Advances in Difference Equations
a SpringerOpen Journal

Analytical properties of the Hurwitz–Lerch zeta function

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Abstract

In the present paper, we aim to extend the Hurwitz–Lerch zeta function $\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;p)$ involving the extension of the beta function (Choi et al. in Honam Math. J. 36(2):357–385, 2014). We also study the basic properties of this extended Hurwitz–Lerch zeta function which comprises various integral formulas, a derivative formula, the Mellin transform, and the generating relation. The fractional kinetic equation for an extended Hurwitz–Lerch zeta function is also obtained from an application point of view. Furthermore, we obtain certain interesting relations in the form of particular cases.

MSC: 33C05; 33C45; 33C47; 33C90

Keywords: Generalized; Generating functions; Rodrigues formula

1 Overture

The familiar Hurwitz–Lerch zeta function $\Phi(\xi,s,\upsilon)$ is defined by (see, e.g., [2, p. 27, Sect. 1.11, Eq. (1)]; see also [3])

$$\Phi(\xi,s,\upsilon) = \sum_{m=0}^{\infty} \frac{\xi^m}{(m+\upsilon)^s}$$

$$(\upsilon \neq \{0,-1,-2,\dots\}; s \in \mathbb{C} \text{ when } |\xi| < 1; \Re(s) > 1 \text{ when } |\xi| = 1). \qquad (1.1)$$

A more detailed exposition of the various generalizations, properties, and applications of the Hurwitz–Lerch zeta functions could be found in the literature (see [3–12]). For example Goyal and Laddha [9], Lin and Srivastava [13] and Garg et al. [7] established certain remarkable extensions of the Hurwitz–Lerch zeta function $\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon)$ given in Eq. (1.1), which are described, respectively, by

$$\Phi_\delta^*(\xi,s,\upsilon) = \sum_{m=0}^{\infty} \frac{(\delta)_m}{m!} \frac{\xi^m}{(m+\upsilon)^s}$$

$$(\delta \in \mathbb{C}; \upsilon \neq \{0,-1,-2,\dots\}, s \in \mathbb{C} \text{ when } |\xi| < 1; \Re(s-\delta) > 1 \text{ when } |\xi| = 1), \qquad (1.2)$$

$$\Phi_{\varsigma,\gamma}^{(\eta,\omega)}(\xi,s,\upsilon) = \sum_{m=0}^{\infty} \frac{(\varsigma)_{\eta m}}{(\gamma)_{\omega m}} \frac{\xi^m}{(m+\upsilon)^s}$$

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$$\left(\varsigma \in \mathbb{C}; \upsilon, \gamma \neq \{0, -1, -2, \ldots\}; \eta, \omega \in \mathbb{R}^+; \eta < \omega \text{ when } s, \xi \in \mathbb{C};\right.$$

$$\left.\eta = \omega \text{ and } s \in \mathbb{C} \text{ when } |\xi| < 1, \eta = \omega \text{ and } \Re(s - \varsigma + \gamma) > 1 \text{ when } |\xi| = 1\right) \tag{1.3}$$

and

$$\Phi_{\delta,\varsigma;\gamma}(\xi, s, \upsilon) = \sum_{m=0}^{\infty} \frac{(\delta)_m (\varsigma)_m}{(\gamma)_m (m + \upsilon)^s} \frac{\xi^m}{m!}$$

$$\left(\delta, \varsigma \in \mathbb{C}; \gamma, \upsilon \neq \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |\xi| < 1;\right.$$

$$\left.\Re(s + \gamma - \delta - \varsigma) > 1 \text{ when } |\xi| = 1\right), \tag{1.4}$$

where $(\delta)_m$ stands for the Pochhammer symbol (for $\delta \in \mathbb{C}$) given by (see [14, p. 22, Eq. (1)])

$$(\delta)_m = \frac{\Gamma(\delta + m)}{\Gamma(\delta)} = \begin{cases} \delta(\delta + 1)(\delta + 2) \cdots (\delta + m - 1) & (m \geq 1, \delta \in \mathbb{C}), \\ 1 & (m = 0, \delta \in \mathbb{C} \setminus \{0\}). \end{cases}$$

The integral representation of Eqs. (1.2), (1.3) and (1.4) are given below, respectively,

$$\Phi_\delta^*(\xi, s, \upsilon) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\upsilon t}}{(1 - \xi e^{-t})^\delta} \, dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(\upsilon-1)t}}{(e^t - \xi)^\delta} \, dt$$

$$\left(\Re(\upsilon) > 0; \Re(s) > 0 \text{ when } |\xi| \leqq 1 \ (\xi \neq 1); \Re(s) > 1 \text{ when } \xi = 1\right), \tag{1.5}$$

$$\Phi_{\varsigma,\gamma}^{(\eta,\omega)}(\xi, s, \upsilon) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\upsilon t} {}_2\psi_1 \begin{bmatrix} (\varsigma; \eta), (1; 1) \; ; \\ (\gamma; \omega) \qquad \; ; \end{bmatrix} \xi e^{-t} \end{bmatrix} dt$$

$$\left(\Re(\upsilon) > 0; \Re(s) > 0 \text{ when } |\xi| \leqq 1 \ (\xi \neq 1); \Re(s) > 1 \text{ when } \xi = 1\right), \tag{1.6}$$

where ${}_2\psi_1$ is the Fox–Wright function defined in [8] and

$$\Phi_{\delta,\varsigma;\gamma}(\xi, s, \upsilon) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\upsilon t} {}_2F_1(\delta, \varsigma; \gamma; \xi e^{-t}) \, dt$$

$$\left(\Re(\upsilon) > 0; \Re(s) > 0 \text{ when } |\xi| \leqq 1 \ (\xi \neq 1); \Re(s) > 1 \text{ when } \xi = 1\right). \tag{1.7}$$

Firstly, in 2014 Parmar and Raina [15] introduced the generalized Hurwitz–Lerch zeta function involving the extended beta function [16] given by

$$\Phi_{\delta,\varsigma;\gamma}(\xi, s, \upsilon; p) = \sum_{m=0}^{\infty} \frac{(\delta)_m}{m!} \frac{B_p(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{\xi^m}{(m + \upsilon)^s}$$

$$\left(p \geqq 0; \delta, \varsigma \in \mathbb{C}; \gamma, \upsilon \neq \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |\xi| < 1;\right.$$

$$\left.\Re(s + \gamma - \delta - \varsigma) > 1 \text{ when } \xi = 1\right), \tag{1.8}$$

and they also established their integral representation as follows:

$$\Phi_{\delta,\varsigma;\gamma}(\xi, s, \upsilon; p) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\upsilon t} F_p(\delta, \varsigma; \gamma; \xi e^{-t}) \, dt$$

$$\left(p \geqq 0; \Re(\upsilon) > 0; \Re(s) > 0 \text{ when } |\xi| \leqq 1 \ (\xi \neq 1); \Re(s) > 1 \text{ when } \xi = 1\right), \tag{1.9}$$

where the extended beta function $B(\delta_1, \delta_2; p)$ appears in (1.8) and extended hypergeometric function $F_p(a, b; c; \xi)$ [17] appears in (1.9) defined, respectively, by

$$
B(\delta_1, \delta_2; p) = \int_0^1 t^{\delta_1 - 1}(1 - t)^{\delta_2 - 1} \exp\left(-\frac{p}{t(1 - t)}\right) dt
$$

$$
\left(\Re(p) > 0, \Re(\delta_1) > 0, \Re(\delta_2) > 0\right), \tag{1.10}
$$

and

$$
F_p(a, b; c; \xi) = \sum_{m=0}^{\infty} (a)_m \frac{B_p(b + m, c - b)}{B(b, c - b)} \frac{(\xi)^m}{m!}
$$

$$
\left(p \geqq 0, |\xi| < 1; \Re(c) > \Re(b) > 0\right). \tag{1.11}
$$

Obviously for $p = 0$, Eq. (1.8) reduces directly to (1.4).

Moreover, Choi et al. [1] established the underlying generalization of extended beta and extended hypergeometric functions given by, respectively,

$$
B_{p,q}(\delta_1, \delta_2) = \int_0^1 t^{\delta_1 - 1}(1 - t)^{\delta_2 - 1} \exp\left(-\frac{p}{t} - \frac{q}{1 - t}\right) dt
$$

$$
\left(\Re(p) > 0; \Re(q) > 0; \Re(\delta_1) > 0; \Re(\delta_2) > 0\right), \tag{1.12}
$$

and

$$
F_{p,q}(a, b; c; \xi) = \sum_{m=0}^{\infty} (a)_m \frac{B_{p,q}(b + m, c - b)}{B(b, c - b)} \frac{(\xi)^m}{m!}
$$

$$
\left(p \geqq 0, q \geqq 0, |\xi| < 1; \Re(c) > \Re(b) > 0\right). \tag{1.13}
$$

It is clearly seen that Eqs. (1.10) and (1.11) are particular cases of Eq. (1.12) and Eq. (1.13), respectively, provided when $p = q$.

Motivated by those various fascinating extensions of Hurwitz–Lerch zeta function, further we establish an extension of generalized Hurwitz–Lerch zeta function involving extended beta function $B(\delta_1, \delta_2; p, q)$.

2 A new extension of the Hurwitz–Lerch zeta function

In this section, we consider a new extension of the generalized Hurwitz–Lerch zeta function involving extended beta function [1] given by

$$
\Phi_{\delta, \varsigma; \gamma}(\xi, s, \upsilon; p, q) = \sum_{m=0}^{\infty} \frac{(\delta)_m}{m!} \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{\xi^m}{(m + \upsilon)^s}
$$

$$
\left(p \geq 0, q \geq 0; \delta, \varsigma \in \mathbb{C}; \gamma, \upsilon \neq \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |\xi| < 1;
\right.
$$

$$
\left. \Re(s + \gamma - \delta - \varsigma) > 1 \text{ when } |\xi| = 1\right). \tag{2.1}
$$

Remark 2.1 We enumerate the following particular and limiting cases of the function $\Phi_{\delta, \varsigma, \gamma}(\xi, s, \upsilon; p, q)$:

(i) In Eq. (2.1) substituting $\delta = 1$, we obtain a new extended form of the generalized Hurwitz–Lerch zeta function found by Lin and Srivastava [13]:

$$\Phi_{\varsigma,\gamma}^{1,1}(\xi, s, \upsilon; p, q) = \Phi_{1,\varsigma,\gamma}(\xi, s, \upsilon; p, q) = \sum_{m=0}^{\infty} \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{\xi^m}{(m + \upsilon)^s}$$

$$\big(p \geq 0, q \geq 0; \varsigma \in \mathbb{C}; \gamma, \upsilon \neq \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |\xi| < 1;$$

$$\Re(s + \gamma - \varsigma) > 1 \text{ when } |\xi| = 1\big). \tag{2.2}$$

(ii) If we set $q = 1$, Eq. (2.2) reduces the extended Hurwitz–Lerch zeta function introduced in [15, p. 160, Eq. (2.2)]:

$$\Phi_{\varsigma,\gamma}^{1,1}(\xi, s, \upsilon; p) = \Phi_{1,\varsigma,\gamma}(\xi, s, \upsilon; p) = \sum_{m=0}^{\infty} \frac{(\delta)_m}{m!} \frac{B_p(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{\xi^m}{(m + \upsilon)^s}$$

$$\big(p \geq 0, \varsigma \in \mathbb{C}; \gamma, \upsilon \neq \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |\xi| < 1;$$

$$\Re(s + \gamma - \varsigma) > 1 \text{ when } |\xi| = 1\big). \tag{2.3}$$

(iii) On taking the values of $\delta = \gamma = 1$ in Eq. (2.1), we find a new particular case of the extended generalized zeta function $\Phi_\delta^*(\xi, s, \upsilon)$ established by Goyal and Laddha [9]:

$$\Phi_\varsigma^*(\xi, s, \upsilon; p, q) = \Phi_{1,\varsigma,1}(\xi, s, \upsilon; p, q) = \sum_{m=0}^{\infty} \frac{B_{p,q}(\varsigma + m, 1 - \varsigma)}{B(\varsigma, 1 - \varsigma)} \frac{\xi^m}{(m + \upsilon)^s}$$

$$\big(\Re(p) > 0, \Re(q) > 0; \varsigma \in \mathbb{C}; \upsilon \neq \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |\xi| < 1;$$

$$\Re(s + \delta - 1) > 1 \text{ when } |\xi| = 1\big). \tag{2.4}$$

(iv) In Eq. (2.4) if we set $p = q = 1$ and $\delta = \gamma = 1$ gives the special case of the Hurwitz–Lerch zeta function introduced in [8, p. 160, Eq. (2.3)]:

$$\Phi_\varsigma^*(\xi, s, \upsilon; 1, 1) = \Phi_{1,\varsigma,1}(\xi, s, \upsilon; 1, 1) = \sum_{m=0}^{\infty} \frac{B_{p,q}(\varsigma + m, 1 - \varsigma)}{B(\varsigma, 1 - \varsigma)} \frac{\xi^m}{(m + \upsilon)^s}$$

$$\big(\varsigma \in \mathbb{C}; \upsilon \neq \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |\xi| < 1;$$

$$\Re(s + \delta - 1) > 1 \text{ when } |\xi| = 1\big). \tag{2.5}$$

(v) The limiting case of new extension of the generalized Hurwitz–Lerch zeta function involving extended beta function $\Phi_{\varsigma;\gamma}^*(\xi, s, \upsilon; p, q)$ is given by

$$\Phi_{\varsigma;\gamma}^*(\xi, s, \upsilon; p, q) = \lim_{|\delta| \to \infty} \left\{ \Phi_{\varsigma;\gamma}^*\left(\frac{\xi}{\delta}, s, \upsilon; p, q \right) \right\}$$

$$= \sum_{m=0}^{\infty} \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma) m!} \frac{\xi^m}{(m + \upsilon)^s}$$

$$\big(p \geq 0, q \geq 0; \varsigma \in \mathbb{C}; \gamma, \upsilon \neq \{0, -1, -2, \ldots\}; s \in \mathbb{C} \text{ when } |\xi| < 1;$$

$$\Re(s + \gamma - \varsigma) > 1 \text{ when } |\xi| = 1\big). \tag{2.6}$$

(vi) The particular cases of Eqs. (2.1) and (2.4) are clearly seen to reduce to Eqs. (1.4) and (1.2), respectively, provided $p = q = 0$, in view of the underlying connection of the functions:

$$\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;0,0) = \Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon) \quad \text{and} \quad \Phi_{1,\varsigma;1}(\xi,s,\upsilon;0,0) = \Phi_{\varsigma}^{*}(\xi,s,\upsilon).$$

Moreover, if $p = q = 0$ in Eq. (2.2) yields the underlying particular case of the generalized Hurwitz–Lerch zeta function of Lin and Srivastava [13] with $(\eta = \omega = 1)$:

$$\Phi_{1,\varsigma;\varsigma}(\xi,s,\upsilon;0,0) = \Phi_{\varsigma;\gamma}^{1,1}(\xi,s,\upsilon).$$

3 Integral representations differential formula

The section deals with the integral representation of the new extension of the generalized Hurwitz–Lerch zeta function involving the extended beta function (2.1) as follows.

Theorem 3.1 *For* $\Re(p) \geqq 0$, $\Re(q) \geqq 0$; $p = 0$, $\Re(\upsilon) > 0$; $\Re(s) > 0$, *when* $|\xi| \leqq 1$; $\Re(s) > 1$, *when* $\xi = 1$, *then*

$$\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;p,q) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-\upsilon t} F_{p,q}\big(\delta,\varsigma;\gamma;\xi e^{-\upsilon t}\big)\,dt. \tag{3.1}$$

Proof We know that the Eulerian integral of the gamma function obeys the following identity [18]:

$$\frac{1}{(m+\upsilon)^{s}} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-(m+\upsilon)t}\,dt \quad \big(\min\{\Re(s),\Re(\upsilon)\} > 0; m \in \mathbb{N}_0\big).$$

Employing the above result in Eq. (2.1) and then interchanging the order of summation and integration (condition above), we obtain

$$\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;p,q) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-\upsilon t} \left(\sum_{m=0}^{\infty} (\delta)_m \frac{B_{p,q}(\varsigma+m,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)} \frac{(\xi e^{-t})^m}{m!}\right) dt.$$

In view of the definition (1.12) and (1.13), we obtain the required result (3.1). □

Theorem 3.2 *For* $\Re(p) > 0$, $\Re(q) > 0$; $p = 0$, $q = 0$, $\Re(\varsigma) > 0$, $\Re(\gamma) > 0$, $\min\{s,\upsilon\} > 0$, *then*

$$\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;p,q) = \frac{e^{-p-q}\Gamma(\gamma)}{\Gamma(\varsigma)\Gamma(\gamma-\varsigma)}$$
$$\times \int_{0}^{\infty} \frac{x^{\varsigma-1}}{(1+x)^{\gamma}} \exp\left(-qx - \frac{p}{x}\right) \Phi_{\delta}^{*}\left(\frac{\xi x}{1+x}, s, \upsilon\right) dx \tag{3.2}$$

and

$$\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;p,q) = \frac{e^{-p-q}\Gamma(\gamma)}{\Gamma(\varsigma)\Gamma(s)\Gamma(\gamma-\varsigma)} \int_{0}^{\infty}\int_{0}^{\infty} \frac{t^{s-1} e^{-\upsilon t} x^{\varsigma-1}}{(1+x)^{\gamma-1}}$$
$$\times \exp\left(-qx - \frac{p}{x}\right)\left(1 - \frac{\xi x e^{-t}}{1+x}\right)^{-\delta}, \tag{3.3}$$

provided the integrals in the right-hand side of Eqs. (3.2) and (3.3) converge.

Proof On setting $\delta_1 = \varsigma + m$ and $\delta_2 = \gamma - \varsigma$ in the underlying integral representation of the extended beta function (see, e.g., [1, p. 361, Eq. (2.6)]):

$$B_{p,q}(\delta_1, \delta_2) = e^{-p-q} \int_0^\infty \frac{x^{\delta_1 - 1}}{(1+x)^{\delta_1 + \delta_2}} \exp\left(-qx - \frac{p}{x}\right) dx, \tag{3.4}$$

we obtain

$$B_{p,q}(\varsigma + m, \gamma - \varsigma) = e^{-p-q} \int_0^\infty \frac{x^{\varsigma + m - 1}}{(1+x)^{\gamma + m}} \exp\left(-qx - \frac{p}{x}\right) dx, \tag{3.5}$$

in view of the above relation and using definition (1.2), Eq. (2.1) clearly gives the first statement of Theorem 3.2.

Moreover, using the integral representation (1.5) in Eq. (3.4), we obtain the required result (3.3). □

Theorem 3.3 *For $\Re(p) > 0$, $\Re(q) > 0$; $p = 0$, $q = 0$, $\Re(\varsigma) > 0$, $\Re(\gamma) > 0$, $\min\{s, \upsilon\} > 0$, then*

$$\Phi_{\delta, \varsigma; \gamma}(\xi, s, \upsilon; p, q) = \frac{2^{1-\gamma} \Gamma(\gamma)}{\Gamma(\varsigma) \Gamma(\gamma - \varsigma)} \int_{-1}^1 (1+x)^{\varsigma - 1} (1-x)^{\gamma - \varsigma - 1}$$
$$\times \exp\left(-\frac{2(p+q) + 2(q-p)}{1-x^2}\right) \Phi_\delta^*\left(\frac{\xi(1+x)}{2}, s, \upsilon\right) dx \tag{3.6}$$

and

$$\Phi_{\delta, \varsigma; \gamma}(\xi, s, \upsilon; p, q) = \frac{2^{1-\gamma} \Gamma(\gamma)}{\Gamma(\varsigma) \Gamma(s) \Gamma(\gamma - \varsigma)} \int_{-1}^1 \int_0^\infty t^{s-1} e^{-\upsilon t} (1+x)^{\varsigma - 1} (1-x)^{\gamma - \varsigma - 1}$$
$$\times \exp\left(-\frac{2(p+q) + 2(q-p)}{1-x^2}\right)\left(1 - \frac{\xi e^{-t}(1+x)}{2}\right)^{-\delta} dx. \tag{3.7}$$

Proof On setting $\delta_1 = \varsigma + m$ and $\delta_2 = \gamma - \varsigma$ in the underlying integral representation of the extended beta function (see, e.g., [1, p. 361, Eq. (2.7)]):

$$B_{p,q}(\delta_1, \delta_2) = 2^{1-\delta_1-\delta_2} \int_{-1}^\infty (1+x)^{\delta_1 - 1}(1-x)^{\delta_2 - 1} \exp\left(-\frac{2(p+q) + 2(q-p)}{1-x^2}\right) dx, \tag{3.8}$$

similarly, it can be easy to prove both the assertions of Theorem 3.3 and of Theorem 3.2. □

Theorem 3.4 *For $\Re(p) > 0$, $\Re(q) > 0$; $p = 0$, $q = 0$, $\Re(\varsigma) > 0$, $\Re(\gamma) > 0$, $\min\{s, \upsilon\} > 0$, then*

$$\Phi_{\delta, \varsigma; \gamma}(\xi, s, \upsilon; p, q) = \frac{(c-a)\Gamma(\gamma)}{\Gamma(\varsigma)\Gamma(\gamma - \varsigma)} \int_a^c (x-a)^{\varsigma - 1}(c-x)^{\gamma - \varsigma - 1}$$
$$\times \exp\left(-\frac{(c-a)}{(x-a)(c-x)}\{(q-p)x + (pc - qa)\}\right)$$
$$\times \Phi_\delta^*\left(\xi\left(\frac{x-a}{c-a}\right), s, \upsilon\right) dx \tag{3.9}$$

and

$$\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;p,q) = \frac{(c-a)\Gamma(\gamma)}{\Gamma(\varsigma)\Gamma(s)\Gamma(\gamma-\varsigma)}\int_a^c (x-a)^{\varsigma-1}(c-x)^{\gamma-\varsigma-1}$$
$$\times \exp\left(-\frac{(c-a)}{(x-a)(c-x)}\{(q-p)x+(pc-qa)\}\right)$$
$$\times \left(1-\xi\left(\frac{x-a}{c-a}\right)e^{-t}\right)^{-\delta}dx. \tag{3.10}$$

Proof On setting $\delta_1 = \varsigma + m$ and $\delta_2 = \gamma - \varsigma$ in the underlying integral representation of the extended beta function (see, e.g., [1, p. 362, Eq. (2.8)]):

$$B_{p,q}(\delta_1,\delta_2) = (c-a)^{1-\delta_1-\delta_2}\int_a^c (x-a)^{\delta_1-1}(c-x)^{\delta_2-1}$$
$$\times \exp\left(-\frac{(c-a)}{(x-a)(c-x)}\{(q-p)x+(pc-qa)\}\right)dx, \tag{3.11}$$

similarly, it is easy to prove both the assertions of Theorem 3.4 and of Theorem 3.2. □

Theorem 3.5 *For $p \geq 0$, $q \geq 0$, $\Re(\delta) > 0$, $\Re(\upsilon) > 0$, $\Re(s) > 0$, when $|\xi| \leqq 1$ ($\xi \neq 1$); $\Re(s) > 1$, when $\xi = 1$, then*

$$\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q) = \frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}e^{-t}\Phi^*_{\varsigma,\gamma;p,q}(\xi t,s;\upsilon;p,q)\,dt, \tag{3.12}$$

where $\Phi^*_{\varsigma,\gamma;p,q}(\xi t,s;\upsilon;p,q)$ is the limiting case in (2.6).

Proof The integral representation of the Pochhammer symbol $(\delta)_m$ is defined as

$$(\delta)_m = \frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta+m-1}e^{-t}\,dt.$$

By making use of the above relation in (2.1) and interchanging the order of summation and integration which may be admissible subject to the condition of Theorem 3.5, we obtain

$$\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q) = \frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}e^{-t}\sum_{m=0}^\infty \frac{B_{p,q}(\varsigma+m,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)}\frac{(\xi t)^m}{m!(m+\upsilon)^s}\,dt. \tag{3.13}$$

Applying (2.6), we get the required integral representation. □

Remark 3.1 On substituting $q = 0$ in Eq. (3.13), we obtain the result earlier obtained by Parmar and Raina [8].

Subsequently, we establish the underlying derivative formula of (2.1).

Theorem 3.6 *The following differential formula for $\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q)$ holds*:

$$\frac{d^m}{d\xi^m}\left\{\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q)\right\} = \frac{(\delta)_m(\varsigma)_m}{(\gamma)_m}\Phi_{\delta+m,\varsigma+m,\gamma+m;p,q}(\xi,s;\upsilon+m;p,q) \quad (m \in \mathbb{N}).$$

Proof Consider the derivative of (2.1) with respect to ξ, we obtain

$$\frac{d}{d\xi}\left\{\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q)\right\} = \sum_{m=0}^{\infty}\frac{(\delta)_m}{(m-1)!}\frac{B_{p,q}(\varsigma+m,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)}\frac{\xi^{m-1}}{(m+\upsilon)^s}, \tag{3.14}$$

which upon replacing m by $m+1$ in Eq. (3.14) and employing the identity

$$B(\varsigma,\gamma-\varsigma) = \frac{\gamma}{\varsigma}B(\varsigma+1,\gamma-\varsigma), \qquad (\upsilon)_{m+1} = \upsilon(\upsilon+1)_m,$$

leads to the derivative formula

$$\frac{d}{d\xi}\left\{\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q)\right\} = \frac{\delta\varsigma}{\gamma}\Phi_{\delta+1,\varsigma+1,\gamma+1;p,q}(\xi,s;\upsilon+1;p,q), \tag{3.15}$$

recursive application of this procedure provides the required result (3.6). $\qquad\square$

4 Mellin transform of the Hurwitz–Lerch zeta function and their relation between $\overline{H}$-function

The Mellin transform of a suitable integrable function $f(\kappa)$ with index φ is defined, generally, by

$$\mathcal{M}\left\{f(\kappa):\kappa\to\varphi\right\} = \int_0^\infty \kappa^{\varphi-1}f(\kappa)\,d\kappa \quad (0<\kappa<\infty). \tag{4.1}$$

Theorem 4.1 *Consider* $\Re(p)>0$, $\Re(q)>0$, $\Re(\varphi)>0$, $\Re(\psi)>0$, $\Re(\varsigma+m+\varphi)>0$, $\Re(\gamma-\varsigma+\psi)>0$, *then the Mellin transform of the function* $\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q)$ *defined by* (2.1) *is given by*

$$\mathcal{M}\left\{\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q):p\to\varphi,q\to\psi\right\}$$
$$= \frac{\Gamma(\varphi)\Gamma(\psi)B(\varsigma+\varphi,\gamma-\varsigma+\psi)}{B(\varsigma,\gamma-\varsigma)}\Phi_{\delta,\varsigma+\varphi;\varphi+\psi+\gamma}(\xi,s;\upsilon). \tag{4.2}$$

Proof Consider the Mellin transform for (2.1) in view of definition (4.1), we obtain

$$\mathcal{M}\left\{\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q):p\to\varphi,q\to\psi\right\}$$
$$= \int_0^\infty\int_0^\infty p^{\varphi-1}q^{\psi-1}\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q)\,dp\,dq$$
$$= \int_0^\infty\int_0^\infty p^{\varphi-1}q^{\psi-1}\sum_{m=0}^{\infty}\frac{B_{p,q}(\varsigma+m,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)}\frac{(\delta)_m\xi^m}{m!(m+\upsilon)^s}\,dp\,dq$$
$$= \sum_{m=0}^{\infty}\frac{(\delta)_m\xi^m}{m!(m+\upsilon)^s}\frac{1}{B(\varsigma,\gamma-\varsigma)}\int_0^\infty\int_0^\infty p^{\varphi-1}q^{\psi-1}B_{p,q}(\varsigma+m,\gamma-\varsigma)\,dp\,dq.$$

Employing the underlying well-known integral representation (see [1, p. 360, Eq. (1.14)])

$$\int_0^\infty\int_0^\infty p^{\varphi-1}q^{\psi-1}B_{p,q}(\varsigma+m,\gamma-\varsigma)\,dp\,dq = \Gamma(\varphi)\Gamma(\psi)B(\varsigma+m+\varphi,\gamma-\varsigma+\psi)$$

$$\left(\Re(\varphi)>0,\Re(\psi)>0\right), \tag{4.3}$$

we get

$$\mathcal{M}\left\{\Phi_{\delta,\varsigma,\gamma;p,q}(\xi,s;\upsilon;p,q):p\to\varphi,q\to\psi\right\}$$

$$=\frac{\Gamma(\varphi)\Gamma(\psi)}{B(\varsigma,\gamma-\varsigma)}\sum_{m=0}^{\infty}\frac{(\delta)_m}{m!}\frac{\xi^m}{(m+\upsilon)^s}B(\varsigma+m+\varphi,\gamma-\varsigma+\psi)$$

$$=\frac{\Gamma(\varphi)\Gamma(\psi)B(\varsigma+\varphi,\gamma-\varsigma+\psi)}{B(\varsigma,\gamma-\varsigma)}\sum_{m=0}^{\infty}\frac{(\delta)_m(\varsigma+\varphi)_m}{m!(\varphi+\gamma+\psi)_m}\frac{\xi^m}{(m+\upsilon)^s}, \tag{4.4}$$

which gives the desired result 4.1. □

Remark 4.1 The generalized Hurwitz–Lerch zeta function can be easily written in terms of $\overline{H}$-function as appears in the literature [19, 20] (see also [7, p. 316, Eq. (3.2)]):

$$\Phi_{\delta,\varsigma,\gamma}(\xi,s;\upsilon)=\frac{\Gamma(\gamma)}{\Gamma(\varsigma)\Gamma(\varsigma-\gamma)}\overline{H}_{3,3}^{1,3}\left[-\xi\left|\begin{array}{c}(1-\delta,1;1),(1-\varsigma,1;1),(1-\upsilon,1;s)\\(0,1),(1-\gamma,1;1),(-\upsilon,1;s)\end{array}\right.\right]. \tag{4.5}$$

By using (4.5) we can easily deduce Eq. (4.2), Mellin representation in terms of the $\overline{H}$-function which is a fascinating result given as a corollary.

Corollary 4.1 *The underlying Mellin representation holds*:

$$\mathcal{M}\left\{\Phi_{\delta,\varsigma,\gamma}(\xi,s;\upsilon;p,q):p\to\varphi,q\to\psi\right\}$$

$$=\frac{\Gamma(\varphi)\Gamma(\psi)\Gamma(\gamma-\varsigma+\psi)}{\Gamma(\delta)B(\varsigma,\gamma-\varsigma)}\overline{H}_{3,3}^{1,3}\left[-\xi\left|\begin{array}{c}(1-\delta,1;1),(1-\varsigma-\varphi,1;1),(1-\upsilon,1;s)\\(0,1),(1-\varphi-\psi-\gamma,1;1),(-\upsilon,1;s)\end{array}\right.\right]. \tag{4.6}$$

Theorem 4.2 *For* $p\geqq0,q\geqq0,\delta\in\mathbb{C}$ *and* $|t|<1$, *the underlying generating function holds*:

$$\sum_{m=0}^{\infty}(\delta)_m\Phi_{\delta+m,\varsigma,\gamma}(\xi,s;\upsilon;p,q)\frac{t^m}{m!}=(1-t)^{-\delta}\Phi_{\delta+m,\varsigma,\gamma}\left(\frac{\xi}{1-t},s,\upsilon;p,q\right). \tag{4.7}$$

Proof Consider the left-hand side of the assertion (4.7) of Theorem 4.2 be denoted by K_1 and in view of definition (2.1), we obtain

$$K_1=\sum_{m=0}^{\infty}(\delta)_m\left\{\sum_{l=0}^{\infty}(\delta+m)_l\frac{B_{p,q}(\varsigma+l,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)}\frac{\xi^l}{l!(m+\upsilon)^s}\right\}\frac{t^m}{m!}.$$

Inverting the order of summation of the above equality and after a little simplification, we get

$$=\sum_{l=0}^{\infty}(\delta)_l\frac{B_{p,q}(\varsigma+l,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)}\left\{\sum_{m=0}^{\infty}(\delta+l)_m\frac{t^m}{m!}\right\}\frac{\xi^l}{l!(m+\upsilon)^s}. \tag{4.8}$$

Now, employing the binomial expansion

$$(1-t)^{-(\delta+l)}=\sum_{m=0}^{\infty}(\delta+l)_m\frac{t^m}{m!}, \tag{4.9}$$

and in view of definition (2.1), we obtain the assertion (4.7) of Theorem 4.2. □

Theorem 4.3 *Let $p,q \geqq 0$, $\delta \in \mathbb{C}$ and $|t| < |\upsilon|$; $s \neq 1$ then the generating functions of $\Phi_{\delta,\varsigma,\gamma}(\xi,s;\upsilon;p,q)$ is given by*

$$\sum_{m=0}^{\infty} \frac{(s)_m}{m!} \Phi_{\delta,\varsigma,\gamma}(\xi,s+m;\upsilon;p,q)t^m = \Phi_{\delta,\varsigma,\gamma}(\xi,s;\upsilon-t;p,q). \tag{4.10}$$

More generally

$$\sum_{m=0}^{\infty} \frac{(\delta)_m}{m!} \Phi_{\delta,\varsigma,\gamma}(\xi,s+m;\upsilon;p,q)t^m$$
$$= \sum_{l=0}^{\infty} \frac{(\delta)_l}{l!} \frac{B_{p,q}(\varsigma+l,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)} \frac{\xi^l}{(l+\upsilon)^{s-\delta}(l+\upsilon-t)^{\delta}} \tag{4.11}$$

and

$$\sum_{m=0}^{\infty} \Phi_{\delta,\varsigma,\gamma}(\xi,s+m;\upsilon;p,q)\frac{t^m}{m!}$$
$$= \sum_{l=0}^{\infty} \frac{(\delta)_l}{l!} \frac{B_{p,q}(\varsigma+l,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)} \frac{\xi^l}{(l+\upsilon)^s} \exp\left(\frac{t}{l+\upsilon}\right). \tag{4.12}$$

Proof Using the definition (2.1) in the right-hand side of (4.10), we have

$$\Phi_{\delta,\varsigma,\gamma}(\xi,s;\upsilon-t;p,q) = \sum_{l=0}^{\infty}(\delta)_l \frac{B_{p,q}(\varsigma+l,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)} \frac{\xi^l}{l!(l+\upsilon-t)^s}$$
$$= \sum_{l=0}^{\infty}(\delta)_l \frac{B_{p,q}(\varsigma+l,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)} \frac{\xi^l}{l!(l+\upsilon)^s}\left(1-\frac{t}{l+\upsilon}\right)^{-s}.$$

In view of expansion (4.9) and some little simplification of the above second equality, we are thus led to the assertion (4.11).

The generating function (4.11) can easily be deduced by substituting $\delta = s$ in Eq. (4.12). □

5 Fractional kinetic equation

This section deals with the fractional kinetic equation (FKE) involving the new extended Hurwitz–Lerch zeta function (2.1). The FKE has great significance in the field of astrophysics and mathematical physics.

The solutions of FKE has many applications in various fields such as renormalization of the non-stationary problem near the phase transition point [21], the theory of turbulence [22], diffusion in porous media [23], and kinetics in viscoelastic media [24], which has been published in the literature of special functions.

In 2000 Haubold and Mathai [25] derived a fascinating result between the rate of change of reaction, the destruction rate, and the production rate given by

$$\frac{d\mathcal{N}}{dt} = -\delta(\mathcal{N}_t) + \mathfrak{p}(\mathcal{N}_t), \tag{5.1}$$

where $\mathcal{N} = \mathcal{N}(t)$ is the rate of reaction, $\delta(\mathcal{N}_t) =: \delta$ is the rate of destruction, $\mathfrak{p} = \mathfrak{p}(\mathcal{N})$ is the rate of production and $\mathcal{N}_t$ signifies the function defined by $\mathcal{N}_t(t^*) = \mathcal{N}(t-t^*)$; $t^* > 0$.

Under spatial fluctuations or homogeneities where the quantity $\mathcal{N}(t)$ is neglected we arrive at a particular case of Eq. (5.1), which is given by (see [25, 26])

$$\frac{d\mathcal{N}_i}{dt} = -c_i \mathcal{N}_i(t), \tag{5.2}$$

with the initial condition $\mathcal{N}_i(t=0) = \mathcal{N}_0$ standing for the number of density of species i at time $t=0$; $c_i > 0$, studied as standard kinetic equation. Integrating both sides of Eq. (5.2) under the condition if the index i is omitted, we get

$$\mathcal{N}(t) - \mathcal{N}_0 = -c_0 \, _0D_t^{-1} - \mathcal{N}(t), \tag{5.3}$$

where $_0D_t^{-1}$ denotes the standard fractional integral operator.

A fractional generalization of standard kinetic Eq. (5.2) is investigated by Haubold and Mathai [25] as follows:

$$\mathcal{N}(t) - \mathcal{N}_0 = -c_0^\omega \, _0D_t^{-\omega} - \mathcal{N}(t), \tag{5.4}$$

where $_0D_t^{-\omega}$ is the familiar Riemann–Liouville fractional integral operator (see [27]) defined as

$$_0D_t^{-\omega}f(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-u)^{\omega-1} f(u)\, du \quad \left(t > 0, \Re(\omega) > 0\right), \tag{5.5}$$

and they obtained the solution of (5.5) as follows:

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{l=0}^\infty \frac{(-1)^l}{\Gamma(\omega l + 1)} (ct)^{\omega k}. \tag{5.6}$$

Moreover, Saxena and Kalla [25] obtained the underlying fractional kinetic equation:

$$\mathcal{N}(t) - \mathcal{N}_0 f(t) = -c_0^\omega \, _0D_t^{-\omega}\mathcal{N}(t) \quad \left(\Re(\omega) > 0\right), \tag{5.7}$$

where $\mathcal{N}(t)$ is the number density of a given species at time t, $\mathcal{N}_0 = \mathcal{N}(0)$ is the number density of that species at time $t=0$, c is a constant and $f \in \mathcal{L}(0, \infty)$.

Applying the Laplace transform to (5.7) (see [28]),

$$L\{\mathcal{N}(t); \mathfrak{p}\} = \mathcal{N}_0 \left(\sum_{m=0}^\infty (-c^\omega)^m \mathfrak{p}^{-m\omega} \right) F(\mathfrak{p})$$

$$\left(m \in \mathbb{N}_0, \left| \frac{c}{\mathfrak{p}} \right| < 1 \right), \tag{5.8}$$

$$F(\mathfrak{p}) = L\{f(t); \mathfrak{p}\} = \int_0^\infty e^{-\mathfrak{p}t} f(t)\, dt \quad \left(\Re(\mathfrak{p}) > 0\right). \tag{5.9}$$

Now, we proceed to obtain the solution of the generalized fractional; kinetic equations by considering our new extended Hurwitz–Lerch zeta function. The result obtained in terms of the generalized Mittag-Leffler function.

Theorem 5.1 *Let $d > 0$, $\omega > 0$ and $\upsilon \neq d$, $c > 0$ then the solution of the equation*

$$\mathcal{N}(t) - \mathcal{N}_0\big\{\Phi_{\delta,\varsigma;\gamma}\big(t^\omega\xi, s, \upsilon; p, q\big)\big\} = -d_0^\omega \,_0D_t^{-\omega}\mathcal{N}(t) \tag{5.10}$$

is given by the following relation:

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{m=0}^\infty \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{\Gamma(m\omega + m + 1)t^{m+m\omega}}{(m + \upsilon)^s}$$

$$\times E_{\omega, m+m\omega+1}\big(d^\omega t^\omega\big), \tag{5.11}$$

where the $E_{\alpha,\beta}(\xi)$ denotes the generalized Mittag-Leffler function [29] *is given by*

$$E_{\alpha,\beta}(\xi) = \sum_{m=0}^\infty \frac{\xi^m}{\Gamma(\alpha m + \beta)} \quad \big(\Re(\alpha) > 0, \Re(\beta) > 0\big).$$

Proof The Laplace transform of the Riemann–Liouville fractional integral operator is given by [30]

$$L\big\{_0D_t^{-\omega}f(t); \mathfrak{p}\big\} = \mathfrak{p}^{-\omega}F(\mathfrak{p}), \tag{5.12}$$

where $F(\mathfrak{p})$ is defined in (5.9).

Now, taking the Laplace transform of both sides of Eq. (5.11), we obtain

$$L\big\{\mathcal{N}(t); \mathfrak{p}\big\} = \mathcal{N}_0 L\big\{\Phi_{\delta,\varsigma;\gamma}\big(t^\omega\xi, s, \upsilon; p, q\big); \mathfrak{p}\big\} - d_0^\omega\big\{_0D_t^{-\omega}\mathcal{N}(t); \mathfrak{p}\big\},$$

$$\mathcal{N}(\mathfrak{p}) = \mathcal{N}_0\left(\int_0^\infty e^{-\mathfrak{p}t} \sum_{m=0}^\infty \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{(t^{\omega+1})^m}{(m + \upsilon)^s} \, dt\right) - d^\omega \mathfrak{p}^{-\omega}\mathcal{N}(\mathfrak{p}),$$

$$\mathcal{N}(\mathfrak{p}) + d^\omega \mathfrak{p}^{-\omega}\mathcal{N}(\mathfrak{p}) = \mathcal{N}_0 \sum_{m=0}^\infty \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{1}{(m + \upsilon)^s} \int_0^\infty e^{-\mathfrak{p}t} t^{\omega m+m} \, dt,$$

$$\mathcal{N}(\mathfrak{p}) + d^\omega \mathfrak{p}^{-\omega}\mathcal{N}(\mathfrak{p}) = \mathcal{N}_0 \sum_{m=0}^\infty \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{1}{(m + \upsilon)^s} \frac{\Gamma(m\omega + m + 1)}{\mathfrak{p}^{m\omega+m+1}},$$

$$\mathcal{N}(\mathfrak{p}) = \mathcal{N}_0 \sum_{m=0}^\infty \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{1}{(m + \upsilon)^s}$$

$$\times \Gamma(m\omega + m + 1)\mathfrak{p}^{-(m\omega+m+1)}\big(1 + d^\omega \mathfrak{p}^{-\omega}\big)^{-1},$$

$$\mathcal{N}(\mathfrak{p}) = \mathcal{N}_0 \sum_{m=0}^\infty \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{1}{(m + \upsilon)^s}$$

$$\times \Gamma(m\omega + m + 1)\mathfrak{p}^{-(m\omega+m+1)} \sum_{r=0}^\infty \big(d^\omega \mathfrak{p}^{-\omega}\big)^r. \tag{5.13}$$

Applying the inverse Laplace transform to (5.13), we obtain

$$L^{-1}\{\mathcal{N}(\mathfrak{p})\} = \mathcal{N}_0 \sum_{m=0}^{\infty} \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{1}{(m + \upsilon)^s}$$

$$\times \Gamma(m\omega + m + 1) L^{-1}\left\{ \sum_{r=0}^{\infty} d^{\omega r} \mathfrak{p}^{-\omega r - m\omega - m - 1} \right\},$$

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{m=0}^{\infty} \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{1}{(m + \upsilon)^s}$$

$$\times \Gamma(m\omega + m + 1) \left\{ \sum_{r=0}^{\infty} d^{\omega r} \frac{t^{\omega r + m\omega + m1}}{\Gamma(\omega r + m\omega + m + 1)} \right\},$$

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{m=0}^{\infty} \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{1}{(m + \upsilon)^s}$$

$$\times \Gamma(m\omega + m + 1) t^{m\omega + m} \left\{ \sum_{r=0}^{\infty} \frac{d^{\omega r} t^{\omega r}}{\Gamma(\omega r + m\omega + m + 1)} \right\},$$

$$\mathcal{N}(t) = \mathcal{N}_0 \sum_{m=0}^{\infty} \frac{B_{p,q}(\varsigma + m, \gamma - \varsigma)}{B(\varsigma, \gamma - \varsigma)} \frac{(\delta)_m}{m!} \frac{1}{(m + \upsilon)^s}$$

$$\times \Gamma(m\omega + m + 1) t^{m\omega + m} E_{\omega, m\omega + m + 1}(d^\omega t^\omega),$$

which is the required result. $\square$

6 Concluding remarks

In the present paper, it seems to be of interest that the extensions of Hurwitz–Lerch zeta function so obtained are very general in nature and, by specific parameters, can yield the previously defined Hurwitz–Lerch zeta function which is shown in this paper. On that account, they become of great importance from an application perspective. For example, here we establish the connection of the new extended Hurwitz–Lerch zeta function with other special functions.

By using the connection of the generalization of the extended beta function with other special functions (see [1, p. 367, Sect. 5]), we can obtain the relation between the new extended Hurwitz–Lerch zeta function with other special functions.

• *Laguerre polynomials*

$$\Phi_{\delta, \varsigma; \gamma}(\xi, s, \upsilon; p, q) = \sum_{m=0}^{\infty} \sum_{n,r=0}^{\infty} \frac{(\delta)_m}{m!} \frac{B(\varsigma + m + n + 1, \gamma - \varsigma + r + 1)}{B(\varsigma, \gamma - \varsigma)}$$

$$\times \frac{\xi^m}{(m + \upsilon)^s} L_n(p) L_r(q)$$

$$\left(\Re(\varsigma + m + n) > -1, \Re(\gamma - \varsigma + r) > -1, \Re(\varsigma) > 0, \Re(\gamma) > \Re(\varsigma) \right). \tag{6.1}$$

- *Meijer's G-function*

$$\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;p,q)$$

$$= \sum_{m=0}^{\infty} \frac{(\delta)_m}{m!} \frac{1}{B(\varsigma,\gamma-\varsigma)} \frac{\xi^m}{(m+\upsilon)^s}$$

$$\times G_{0,0:2,0;2,0}^{1,0:0,2;0,2} \left[\begin{matrix} (\gamma+m;1,1): -; & -; \\ -,(0,1),(\varsigma+m,1);(0,1),(\gamma-\varsigma,1); \end{matrix} p,q \right], \qquad (6.2)$$

where G represents Meijer's G-function (see [31], p. 7, Eq. (1.2.3) and p. 88, Eq. (6.4.1)]).

- *Appell series*

$$\Phi_{\delta,\varsigma;\gamma}(\xi,s,\upsilon;p,q)$$

$$= \sum_{m=0}^{\infty} \frac{(\delta)_m}{m!} \frac{1}{B(\varsigma,\gamma-\varsigma)} \frac{\xi^m}{(m+\upsilon)^s}$$

$$\times B(\varsigma+m,\gamma-\varsigma)F_2[1-m-\gamma,-,-;1-\varsigma-m,1-\gamma+\varsigma;-p,-q], \qquad (6.3)$$

where $F_2[\cdot]$ represents one of the four Appell series F_j ($j=1,2,3,4$) (see [32], pp. 22–23]).

Similarly, we can further obtain the connection with the Macdonald function and the Whittaker function.

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