambda_n\langle y_n-u_n,Ay_n-Ap\rangle\\
&\quad -\lambda_n^2\|Ay_n-Ap\|^2\big],
\end{aligned}$$

from which one deduces that

$$\|u_n-p\|^2\le \|y_n-p\|^2-\|y_n-u_n\|^2+2\lambda_n\|y_n-u_n\|\|Ay_n-Ap\|. \tag{19}$$

Using Lemma 1.2 and (4), we have

$$\begin{aligned}
\|y_n-p\|^2 &= \big\|T_{r_n}(x_n-r_nMx_n)-T_{r_n}(p-r_nMp)\big\|^2\\
&\le \big\langle (x_n-r_nMx_n)-(p-r_nMp),y_n-p\big\rangle\\
&= \frac{1}{2}\big\|(x_n-r_nMx_n)-(p-r_nMp)\big\|^2+\frac{1}{2}\|y_n-p\|^2\\
&\quad -\frac{1}{2}\big\|(x_n-r_nMx_n)-(p-r_nMp)-(y_n-p)\big\|^2\\
&\le \frac{1}{2}\big[\|x_n-p\|^2+\|y_n-p\|^2-\big\|(x_n-y_n)-2r_n(Mx_n-Mp)\big\|^2\big]\\
&= \frac{1}{2}\big[\|x_n-p\|^2+\|y_n-p\|^2-\|x_n-y_n\|^2+2r_n\langle x_n-y_n,Mx_n-Mp\rangle\\
&\quad -r_n^2\|Mx_n-Mp\|^2\big].
\end{aligned}$$

It follows that

$$\|y_n-p\|^2\le \|x_n-p\|^2-\|x_n-y_n\|^2+2r_n\langle x_n-y_n,Mx_n-Mp\rangle. \tag{20}$$

We have from (7), (12), (14), (19) and (20)

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \big\|b_nf(x_n)+(1-b_n)v_n-p\big\|^2\\
&\le b_n\big\|f(x_n)-p\big\|^2+(1-b_n)\|v_n-p\|^2\\
&\le b_n\big\|f(x_n)-p\big\|^2+(1-b_n)\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\big[\|x_n-p\|^2-\|y_n-u_n\|^2\\
&\quad +2\lambda_n\|y_n-u_n\|\|Ay_n-Ap\|-\|x_n-y_n\|^2+2r_n\langle x_n-y_n,Mx_n-Mp\rangle\big]\\
&\le b_n\big(\big\|f(x_n)-f(p)\big\|+\big\|f(p)-p\big\|\big)^2+(1-b_n)\left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\big[\|x_n-p\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \|y_n - u_n\|^2 + 2\lambda_n \|y_n - u_n\| \|Ay_n - Ap\| - \|x_n - y_n\|^2 \\
&+ 2r_n \langle x_n - y_n, Mx_n - Mp \rangle \big] \\
\le\ & b_n \big(\rho \|x_n - p\| + \big\|f(p) - p\big\|\big)^2 + (1 - b_n)\left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\big[\|x_n - p\|^2 \\
&- \|y_n - u_n\|^2 + 2\lambda_n \|y_n - u_n\| \|Ay_n - Ap\| - \|x_n - y_n\|^2 \\
&+ 2r_n \langle x_n - y_n, Mx_n - Mp \rangle \big] \\
\le\ & b_n \big(\rho^2 \|x_n - p\|^2 + \big\|f(p) - p\big\|^2 + 2\rho \|x_n - p\| \big\|f(p) - p\big\|\big) \\
&+ (1 - b_n)\left[\left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|^2 - \|y_n - u_n\|^2 \right. \\
&+ 2\lambda_n \|y_n - u_n\| \|Ay_n - Ap\| \\
&\left. - \|x_n - y_n\|^2 + 2r_n \langle x_n - y_n, Mx_n - Mp \rangle \right] \\
\le\ & b_n \big(\rho(1 + \rho) \|x_n - p\|^2 + (1 + \rho)\big\|f(p) - p\big\|^2\big) \\
&+ (1 - b_n)\left[\left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|^2 - \|y_n - u_n\|^2 \right. \\
&+ 2\lambda_n \|y_n - u_n\| \|Ay_n - Ap\| \\
&\left. - \|x_n - y_n\|^2 + 2r_n \langle x_n - y_n, Mx_n - Mp \rangle \right] \\
\le\ & \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\big(1 - b_n\big(1 - \rho(1 + \rho)\big)\big)\|x_n - p\|^2 + b_n(1 + \rho)\big\|f(p) - p\big\|^2 \\
&+ (1 - b_n)\left[\left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)2\lambda_n \|y_n - u_n\| \|Ay_n - Ap\| - \|y_n - u_n\|^2 \right. \\
&\left. + \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)2r_n \langle x_n - y_n, Mx_n - Mp \rangle - \|x_n - y_n\|^2 \right].
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
&(1 - b_n)\big(\|y_n - u_n\|^2 + \|x_n - y_n\|^2\big) \\
&\quad \le \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\big(1 - b_n\big(1 - \rho(1 + \rho)\big)\big)\|x_n - p\|^2 \\
&\qquad - \|x_{n+1} - p\|^2 + b_n(1 + \rho)\big\|f(p) - p\big\|^2 \\
&\qquad + \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\big[2\lambda_n \|y_n - u_n\| \|Ay_n - Ap\| \\
&\qquad + 2r_n \langle x_n - y_n, Mx_n - Mp \rangle \big].
\end{aligned}
$$

We find from (17) and the restrictions (d_3) and (d_4) that

$$
\lim_{n \to \infty} \|y_n - u_n\| = \lim_{n \to \infty} \|x_n - y_n\| = 0. \tag{21}
$$

We have from $\|x_n - x_{n+1}\| \le \|x_n - y_n\| + \|y_n - u_n\| + \|u_n - x_{n+1}\|$ and (18) that

$$
\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0.
$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle \le 0,$$

where $p = P_\Omega f(p)$. The existence of q is justified since P_Ω is nonexpansive and f is a contraction, then P_Ω is a contraction so it has a fixed point. To show it, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle = \lim_{j\to\infty}\langle f(p)-p,x_{n_j}-p\rangle. \tag{22}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$, converges weakly to u. Without loss of generality, we assume that $x_{n_j} \rightharpoonup u$. Since $\|x_n - y_n\| \to 0$ as $n \to \infty$ we obtain $y_{n_j} \rightharpoonup u$. Since $\{y_{n_j}\} \subset C$ and C is closed and convex, we obtain $u \in C$. First, we show that $u \in F(S)$. Then, from (17) and Lemma 1.4, we have $u \in F(S)$. We now show $u \in \mathrm{GEP}(F,M)$. By $y_n = T_{r_n}(x_n - r_n Mx_n)$, we know that

$$F(y_n,y) + \langle Mx_n, y-y_n\rangle + \frac{1}{r_n}\langle y-y_n, y_n-x_n\rangle \ge 0, \quad \forall y \in C.$$

It follows from (A_2) that

$$\langle Mx_n, y-y_n\rangle + \frac{1}{r_n}\langle y-y_n, y_n-x_n\rangle \ge F(y,y_n), \quad \forall y \in C.$$

Hence,

$$\langle Mx_{n_j}, y-y_{n_j}\rangle + \left\langle y-y_{n_j}, \frac{y_{n_j}-x_{n_j}}{r_{n_j}}\right\rangle \ge F(y,y_{n_j}), \quad \forall y \in C. \tag{23}$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)u$. Since $y \in C$ and $u \in C$, we obtain $y_t \in C$. So, from (23) we have

$$\begin{aligned}
\langle y_t - y_{n_j}, My_t\rangle &\ge \langle y_t - y_{n_j}, My_t\rangle - \langle y_t - y_{n_j}, Mx_{n_j}\rangle \\
&\quad - \left\langle y_t - y_{n_j}, \frac{y_{n_j}-x_{n_j}}{r_{n_j}}\right\rangle + F(y_t,y_{n_j}) \\
&= \langle y_t - y_{n_j}, My_t - My_{n_j}\rangle + \langle y_t - y_{n_j}, My_{n_j} - Mx_{n_j}\rangle \\
&\quad - \left\langle y_t - y_{n_j}, \frac{y_{n_j}-x_{n_j}}{r_{n_j}}\right\rangle + F(y_t,y_{n_j}).
\end{aligned}$$

Since $\|y_{n_j} - x_{n_j}\| \to 0$, we have $\|My_{n_j} - Mx_{n_j}\| \to 0$. Further, from the inverse strongly monotonicity of M, we have $\langle y_t - y_{n_j}, My_t - My_{n_j}\rangle \ge 0$. It follows from A_4 and $\frac{y_{n_j}-x_{n_j}}{r_{n_j}} \to 0$ and $y_{n_j} \rightharpoonup u$ that we have

$$\langle y_t - v, My_t\rangle \ge F(y_t,u),$$

as $j \to \infty$. From (A_1), (A_4) we have

$$\begin{aligned} 0 &= F(y_t, y_t) \\ &= tF(y_t, y) + (1-t)F(y_t, u) \\ &\leq tF(y_t, y) + (1-t)\langle y_t - u, My_t\rangle \\ &= tF(y_t, y) + (1-t)t\langle y - u, My_t\rangle, \end{aligned}$$

and hence

$$0 \leq F(y_t, y) + (1-t)\langle y - u, My_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(u, y) + (1-t)\langle y - u, Mu\rangle \geq 0.$$

This implies that $u \in \mathrm{GEP}(F, M)$. Next we show $u \in (A+B)^{-1}(0)$. Due to (a), there is a subsequence $\{\lambda_{n_{j_k}}\}$ of $\{\lambda_{n_j}\}$ such that $\lambda_{n_{j_k}} \to \lambda^* \in [c, d]$. Without loss of generality, we assume that $\lambda_{n_j} \to \lambda^*$. From Lemma 1.7, we have

$$\begin{aligned} &\bigl\|x_{n_j} - J^B_{\lambda^*}\bigl(I - \lambda^* A\bigr)x_{n_j}\bigr\| \\ &\quad\leq \|x_{n_j} - u_{n_j}\| + \bigl\|J^B_{\lambda_{n_j}}(I - \lambda_{n_j}A)x_{n_j} - J^B_{\lambda^*}\bigl(I - \lambda^* A\bigr)x_{n_j}\bigr\| \\ &\quad\leq \|x_{n_j} - u_{n_j}\| + \bigl\|J^B_{\lambda_{n_j}}(I - \lambda_{n_j}A)x_{n_j} - J^B_{\lambda_{n_j}}\bigl(I - \lambda^* A\bigr)x_{n_j}\bigr\| \\ &\qquad + \bigl\|J^B_{\lambda_{n_j}}\bigl(I - \lambda^* A\bigr)x_{n_j} - J^B_{\lambda^*}\bigl(I - \lambda^* A\bigr)x_{n_j}\bigr\| \\ &\quad\leq \|x_{n_j} - u_{n_j}\| + \bigl|\lambda_{n_j} - \lambda^*\bigr|\|Ax_{n_j}\| \\ &\qquad + \left|\frac{\lambda_{n_j} - \lambda^*}{\lambda^*}\right|\bigl\|J^B_{\lambda^*}\bigl(I - \lambda^* A\bigr)x_{n_j} - \bigl(I - \lambda^* A\bigr)x_{n_j}\bigr\|. \end{aligned}$$

This implies that

$$\lim_{j\to\infty}\bigl\|x_{n_j} - J^B_{\lambda^*}\bigl(I - \lambda^* A\bigr)x_{n_j}\bigr\| = 0.$$

Since $J^B_{\lambda^*}(I - \lambda^* M)$ is nonexpansive, the demiclosedness for a nonexpansive mapping implies that $u \in F(J^B_{\lambda^*}(I - \lambda^* A))$, that is, $u \in (A+B)^{-1}(0)$. Finally we show $u \in N^{-1}(0)$. Since $\|e_n\| \to 0$ and $\|x_n - v_n\| = \|u_n - v_n\| \to 0$ as $n \to \infty$, we have $v_{n_j} \rightharpoonup u$ and

$$\bigl\|x_{n_j} + e_{n_j} - J^N_{s_n}(x_{n_j} + e_{n_j})\bigr\| \leq \|x_{n_j} - v_{n_j}\| + \|e_{n_j}\| \to 0.$$

From Lemma 1.1, we have $0 \in N(u)$. This implies $u \in \Omega$. Due to (22), we arrive at

$$\limsup_{n\to\infty}\bigl\langle f(p) - p, x_n - p\bigr\rangle = \lim_{j\to\infty}\bigl\langle f(p) - p, x_{n_j} - p\bigr\rangle = \bigl\langle f(p) - p, u - p\bigr\rangle \leq 0.$$

Since $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} b_n = \infty$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\sum_{n=0}^{+\infty} \varepsilon_n < +\infty$, we obtain from Lemma 1.5 and (16)

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

Consequently, $x_n \to p = P_C f(p)$.

Case 2: Assume that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\|x_{n_j} - p\| \leq \|x_{n_j+1} - p\|$$

for all $j \in \mathbb{N}$. From Lemma 1.9 there exists a nondecreasing sequence $\{t_k\} \subset \mathbb{N}$ such that $t_k \to \infty$ and

$$\|x_{t_k} - p\| \leq \|x_{t_k+1} - p\| \quad \text{and} \quad \|x_k - p\| \leq \|x_{t_k+1} - p\| \tag{24}$$

for all $k \in \mathbb{N}$. Since $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{+\infty} \varepsilon_n < +\infty$ we can obtain from (17), (18) and (21)

$$\begin{aligned}\lim_{k\to\infty} \|x_{t_k} - z_{t_k}\| &= \lim_{k\to\infty} \|x_{t_k} - y_{t_k}\| = \lim_{k\to\infty} \|x_{t_k} - v_{t_k}\| \\ &= \lim_{k\to\infty} \|z_{t_k} - Tz_{t_k}\| = \lim_{k\to\infty} \|x_{t_k+1} - x_{t_k}\| = 0.\end{aligned}$$

From Case 1, we also have

$$\limsup_{k\to\infty} \langle f(p) - p, x_{t_k} - p\rangle \leq 0 \tag{25}$$

Using (16) and following the methods used to get (16), we obtain

$$\begin{aligned}&= \|x_{t_k+1} - p\|^2 \leq (1 - b_{t_k})\left(1 + \frac{2\varepsilon_{t_k}}{1 - 2\varepsilon_{t_k}}\right)\|x_{t_k} - p\|^2 + 2b_{t_k}\langle f(p) - p, x_{t_k} - p\rangle \\ &\quad + 2b_{t_k}\left(\|x_{t_k+1} - x_{t_k}\| \left\|f(x_{t_k}) - p\right\| + \rho\|x_{t_k} - p\|^2\right) \\ &\leq \left(1 - b_{t_k}(1 - 2\rho)\right)\|x_{t_k} - p\|^2 + \frac{2\varepsilon_{t_k}}{1 - 2\varepsilon_{t_k}}L + 2b_{t_k}\langle f(p) - p, x_{t_k} - p\rangle \\ &\quad + 2b_{t_k}\|x_{t_k+1} - x_{t_k}\| \left\|f(x_{t_k}) - p\right\| \end{aligned} \tag{26}$$

where $L > 0$ is a sufficiently large number. This implies that

$$\begin{aligned}b_{t_k}(1 - 2\rho)\|x_{t_k} - p\|^2 &\leq \|x_{t_k} - p\|^2 - \|x_{t_k+1} - p\|^2 + \frac{2\varepsilon_{t_k}}{1 - 2\varepsilon_{t_k}}L \\ &\quad + 2b_{t_k}\langle f(p) - p, x_{t_k} - p\rangle + 2b_{t_k}\|x_{t_k+1} - x_{t_k}\| \left\|f(x_{t_k}) - p\right\|.\end{aligned}$$

Since $b_{t_k} > 0$, we get from (24)

$$(1 - \rho)\|x_{t_k} - p\|^2 \leq \frac{2\varepsilon_{t_k}}{1 - 2\varepsilon_{t_k}}L + 2\langle f(p) - p, x_{t_k} - p\rangle + 2\|x_{t_k+1} - x_{t_k}\| \left\|f(x_{t_k}) - p\right\|.$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\sum_{n=0}^{+\infty}\varepsilon_n<+\infty$, we obtain from (25) $\|x_{t_k}-p\|\to 0$ as $k\to\infty$. From (26) we have $\|x_{t_k+1}-p\|\to 0$ as $k\to\infty$. Using (24), we obtain $\lim_{k\to\infty}\|x_k-p\|=0$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to a point $p=P_\Omega f(p)$, which satisfies the variational inequality $\langle (I-f)p, x-p\rangle\geq 0$, for all $x\in\Omega$. The proof is complete. □

If $f(x)=u\in C$ in Theorem 2.1, then we have the following result.

Corollary 2.2 *Assume that C is a nonempty, closed and convex subset of H, F is a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), M is an α-inverse strongly monotone mapping from C into H, A is a β-inverse strongly monotone map from C into H, B and N are two maximal monotone operators on H such that their domains contained in C and $S:C\to C$ is a Lipschitz pseudo-contractive mapping with Lipschitz constants K such that $\Omega=F(S)\cap(A+B)^{-1}(0)\cap N^{-1}(0)\cap\mathrm{GEP}(F,M)\neq\emptyset$. Assume that $\{b_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are some sequences in $(0,1)$ and $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$ are the sequences generated by*

$$\begin{cases} x_0\in C,\\ F(y_n,y)+\langle Mx_n,y-y_n\rangle+\frac{1}{r_n}\langle y-y_n,y_n-x_n\rangle\geq 0, \quad \forall y\in C,\\ u_n=J^B_{\lambda_n}(y_n-\lambda_n Ay_n),\\ z_n=b_n u+(1-b_n)J^N_{s_n}(u_n+e_n),\\ x_{n+1}=(1-\beta_n)z_n+\beta_n S(\delta_n z_n+(1-\delta_n)Sz_n), \quad \forall n\geq 0.\end{cases}$$

If the conditions (d_1)–(d_4) hold, then the sequence $\{x_n\}$ converges strongly to a point $p\in\Omega$ which is the unique solution of the variational inequality $\langle p-u,x-p\rangle\geq 0$ for all $x\in\Omega$.

Now, we discuss weak convergence of the sequence in the new iteration.

Theorem 2.3 *Assume that C is a nonempty, closed and convex subset of H, F is a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), M is an α-inverse strongly monotone mapping from C into H, A is a β-inverse strongly monotone map from C into H, B and N are two maximal monotone operators on H such that their domains contained in C and $S:C\to C$ is a Lipschitz pseudo-contractive mapping with Lipschitz constants k such that $\Omega=F(S)\cap(A+B)^{-1}(0)\cap N^{-1}(0)\cap\mathrm{GEP}(F,M)\neq\emptyset$. Assume that $\{b_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are some sequences in $(0,1)$ and $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$ are the sequences generated by*

$$\begin{cases} x_0\in C,\\ F(y_n,y)+\langle Mx_n,y-y_n\rangle+\frac{1}{r_n}\langle y-y_n,y_n-x_n\rangle\geq 0, \quad \forall y\in C,\\ u_n=J^B_{\lambda_n}(y_n-\lambda_n Ay_n),\\ z_n=b_n x_n+(1-b_n)J^N_{s_n}(u_n+e_n),\\ x_{n+1}=(1-\beta_n)z_n+\beta_n S(\delta_n z_n+(1-\delta_n)Sz_n) \quad \forall n\geq 0.\end{cases} \tag{27}$$

If

(d_1) $0<c\leq\lambda_n\leq d<2\beta$, $0<a\leq r_n\leq b<2\alpha$,

(d_2) $0<c<\beta_n\leq\delta_n<d<\frac{1}{\sqrt{1+k^2}+1}$, $s_n>s>0$,

(d_3) $\|e_n\| \le \frac{\varepsilon_n}{2} \max\{\|u_n - J_{s_n}^N(u_n + e_n)\|, \|J_{s_n}^N(u_n + e_n) - p\|\}$ *with* $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, *then* $\{x_n\}$ *converges weakly to an element* $p \in \Omega$.

Proof Let $p \in \Omega$. Similarly, from (6) and (7) we obtain

$$\begin{aligned}\|y_n - p\|^2 &\le \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Mx_n - Mp\|^2 \\ &\le \|x_n - p\|^2\end{aligned}$$

and

$$\begin{aligned}\|u_n - p\|^2 &\le \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\beta)\|Ay_n - Ap\|^2 \\ &\le \|x_n - p\|^2.\end{aligned}$$

We also conclude from (11) and (12) that

$$\begin{aligned}\|x_{n+1} - p\|^2 \le{}& \|z_n - p\|^2 - \delta_n\bigl(1 - 2\delta_n - \delta_n^2K^2\bigr)\|z_n - Sz_n\|^2 \\ &+ \beta_n(\beta_n - \delta_n)\|z_n - St_n\|^2. \end{aligned} \tag{28}$$

Put $v_n = J_{\lambda_n}^N(u_n + e_n)$ for all $n \ge 0$. From (14), we have

$$\begin{aligned}\|v_n - p\|^2 &\le \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|^2 - \|u_n - v_n\|^2 \\ &\le \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|^2.\end{aligned}$$

These have already been proved in Theorem 2.1. Since $0 < c < \beta_n \le \delta_n < d < \frac{1}{\sqrt{1+k^2}+1}$ for all $n \ge 1$, we conclude from (27), (28) and Lemma 1.6 that

$$\begin{aligned}&= \|x_{n+1} - p\|^2 \\ &\le \|z_n - p\|^2 - \delta_n\bigl(1 - 2\delta_n - \delta_n^2K^2\bigr)\|z_n - Sz_n\|^2 \\ &\quad + \beta_n(\beta_n - \delta_n)\|z_n - St_n\|^2 \\ &= \bigl\|b_nx_n + (1 - b_n)v_n - p\bigr\|^2 - \delta_n\bigl(1 - 2\delta_n - \delta_n^2K^2\bigr)\|z_n - Sz_n\|^2 \\ &\quad + \beta_n(\beta_n - \delta_n)\|z_n - St_n\|^2 \\ &\le b_n\|x_n - p\|^2 + (1 - b_n)\|v_n - p\|^2 - (1 - b_n)b_n\|x_n - v_n\|^2 \\ &\quad - \delta_n\bigl(1 - 2\delta_n - \delta_n^2K^2\bigr)\|z_n - Sz_n\|^2 + \beta_n(\beta_n - \delta_n)\|z_n - St_n\|^2 \\ &\le b_n\|x_n - p\|^2 + (1 - b_n)\left[\left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|^2 - \|u_n - v_n\|^2\right] \\ &\quad - (1 - b_n)b_n\|x_n - v_n\|^2 - \delta_n\bigl(1 - 2\delta_n - \delta_n^2K^2\bigr)\|z_n - Sz_n\|^2 \\ &\quad + \beta_n(\beta_n - \delta_n)\|z_n - St_n\|^2 \\ &\le \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}(1 - b_n)\right)\|x_n - p\|^2 - (1 - b_n)\bigl(\|u_n - v_n\|^2 + b_n\|x_n - v_n\|^2\bigr) \\ &\quad - \delta_n\bigl(1 - 2\delta_n - \delta_n^2K^2\bigr)\|z_n - Sz_n\|^2 + \beta_n(\beta_n - \delta_n)\|z_n - St_n\|^2\end{aligned}$$

$$\begin{aligned}&\leq \left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2-(1-b_n)\left(\|u_n-v_n\|^2+b_n\|x_n-v_n\|^2\right)\\&\quad-\delta_n\left(1-2\delta_n-\delta_n^2K^2\right)\|z_n-Sz_n\|^2+\beta_n(\beta_n-\delta_n)\|z_n-St_n\|^2\\&\leq \left(1+\frac{2\varepsilon_n}{1-2\varepsilon_n}\right)\|x_n-p\|^2.\end{aligned}\tag{29}$$

For every $n=0,1,2,\ldots$, since $\sum_{n=0}^{\infty}\varepsilon_n^2<\infty$, we obtain

$$M_0:=\sum_{n=m_0}^{\infty}\frac{2\varepsilon_n^2}{1-2\varepsilon_n^2}<\infty \quad \text{and} \quad M_1:=\prod_{n=m_0}^{\infty}\left(1+\frac{2\varepsilon_n^2}{1-2\varepsilon_n^2}\right)<\infty.$$

Hence for each integer $n\geq m_0$,

$$\begin{aligned}\|x_{n+1}-p\|^2&\leq \left(1+\frac{2\varepsilon_n^2}{1-2\varepsilon_n^2}\right)\|x_n-p\|^2\\&\leq \left(1+\frac{2\varepsilon_n^2}{1-2\varepsilon_n^2}\right)\left(1+\frac{2\varepsilon_{n-1}^2}{1-2\varepsilon_{n-1}^2}\right)\|x_{n-1}-p\|^2\\&\ \vdots\\&\leq \prod_{i=m_0}^{n}\left(1+\frac{2\varepsilon_i^2}{1-2\varepsilon_i^2}\right)\|x_{m_0}-p\|^2\\&\leq \prod_{i=m_0}^{\infty}\left(1+\frac{2\varepsilon_i^2}{1-2\varepsilon_i^2}\right)\|x_{m_0}-p\|^2=K_1\|x_{m_0}-p\|^2.\end{aligned}$$

Therefore, $\{\|x_n-p\|\}$ is bounded. So, $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$. Setting $K:=\sup_{n\geq 0}\|x_n-p\|$, we obtain from (29)

$$\|x_{n+1}-p\|^2\leq \|x_n-p\|^2+\frac{2\varepsilon_n^2}{1-2\varepsilon_n^2}K^2, \quad \forall n\geq m_0.$$

Thus it follows that, for all $n,m\geq m_0$,

$$\begin{aligned}\|x_{n+m+1}-p\|^2&\leq \|x_{n+m}-p\|^2+\frac{2\varepsilon_{n+m}^2}{1-2\varepsilon_{n+m}^2}K^2\\&\leq \|x_{n+m-1}-p\|^2+\frac{2\varepsilon_{n+m-1}^2}{1-2\varepsilon_{n+m-1}^2}K^2+\frac{2\varepsilon_{n+m}^2}{1-2\varepsilon_{n+m}^2}K^2\\&\ \vdots\\&\leq \|x_n-p\|^2+\sum_{n=m_0}^{\infty}\frac{2\varepsilon_n^2}{1-2\varepsilon_n^2}K^2.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\frac{2\varepsilon_n^2}{1-2\varepsilon_n^2}<\infty$ we obtain

$$\limsup_{m\to\infty}\|x_m-p\|^2\leq \|x_n-p\|^2+\sum_{i=0}^{\infty}\frac{2\varepsilon_i^2}{1-2\varepsilon_i^2}K^2.$$

This implies that for every $p \in \Omega$, $\lim_{n\to\infty} \|x_n - p\|^2$ exists. From (29), we have

$$\begin{aligned}&\delta_n\bigl(1 - 2\delta_n - \delta_n^2K^2\bigr)\|z_n - Sz_n\|^2 + \beta_n(\delta_n - \beta_n)\|z_n - St_n\|^2\\&\quad + (1 - b_n)\bigl(\|u_n - v_n\|^2 + b_n\|x_n - v_n\|^2\bigr)\\&\quad\leq \left(1 + \frac{2\varepsilon_n}{1 - 2\varepsilon_n}\right)\|x_n - p\|^2 - \|x_{n+1} - p\|^2.\end{aligned}$$

We find from the restrictions (d_1)–(d_3) that

$$\lim_{n\to\infty} \|u_n - v_n\| = \lim_{n\to\infty} \|x_n - v_n\| = \lim_{n\to\infty} \|z_n - St_n\| = \lim_{n\to\infty} \|z_n - Sz_n\| = 0. \tag{30}$$

From $\|x_{n+1} - u_n\| \leq \|x_{n+1} - z_n\| + \|z_n - u_n\|$, $\|z_n - u_n\| \leq b_n\|x_n - u_n\| + (1 - b_n)\|v_n - u_n\|$, $\|x_{n+1} - z_n\| \leq \|z_n - St_n\|$ and $\|x_n - u_n\| \leq \|x_n - v_n\| + \|v_n - u_n\|$ we get

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = \lim_{n\to\infty} \|z_n - u_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{31}$$

Also from $\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - x_n\|$ and $\|x_n - z_n\| \leq \|x_n - u_n\| + \|u_n - z_n\|$ we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{32}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to u. Since $\|x_n - y_n\| \to 0$ as $n \to \infty$ we obtain $y_{n_j} \rightharpoonup u$. Since $\{y_{n_j}\} \subset C$ and C is closed and convex, we obtain $u \in C$. First, we show that $u \in F(S)$. Then, from (30) and Lemma 1.4, we have $u \cap F(S)$. Using the same argument we had in Theorem 2.1, we get $u \in \mathrm{GEP}(F, M)$ and $u \in (A + B)^{-1}(0)$. In a similar way, we have $0 \in N(u)$. This implies $u \in \Omega$.

Let us consider the uniqueness of the weak cluster point of $\{x_n\}$. Suppose there exist two weak cluster points $\hat{u}$ and $\bar{u}$ of the sequence $\{x_n\}$, then $\hat{u}$ and $\bar{u}$ belong to Ω and the sequences $\{\|\hat{u} - x_n\|\}$ and $\{\|\bar{u} - x_n\|\}$ converge; i.e., there exist $\hat{\beta}, \bar{\beta} \in \mathbb{R}^+$ such that

$$\lim_{n\to+\infty} \|\hat{u} - x_n\| = \hat{\beta}, \qquad \lim_{n\to+\infty} \|\hat{u} - x_n\| = \bar{\beta}. \tag{33}$$

Since

$$\|\hat{u} - x_n\|^2 = \|\bar{u} - x_n\|^2 + 2\langle x_n - \hat{u}, \hat{u} - \bar{u}\rangle + \|\hat{u} - \bar{u}\|^2,$$

from (33), we have

$$\lim_{n\to+\infty} \langle x_n - \hat{u}, \hat{u} - \bar{u}\rangle = \frac{1}{2}\bigl(\bar{\beta}^2 - \hat{\beta}^2 - \|\hat{u} - \bar{u}\|^2\bigr). \tag{34}$$

Because $\hat{u}$ is a weak cluster point of $\{x_n\}$, which implies that

$$\bar{\beta}^2 - \hat{\beta}^2 = \|\hat{u} - \bar{u}\|^2. \tag{35}$$

Reversing the roles of $\bar{p}$ and $\hat{p}$, hence $\hat{\beta}^2 - \bar{\beta}^2 = \|\hat{u} - \bar{u}\|^2$, Combining this with (35), we have $\|\hat{u} - \bar{u}\| = 0$, i.e., $\hat{u} = \bar{u}$, which is a contradiction. Therefore, there exists an unique weak cluster point of $\{x_n\}$.Then $\{x_n\}$ is weakly convergent to an element of Ω, and this completes the proof of Theorem 2.3. □

Remark 2.1 Theorem 2.1 and Theorem 2.3 improves and extends the result in Ceng, Wu, Yao ([21]), Han, He ([20]) and Tian, Wang ([22]).

Let I_C be the indicator function of C defined by $I_C(x) = 0$ whenever $x \in C$ and $I_C(x) = \infty$ otherwise. Recall that the subdifferential ∂I_C of I_C is a maximal monotone operator since I_C is a proper lower semi-continuous convex function on H. The resolvent $J_r^{\partial I_C}$ of ∂I_C for r is P_C and $\mathrm{VI}(C, A) = (A + \partial I_C)^{-1}(0)$, where A is an inverse strongly monotone mapping of C into H ([40]). We obtain the following result.

Theorem 2.4 *Suppose that C is a nonempty, closed and convex subset of H, F is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), M is an α-inverse strongly monotone mapping from C into H, A is a β-inverse strongly monotone map from C into H, B and N are two maximal monotone operators on H such that their domains contained in C, $f : C \to C$ is a ρ-contractive map with $\rho \in (0, \frac{1}{2})$ and $S : C \to C$ a Lipschitz pseudo-contractive mapping with Lipschitz constants K such that $\Omega = F(S) \cap \mathrm{VI}(C, A) \cap N^{-1}(0) \cap \mathrm{GEP}(F, M) \neq \emptyset$. Assume that $\{b_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are some sequences in $(0, 1)$ and $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$ are the sequences generated by*

$$\begin{cases} x_0 \in C, \\ F(y_n, y) + \langle Mx_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ u_n = P_C(y_n - \lambda_n Ay_n), \\ z_n = b_n f(x_n) + (1 - b_n)J_{s_n}^N(u_n + e_n), \\ x_{n+1} = (1 - \beta_n)z_n + \beta_n S(\delta_n z_n + (1 - \delta_n)Sz_n) \quad \forall n \geq 0. \end{cases}$$

If the conditions (d_1)–(d_4) hold, then, $\{x_n\}$ converges strongly to a point $p \in \Omega$ which is the unique solution of the variational inequality $\langle (I - f)p, x - p\rangle \geq 0$ for all $x \in \Omega$.

Proof Putting $B = \partial I_C$ in Theorem 2.1, we know that $J_{\lambda_n} = P_C$ for all $\lambda_n > 0$, we obtain the desired result. □

Remark 2.2 Theorem 2.4 improves and extends the result in Takahashi, Takahashi ([13]) and Su, Shang, Qin ([41]).

Theorem 2.5 *Suppose that C is a nonempty, closed and convex subset of H, F is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), M is an α-inverse strongly monotone mapping from C into H, $\psi : C \to C$ is a β-strict pseudo-contraction, N is a maximal monotone operator on H such that its domains contained in C, $f : C \to C$ is a ρ-contractive map with $\rho \in (0, \frac{1}{2})$ and $S : C \to C$ is a Lipschitz pseudo-contractive mapping with Lipschitz constants K such that $\Omega = F(S) \cap F(\psi) \cap N^{-1}(0) \cap \mathrm{GEP}(F, M) \neq \emptyset$. Assume that $\{b_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are some sequences in $(0, 1)$ and $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{z_n\}$ are the sequences generated* *by*

$$\begin{cases} x_0 \in C, \\ F(y_n, y) + \langle Mx_n, y - y_n \rangle + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ u_n = (1 - \lambda_n) y_n + \lambda_n \psi y_n, \\ z_n = b_n f(x_n) + (1 - b_n) J_{s_n}^N (u_n + e_n), \\ x_{n+1} = (1 - \beta_n) z_n + \beta_n S(\delta_n z_n + (1 - \delta_n) S z_n) \quad \forall n \geq 0. \end{cases}$$

If the conditions (d_1)–(d_4) *hold*, $0 < c < \lambda_n < d < 1 - \beta$, *then* $\{x_n\}$ *converges strongly to a point* $p \in \Omega$ *which is the unique solution of the variational inequality* $\langle (I - f)p, x - p \rangle \geq 0$ *for all* $x \in \Omega$.

Proof Putting $B = \partial I_C$, $A = I - \psi$, we see that A is $\frac{1-\beta}{2}$-inverse strongly monotone. We also have $J_{\lambda_n} = P_C$ for all $\lambda_n > 0$, $F(\psi) = VI(C, A)$ and $P_C(y_n - \lambda_n A y_n) = (1 - \lambda_n) y_n + \lambda_n \psi y_n$, by Theorem 2.1 we obtain the desired result. □

Now, we provide an example to illustrate our first result.

Example 2.1 Let $H = \mathbb{R}$ with Euclidean norm and usual Euclidean inner product. Let $C := (-\infty, 1]$, $Sx = \frac{x}{x-2}$, $Bx = \log(1 - x)$, $Ax = 2x$, $\beta \leq \frac{1}{2}$, $F(x, y) = y - x$, $N(x) = \log(1 - x^3)$, $\alpha \leq 1$ and $Mx = x - 1$. Clearly, S is a Lipschitz pseudo-contractive mapping with Lipschitz constants $K \leq \frac{1}{10}$, A a β-inverse strongly monotone mapping, B, N maximal monotone operators, F a bi-function from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), M an α-inverse strongly monotone mapping and $0 \in N^{-1}(0) \cap F(S) \cap (A + B)^{-1}(0) \cap \mathrm{GEP}(F, M)$.

3 Conclusion

As is well known, many things need to be optimized. Numerous techniques and methods have been used to optimize a variety of issues. This has even been used to solve some differential equations. In this work, we introduced a new iterative method for finding a common element of the set of fixed points of a pseudo-contractive mapping, the set of solutions to a variational inclusion and the set of solutions to a generalized equilibrium problem in a real Hilbert space. We provided some strong and weak convergence results as regards the common solutions. Finally, we provided an example to illustrate our first main result.

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