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On a hybrid fractional Caputo–Hadamard boundary value problem with hybrid Hadamard integral boundary value conditions

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Abstract

In the present research article, we find some important criteria on the existence of solutions for a class of the hybrid fractional Caputo–Hadamard differential equations and its corresponding inclusion problem supplemented with hybrid Hadamard integral boundary conditions. In this direction, we utilize some theorems due to Dhage's fixed point results in our proofs. Finally, we demonstrate two numerical examples to confirm the validity of the main obtained results.

MSC: Primary 34A08; secondary 34A12

Keywords: Fractional hybrid integro-differential equation; Hadamard integral; The Dhage fixed point results; The fractional Caputo–Hadamard derivative

1 Introduction

One way mathematics helps economics is to become more powerful in modeling theory so that different types of processes with distinct parameters can be written in mathematical formulas. In this case, different software can be developed to allow for more cost-free testing and less material consumption. One of basic methods in this way is working with fractional calculus. Nowadays, many researchers are studying advanced fractional modelings and their related existence results and qualitative behaviors of solutions for distinct fractional problems (see, for example, [1–5]). In recent decades, fractional hybrid differential equations and inclusions with complicated boundary value conditions have achieved a great deal of interest and attention of many researchers (see, for example, [6–21]). Also, there are many works on the fractional Hadamard derivative and its applications in different fields (see, for example, [22–26]).

In 2010, Dhage and Lakshmikantham [27] formulated a new category of differential equations called hybrid differential equations and studied properties of the solution for this kind of differential equation. In 2011, Zhao et al. [28] extended Dhage's work to fractional order and studied the corresponding hybrid fractional differential equations. After that, Baleanu et al. [29] derived some existence criteria and the dimension of the solution set for a novel category of fractional hybrid inclusion problem

$$ {}^{c}\mathcal{D}^{\nu}_{0^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t),\mathcal{I}^{\alpha_1}_{0^+}\varrho(t),\ldots,\mathcal{I}^{\alpha_n}_{0^+}\varrho(t))}\right) \in \Psi\big(t,\varrho(t),\mathcal{I}^{\beta_1}_{0^+}\varrho(t),\ldots,\mathcal{I}^{\beta_m}_{0^+}\varrho(t)\big), \quad \big(t\in[0,1]\big) $$

furnished with terminal conditions $\varrho(0)=\varrho_0^*$ and $\varrho(1)=\varrho_1^*$ so that $\nu\in(1,2]$, ${}^{c}\mathcal{D}^{\nu}_{0^+}$ and $\mathcal{I}^{\gamma}_{0^+}$ represent the Caputo derivative operator of order ν and the Riemann–Liouville integral operator of order $\gamma\in\{\alpha_i,\beta_j\}\subset(0,\infty)$ for $i=1,\ldots n$ and $j=1,\ldots,m$, respectively.

Some years later, Ullah et al. [30] derived a new existence result for the fractional hybrid BVP formulated as follows:

$$ \begin{cases} \mathcal{D}^{\alpha}_{0^+}\big(\frac{\varrho(t)-f(t,\varrho(t))}{h(t,\varrho(t))}\big)=g(t,\varrho(t)), \quad (t\in[0,1]),\\ \big(\frac{\varrho(t)-f(t,\varrho(t))}{h(t,\varrho(t))}\big)\big|_{t=0}=0, \qquad \big(\frac{\varrho(t)-f(t,\varrho(t))}{h(t,\varrho(t))}\big)\big|_{t=1}=0, \end{cases} $$

so that $h\in C_{\mathbb{R}\setminus\{0\}}([0,1]\times\mathbb{R})$, f and g are continuous real-valued functions on $[0,1]\times\mathbb{R}$, and $\mathcal{D}^{\alpha}_{0^+}$ illustrates the Riemann–Liouville derivative of order $\alpha\in(0,1]$.

By utilizing the ideas of the aforementioned articles, we design the Caputo–Hadamard fractional hybrid differential equation

$$ {}^{CH}\mathcal{D}^{\gamma}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)=\Theta\big(t,\varrho(t)\big), \quad \big(t\in[1,e]\big), \tag{1} $$

endowed with the hybrid fractional Hadamard integral boundary conditions

$$ \begin{cases} \big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=1}={}^{CH}\mathcal{D}_{1^+}\big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=1},\\ {}^{CH}\mathcal{D}_{1^+}\big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=e}={}^{CH}\mathcal{D}^{2}_{1^+}\big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=e},\\ {}^{H}\mathcal{I}^{\mu}_{1^+}\big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=e}=\frac{1}{\Gamma(\mu)}\int_1^e(\ln\frac{e}{s})^{\mu-1}\big(\frac{\varrho(s)}{\Lambda(s,\varrho(s))}\big)\frac{\mathrm{d}s}{s}=0, \end{cases} \tag{2} $$

so that $\gamma\in(2,3]$, $\mu>0$, ${}^{H}\mathcal{I}^{\mu}_{1^+}$ illustrates the Hadamard integral of order μ and the function $\Theta:[1,e]\times\mathbb{R}\to\mathbb{R}$ is continuous and $\Lambda\in C_{\mathbb{R}\setminus\{0\}}([1,e]\times\mathbb{R})$. In the following, we review the corresponding hybrid fractional Caputo–Hadamard inclusion problem

$$ {}^{CH}\mathcal{D}^{\gamma}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\in\Psi\big(t,\varrho(t)\big), \quad \big(t\in[1,e]\big), \tag{3} $$

furnished with hybrid fractional Hadamard integral boundary conditions

$$ \begin{cases} \big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=1}={}^{CH}\mathcal{D}_{1^+}\big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=1},\\ {}^{CH}\mathcal{D}_{1^+}\big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=e}={}^{CH}\mathcal{D}^{2}_{1^+}\big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=e},\\ {}^{H}\mathcal{I}^{\mu}_{1^+}\big(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\big)\big|_{t=e}=\frac{1}{\Gamma(\mu)}\int_1^e(\ln\frac{e}{s})^{\mu-1}\big(\frac{\varrho(s)}{\Lambda(s,\varrho(s))}\big)\frac{\mathrm{d}s}{s}=0, \end{cases} \tag{4} $$

so that $\Psi:[1,e]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a set-valued map equipped with some required properties. To achieve the main goals of this manuscript, the techniques of the fixed point theory are employed to prove the theoretical results. Our investigation involves two folds in which we first deal with a hybrid differential equation and then with its corresponding hybrid differential inclusion. It is worth mentioning that the proposed hybrid problems (1)–(2)

and (3)–(4) differ from the newly defined ones. We believe that our hybrid problems involve some types of special cases and this can extend to more general hybrid problems. The fractional hybrid modelings are of great significance in different engineering fields, and it can be a unique idea for the future research between various applied sciences.

The content of this article is arranged as follows. In Sect. 2, some required concepts in this regard are recalled. Section 3 is devoted to proving the main theorems relying on some mathematical inequalities and two versions of fixed point theorems due to Dhage. At the end of the paper, we give two numerical examples to support the applicability of our findings.

2 Preliminaries

Prior to proceeding to reach the main purposes, we first recall some essential auxiliary concepts which are needed throughout the paper. Let $\gamma \geq 0$ and assume that the real-valued function ϱ is integrable on (a,b). In this case, the Hadamard fractional integral of a continuous function $\varrho : (a,b) \to \mathbb{R}$ of order γ is defined by ${}^{H}\mathcal{I}^{0}_{a^+}(\varrho(t)) = \varrho(t)$ and

$$ {}^{H}\mathcal{I}^{\gamma}_{a^+}\bigl(\varrho(t)\bigr) = \frac{1}{\Gamma(\gamma)} \int_a^t \left(\ln\frac{t}{s}\right)^{(\gamma-1)} \varrho(s)\frac{\mathrm{d}s}{s} $$

provided that the RHS integral is finite-valued [31, 32]. Note that, for each $\gamma_1, \gamma_2 \in \mathbb{R}^+$, we have ${}^{H}\mathcal{I}^{\gamma_1}_{a^+}{}^{H}\mathcal{I}^{\gamma_2}_{a^+}\varrho(t) = {}^{H}\mathcal{I}^{\gamma_1+\gamma_2}_{a^+}\varrho(t)$ and ${}^{H}\mathcal{I}^{\gamma_1}_{a^+}(\ln\frac{t}{a})^{\gamma_2} = \frac{\Gamma(\gamma_2+1)}{\Gamma(\gamma_1+\gamma_2+1)}(\ln\frac{t}{a})^{\gamma_1+\gamma_2}$ for $t > a$ [32]. It is evident that

$$ {}^{H}\mathcal{I}^{\gamma_1}_{a^+}1 = \frac{1}{\Gamma(\gamma_1+1)}\left(\ln\frac{t}{a}\right)^{\gamma_1} $$

for all $t > a$ by letting $\gamma_2 = 0$ [32]. Now, let $n = [\gamma] + 1$ or $n - 1 \leq \gamma < n$. The Hadamard fractional derivative of order γ for a function $\varrho : (a,b) \to \mathbb{R}$ is defined by

$$ {}^{H}\mathcal{D}^{\gamma}_{a^+}\bigl(\varrho(t)\bigr) = \frac{1}{\Gamma(n-\gamma)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \left(\ln\frac{t}{s}\right)^{(n-\gamma-1)} \varrho(s)\frac{\mathrm{d}s}{s} $$

provided that the RHS integral has finite values [31, 32]. The Caputo–Hadamard fractional derivative of order γ for an absolutely continuous function $\varrho \in AC^n_{\mathbb{R}}([a,b])$ is defined by

$$ {}^{CH}\mathcal{D}^{\gamma}_{a^+}\bigl(\varrho(t)\bigr) = \frac{1}{\Gamma(n-\gamma)} \int_a^t \left(\ln\frac{t}{s}\right)^{(n-\gamma-1)} \left(s\frac{\mathrm{d}}{\mathrm{d}s}\right)^n \varrho(s)\frac{\mathrm{d}s}{s} $$

if the RHS integral exists [31, 32]. Again, let $\varrho \in AC^n_{\mathbb{R}}([a,b])$ so that $n - 1 < \gamma \leq n$. In [31, 32], it has been verified that the solution of the Caputo–Hadamard fractional differential equation ${}^{CH}\mathcal{D}^{\gamma}_{a^+}(\varrho(t)) = 0$ has general solutions of the form $\varrho(t) = \sum_{i=0}^{n-1} c_i(\ln\frac{t}{a})^i$, and we have

$$ {}^{H}\mathcal{I}^{\gamma}_{a^+}{}^{CH}\mathcal{D}^{\gamma}_{a^+}\varrho(t) = \varrho(t) + c_0 + c_1\left(\ln\frac{t}{a}\right) + c_2\left(\ln\frac{t}{a}\right)^2 + \cdots + c_{n-1}\left(\ln\frac{t}{a}\right)^{n-1} $$

for any $t > a$.

Here, consider the normed space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$. Then all subsets of $\mathcal{X}$, all closed subsets of $\mathcal{X}$, all bounded subsets of $\mathcal{X}$, all convex subsets of $\mathcal{X}$, and all compact subsets of $\mathcal{X}$ are denoted by collections $\mathfrak{P}(\mathcal{X})$, $\mathfrak{P}_{cls}(\mathcal{X})$, $\mathfrak{P}_{bnd}(\mathcal{X})$, $\mathfrak{P}_{cvx}(\mathcal{X})$, and $\mathfrak{P}_{cmp}(\mathcal{X})$, respectively. A set-valued map Ψ is convex-valued if, for each $\varrho \in \mathcal{X}$, the set $\Psi(\varrho)$ is convex. The set-valued map Ψ has an upper semi-continuity property whenever, for every $\varrho^* \in \mathcal{X}$, $\Psi(\rho^*)$ belongs to $\mathfrak{P}_{cls}(\mathcal{X})$ and, for each open set $\mathcal{O}$ with $\Psi(\varrho^*) \subset \mathcal{O}$, there is at least a neighborhood $\mathcal{V}_0^*$ of ϱ^* provided that $\Psi(\mathcal{V}_0^*) \subseteq \mathcal{O}$ [33]. Moreover, $\varrho^* \in \mathcal{X}$ is a fixed point for the set-valued map $\Psi : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ whenever $\varrho^* \in \Psi(\varrho^*)$ [33]. The notation $\mathfrak{FIX}(\Psi)$ represents the set of all fixed points of Ψ [33]. Consider the metric space $\mathcal{X}$ furnished with the metric $d_{\mathcal{X}}$. For every $E_1, E_2 \in \mathfrak{P}(\mathcal{X})$, the Pompeiu–Hausdorff metric $\mathrm{PH}_d : \mathfrak{P}(\mathcal{X}) \times \mathfrak{P}(\mathcal{X}) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathrm{PH}_{d_{\mathcal{X}}}(E_1, E_2) = \max\Big\{ \sup_{a_1 \in E_1} d_{\mathcal{X}}(a_1, E_2), \sup_{a_2 \in E_2} d_{\mathcal{X}}(E_1, a_2) \Big\},$$

where $d_{\mathcal{X}}(E_1, a_2) = \inf_{a_1 \in E_1} d_{\mathcal{X}}(a_1, a_2)$ and $d_{\mathcal{X}}(a_1, E_2) = \inf_{a_2 \in E_2} d_{\mathcal{X}}(a_1, a_2)$ [33]. We say that the set-valued function $\Psi : \mathcal{X} \to \mathfrak{P}_{cls}(\mathcal{X})$ is Lipschitzian if $\mathrm{PH}_{d_{\mathcal{X}}}(\Psi(\varrho_1), \Psi(\varrho_2)) \le l^* d_{\mathcal{X}}(\varrho_1, \varrho_2)$ holds for each $\varrho_1, \varrho_2 \in \mathcal{X}$, where $l^* > 0$ is a Lipschitz constant. A Lipschitz map Ψ is said to be a contraction whenever $0 < l^* < 1$ [33]. Furthermore, $\Psi : [1, e] \to \mathfrak{P}_{cls}(\mathbb{R})$ is a measurable function if the mapping $t \longmapsto d_{\mathcal{X}}(r, \Psi(t))$ is measurable for all $r \in \mathbb{R}$ [33, 34]. The graph of $\Psi : \mathcal{X} \to \mathfrak{P}_{cls}(\mathcal{Q})$ is defined by $\mathrm{Graph}(\Psi) = \{(\varrho_1, \varrho_2) \in \mathcal{X} \times \mathcal{Q} : s^* \in \Psi(\varrho)\}$ [33]. Note that the graph of Ψ is closed if, for arbitrary sequences $\{\varrho_n\}_{n\ge1}$ belonging to $\mathcal{X}$ and $\{s_n\}_{n\ge1}$ belonging to $\mathcal{Q}$ with $\varrho_n \to z_0$, $s_n \to s_0$, and $s_n \in \Psi(\varrho_n)$, we have $s_0 \in \Psi(\varrho_0)$ [34].

A set-valued operator Ψ has the complete continuity property if the set $\Psi(\mathcal{W})$ has the relative compactness property for all $\mathcal{W} \in \mathfrak{P}_{bnd}(\mathcal{X})$. Let $\Psi : \mathcal{X} \to \mathfrak{P}_{cls}(\mathcal{Q})$ have the upper semi-continuity property. Then $\mathrm{Graph}(\Psi) \subseteq \mathcal{X} \times \mathcal{Q}$ is a closed set. On the other hand, assume that Ψ has a closed graph with the complete continuity property. Then Ψ has the upper semi-continuity property [33]. We say that $\Psi : [1, e] \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is a Caratheodory set-valued map if the mapping $\varrho \mapsto \Psi(t, \varrho)$ is upper semi-continuous for almost all $t \in [1, e]$ and the mapping $t \mapsto \Psi(t, \varrho)$ is measurable for each $\varrho \in \mathbb{R}$ [33, 34]. In addition, a Caratheodory set-valued map $\Psi : [1, e] \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is called $\mathcal{L}^1$-Caratheodory if for each $r > 0$ there is $\phi_r \in \mathcal{L}^1_{\mathbb{R}^+}([1, e])$ provided that

$$\big\|\Psi(t, \varrho)\big\| = \sup_{t \in [1,e]} \big\{ |q| : q \in \Psi(t, \varrho) \big\} \le \phi_r(t)$$

for almost all $t \in [1, e]$ and for each $|\varrho| \le r$ [33, 34]. All selections of Ψ at $\varrho \in C_{\mathbb{R}}([1, e])$ are defined by the following set:

$$(\mathcal{SEL})_{\Psi,\varrho} := \big\{ \vartheta \in \mathcal{L}^1_{\mathbb{R}}\big([1, e]\big) : \vartheta(t) \in \Psi\big(t, \varrho(t)\big), \text{ a.e. } t \in [1, e] \big\}$$

[33, 34]. As it has been verified before in [33], we have $(\mathcal{SEL})_{\Psi,\varrho} \neq \emptyset$ for all $\varrho \in C_{\mathcal{X}}([1, e])$ whenever $\dim \mathcal{X} < \infty$. We need next results.

Theorem 1 ([35]) *Consider the Banach algebra $\mathcal{X}$. For all $\rho \in \mathbb{R}^+$, consider the open ball $\mathcal{V}_\rho(0)$ and its closure $\overline{\mathcal{V}}_\rho(0)$. Assume that $\Phi_1 : \mathcal{X} \to \mathcal{X}$ and $\Phi_2 : \overline{\mathcal{V}}_\rho(0) \to \mathcal{X}$ are two operators satisfying*:

(i) *Φ_1 is Lipschitzian so that l^* is a Lipschitz constant,*
(ii) *Φ_2 is completely continuous,*
(iii) $l^* \hat{\Delta} < 1$, *where* $\hat{\Delta} = \|\Phi_2(\overline{\mathcal{V}}_\rho(0))\|_{\mathcal{X}} = \sup\{\|\Phi_2 k\|_{\mathcal{X}} : k \in \overline{\mathcal{V}}_\rho(0)\}$.

Then either ($a1$) *the operator equation* $\Phi_1 k \Phi_2 k = k$ *has a solution belonging to* $\overline{\mathcal{V}}_\rho(0)$ *or* ($a2$) *there exists* $v^* \in \mathcal{X}$ *with* $\|v^*\|_{\mathcal{X}} = \rho$ *so that* $\alpha_0 \Phi_1 v^* \Phi_2 v^* = v^*$ *for some* $\alpha_0 \in (0,1)$.

Theorem 2 ([36]) *Consider the separable Banach space* $\mathcal{X}$, *an* $\mathcal{L}^1$-*Carathéodory set-valued map* $\Psi : [1,e] \times \mathcal{X} \to \mathfrak{P}_{cmp,cvx}(\mathcal{X})$, *and the linear continuous map* $\Xi : \mathcal{L}^1_{\mathcal{X}}([1,e]) \to C_{\mathcal{X}}([1,e])$. *Then*

$$\Xi \circ (\mathcal{SEL})_\Psi : C_{\mathcal{X}}\big([1,e]\big) \to \mathfrak{P}_{cmp,cvx}\big(C_{\mathcal{X}}\big([1,e]\big)\big)$$

is an operator which belongs to $C_{\mathcal{X}}([1,e]) \times C_{\mathcal{X}}([1,e])$ *defined by* $\varrho \mapsto (\Xi \circ (\mathcal{SEL})_\Psi)(\varrho) = \Xi((\mathcal{SEL})_{\Psi,\varrho})$ *having the closed graph.*

Theorem 3 ([37]) *Consider the Banach algebra* $\mathcal{X}$. *Assume that there are a set-valued map* $\Phi_2 : \mathcal{X} \to \mathfrak{P}_{cmp,cvx}(\mathcal{X})$ *and a single-valued map* $\Phi_1 : \mathcal{X} \to \mathcal{X}$ *satisfying*:

(i) Φ_1 *is Lipschitzian where* l^* *is Lipschitz constant,*

(ii) Φ_2 *is compact and upper semi-continuous,*

(iii) $2l^* \hat{\Delta} < 1$ *with* $\hat{\Delta} = \|\Phi_2(\mathcal{X})\|$.

Then either ($a'1$) *there is a solution belonging to* $\mathcal{X}$ *for the inclusion* $k \in \Phi_1 k \Phi_2 k$ *or* ($a'2$) $\mathcal{O}^* = \{v^* \in \mathcal{X} | \alpha_0 v^* \in \Phi_1 v^* \Phi_2 v^*, \alpha_0 > 1\}$ *is an unbounded set.*

3 Main results

In this part of the paper, we intend to state our main theoretical findings on the existence results. To reach this aim, we consider $\mathcal{X} = \{\varrho(t) : \varrho(t) \in C_{\mathbb{R}}([1,e])\}$ equipped with the supremum norm $\|\varrho\|_{\mathcal{X}} = \sup_{t\in[1,e]} |\varrho(t)|$ and the multiplication action on the space $\mathcal{X}$ defined by $(\varrho \cdot \varrho')(t) = \varrho(t)\varrho'(t)$ for all $\varrho, \varrho' \in \mathcal{X}$. Then an ordered triple $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}, \cdot)$ is a Banach algebra. In this moment, we present an essential lemma which converts fractional BVP (1)–(2) into integral equation.

Lemma 4 *Assume that* $\breve{\alpha}$ *belongs to* $\mathcal{X}$. *Then* ϱ_0 *is a solution for the hybrid Caputo–Hadamard equation*

$$^{CH}\mathcal{D}^{\gamma}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right) = \breve{\alpha}(t), \quad (t \in [1,e], \gamma \in (2,3]) \tag{5}$$

furnished with hybrid Hadamard integral boundary value conditions

$$\begin{aligned} \left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=1} &= {}^{CH}\mathcal{D}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=1}, \\ {}^{CH}\mathcal{D}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=e} &= {}^{CH}\mathcal{D}^2_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=e}, \\ {}^{H}\mathcal{I}^{\mu}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=e} &= \frac{1}{\Gamma(\mu)}\int_1^e \left(\ln\frac{e}{s}\right)^{\mu-1}\left(\frac{\varrho(s)}{\Lambda(s,\varrho(s))}\right)\frac{\mathrm{d}s}{s} = 0 \end{aligned} \tag{6}$$

iff the function ϱ_0 *is a solution for the following Hadamard integral equation*:

$$\varrho(t) = \Lambda\big(t,\varrho(t)\big)\left[\frac{1}{\Gamma(\gamma)}\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}\right.$$

$$
\begin{aligned}
&+\frac{(2+\mu)^2(\ln t)^2-2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-2}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}\\
&+\frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}\\
&-\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}\Bigg].
\end{aligned}
\tag{7}
$$

Proof Let ϱ_0 be a solution for hybrid equation (5). Then the general solution of homogeneous equation (5) is obtained by the equality $\frac{\varrho_0(t)}{\Lambda(t,\varrho_0(t))} = {}^{H}\mathcal{I}^{\gamma}_{1^+}\breve{\alpha}(t)+\breve{m}^*_0+\breve{m}^*_1(\ln t)+\breve{m}^*_2(\ln t)^2$ where $\breve{m}^*_0,\breve{m}^*_1,\breve{m}^*_2\in\mathbb{R}$. That is,

$$
\varrho_0(t)=\Lambda\big(t,\varrho_0(t)\big)\left[\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}+\breve{m}^*_0+\breve{m}^*_1(\ln t)+\breve{m}^*_2(\ln t)^2\right].
\tag{8}
$$

Now, we employ the following integro-derivative operators of arbitrary orders on both sides of equation (8), and we get

$$
\begin{aligned}
{}^{CH}\mathcal{D}_{1^+}\left(\frac{\varrho_0(t)}{\Lambda(t,\varrho_0(t))}\right)&=\frac{1}{\Gamma(\gamma-1)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-2}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}+\breve{m}^*_1+2\breve{m}^*_2(\ln t),\\
{}^{CH}\mathcal{D}^2_{1^+}\left(\frac{\varrho_0(t)}{\Lambda(t,\varrho_0(t))}\right)&=\frac{1}{\Gamma(\gamma-2)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-3}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}+2\breve{m}^*_2,\\
{}^{H}\mathcal{I}^{\mu}_{1^+}\left(\frac{\varrho_0(t)}{\Lambda(t,\varrho_0(t))}\right)&=\frac{1}{\Gamma(\gamma+\mu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma+\mu-1}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}+\breve{m}^*_0\frac{(\ln t)^{\mu}}{\Gamma(1+\mu)}\\
&\quad+\breve{m}^*_1\frac{(\ln t)^{\mu+1}}{\Gamma(2+\mu)}+\breve{m}^*_2\frac{2(\ln t)^{\mu+2}}{\Gamma(3+\mu)}.
\end{aligned}
$$

Corresponding to the boundary value conditions, we obtain

$$
\breve{m}^*_0=\breve{m}^*_1=\frac{1}{\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}-\frac{1}{\Gamma(\gamma-1)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-2}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}
$$

and

$$
\begin{aligned}
\breve{m}^*_2&=\frac{(2+\mu)^2}{2\Gamma(\gamma-1)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-2}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}-\frac{(2+\mu)^2}{2\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}\\
&\quad-\frac{\Gamma(3+\mu)}{2\Gamma(\gamma+\mu)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

By inserting the values $\breve{m}^*_0$, $\breve{m}^*_1$, and $\breve{m}^*_2$ into (8), we get

$$
\begin{aligned}
\varrho_0(t)=\Lambda\big(t,\varrho_0(t)\big)\Bigg[&\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}\\
&+\frac{(2+\mu)^2(\ln t)^2-2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-2}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}\\
&+\frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}\breve{\alpha}(s)\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$- \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma+\mu-1} \breve{\alpha}(s) \frac{ds}{s} \Bigg].$$

This means that ϱ_0 is a solution for integral equation (7). On the contrary, it is easy to check that ϱ_0 satisfies fractional hybrid BVP (5)–(6) if ϱ_0 is a solution for the integral equation of fractional order (7). □

Now, we derive our first result about the existence of solutions of problem (1)–(2).

Theorem 5 *Suppose that Λ is a nonzero continuous real-valued function on $[1,e] \times \mathbb{R}$ and $\Theta \in C_{\mathbb{R}}([1,e] \times \mathbb{R})$. Furthermore, assume that the following statements hold:*

($\mathcal{C}1$) *There exists a bounded real-valued map $\theta : [1,e] \to \mathbb{R}^+$ so that, for all $\varrho_1, \varrho_2 \in \mathbb{R}$, we have $|\Lambda(t,\varrho_1) - \Lambda(t,\varrho_2)| \leq \theta(t)|\varrho_1 - \varrho_2|$;*

($\mathcal{C}2$) *There exist a continuous function $\psi : [1,e] \to \mathbb{R}^+$ and a continuous nondecreasing map $\xi : [0,\infty) \to (0,\infty)$ provided that $|\Theta(t,\varrho)| \leq \psi(t)\xi(\|\varrho\|)$ for $t \in [1,e]$ and for any $\varrho \in \mathbb{R}$;*

($\mathcal{C}3$) *There exists a number $\rho \in \mathbb{R}^+$ so that*

$$\rho > \frac{\Lambda^* \tilde{M} \psi^* \xi(\|\varrho\|)}{1 - \theta^* \tilde{M} \psi^* \xi(\|\varrho\|)}, \tag{9}$$

where $\Lambda^* = \sup_{t\in[1,e]} |\Lambda(t,0)|$, $\psi^* = \sup_{t\in[1,e]} |\psi(t)|$, $\theta^* = \sup_{t\in[1,e]} |\theta(t)|$, and

$$\tilde{M} = \frac{1}{\Gamma(\gamma+1)} + \frac{(2+\mu)^2 + 4}{2\Gamma(\gamma)} + \frac{4 + (2+\mu)^2}{2\Gamma(\gamma-1)} + \frac{\Gamma(3+\mu)}{2\Gamma(\gamma+\mu+1)}. \tag{10}$$

If $\theta^* \tilde{M} \psi^* \xi(\|\varrho\|) < 1$, then hybrid BVP (1)–(2) has a solution on $[1,e]$.

Proof Construct the closed ball $\overline{\mathcal{V}}_\rho(0) := \{\varrho(t) \in \mathcal{X} : \|\varrho\|_{\mathcal{X}} \leq \rho\}$, where ρ satisfies (9). In view of Lemma 4, we define operators $\Phi_1, \Phi_2 : \overline{\mathcal{V}}_\rho(0) \to \mathcal{X}$ by $(\Phi_1\varrho)(t) = \Lambda(t,\varrho(t))$ and

$$\begin{aligned}
(\Phi_2\varrho)(t) = {} & \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} \Theta\big(s,\varrho(s)\big) \frac{ds}{s} \\
& + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma-2} \Theta\big(s,\varrho(s)\big) \frac{ds}{s} \\
& + \frac{2(1+\ln t) - (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma-3} \Theta\big(s,\varrho(s)\big) \frac{ds}{s} \\
& - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma+\mu-1} \Theta\big(s,\varrho(s)\big) \frac{ds}{s}.
\end{aligned}$$

Obviously, $\varrho \in \mathcal{X}$ as a solution for hybrid BVP (1)–(2) satisfies equation $\Phi_1\varrho\Phi_2\varrho = \varrho$. By considering the assumptions of Theorem 1, we prove that such a solution function exists. First, we want to show that Φ_1 is Lipschitzian with constant $\theta^* = \sup_{t\in[1,e]} |\theta(t)|$. Let $\varrho_1, \varrho_2 \in \overline{\mathcal{V}}_\rho(0)$. Hypothesis ($\mathcal{C}1$) yields

$$\big|(\Phi_1\varrho_1)(t) - (\Phi_1\varrho_2)(t)\big| = \big|\Lambda\big(t,\varrho_1(t)\big) - \Lambda\big(t,\varrho_2(t)\big)\big| \leq \theta(t)\big|\varrho_1(t) - \varrho_2(t)\big|$$

for any $t \in [1,e]$. Hence, we get $\|\Phi_1\varrho_1 - \Phi_1\varrho_2\|_{\mathcal{X}} \leq \theta^* \|\varrho_1 - \varrho_2\|_{\mathcal{X}}$ for every $\varrho_1, \varrho_2 \in \overline{\mathcal{V}}_\rho(0)$. This means that the operator Φ_1 is Lipschitzian with constant θ^*. Now, we establish the complete continuity of the operator Φ_2 on $\overline{\mathcal{V}}_\rho(0)$. We first need to check that Φ_2 is continuous on $\overline{\mathcal{V}}_\rho(0)$. Let $\{\varrho_n\}$ be a convergent sequence belonging to $\overline{\mathcal{V}}_\rho(0)$ so that $\varrho_n \to \varrho$, where $\varrho \in \overline{\mathcal{V}}_\rho(0)$. Because of the continuity of the function Θ on $[1,e] \times \mathbb{R}$, we conclude that $\lim_{n\to\infty} \Theta(t, \varrho_n(t)) = \Theta(t, \varrho(t))$. By utilizing the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} (\Phi_2\varrho_n)(t) &= \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} \lim_{n\to\infty} \Theta\bigl(s, \varrho_n(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2} \lim_{n\to\infty} \Theta\bigl(s, \varrho_n(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{2(1+\ln t) - (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} \lim_{n\to\infty} \Theta\bigl(s, \varrho_n(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\quad - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \lim_{n\to\infty} \Theta\bigl(s, \varrho_n(s)\bigr) \frac{\mathrm{d}s}{s} \\
&= \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} \Theta\bigl(s, \varrho(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2} \Theta\bigl(s, \varrho(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{2(1+\ln t) - (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} \Theta\bigl(s, \varrho(s)\bigr) \frac{\mathrm{d}s}{s} \\
&\quad - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \Theta\bigl(s, \varrho(s)\bigr) \frac{\mathrm{d}s}{s} \\
&= (\Phi_2\varrho)(t)
\end{aligned}$$

for any $t \in [1,e]$. Therefore, $\Phi_2\varrho_n \to \Phi_2\varrho$ as $n \to \infty$ and thus Φ_2 is continuous on $\overline{\mathcal{V}}_\rho(0)$. In the sequel, we must prove that Φ_2 is uniformly bounded on $\overline{\mathcal{V}}_\rho(0)$. To do this, let $\varrho \in \overline{\mathcal{V}}_\rho(0)$. In view of assumption $(\mathcal{C}2)$, we have

$$\begin{aligned}
\bigl|(\Phi_2\varrho)(t)\bigr| &\leq \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} \bigl|\Theta\bigl(s, \varrho(s)\bigr)\bigr| \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(2+\mu)^2(\ln t)^2 + 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2} \bigl|\Theta\bigl(s, \varrho(s)\bigr)\bigr| \frac{\mathrm{d}s}{s} \\
&\quad + \frac{2(1+\ln t) + (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} \bigl|\Theta\bigl(s, \varrho(s)\bigr)\bigr| \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \bigl|\Theta\bigl(s, \varrho(s)\bigr)\bigr| \frac{\mathrm{d}s}{s} \\
&\leq \frac{(\ln t)^\gamma}{\Gamma(\gamma+1)} \psi(t)\xi\bigl(\|\varrho\|\bigr) + \frac{(2+\mu)^2(\ln t)^2 + 2(1+\ln t)}{2\Gamma(\gamma)} \psi(t)\xi\bigl(\|\varrho\|\bigr) \\
&\quad + \frac{2(1+\ln t) + (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-1)} \psi(t)\xi\bigl(\|\varrho\|\bigr)
\end{aligned}$$

$$+\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu+1)}\psi(t)\xi\big(\|\varrho\|\big)$$

for each $t\in[1,e]$. Hence $\|\Phi_2\varrho\|_{\mathcal{X}}\leq\psi^*\xi(\|\varrho\|)\tilde{M}$, where $\tilde{M}$ is represented in (10). This implies that $\Phi_2(\overline{\mathcal{V}}_\rho(0))$ is a uniformly bounded subset of $\mathcal{X}$. Moreover, we show that Φ_2 is equicontinuous. Let $t_1,t_2\in[1,e]$ so that $t_1<t_2$ and $\varrho\in\overline{\mathcal{V}}_\rho(0)$. Thus, we obtain

$$\begin{aligned}
&\big|(\Phi_2\varrho)(t_2)-(\Phi_2\varrho)(t_1)\big|\\
&=\frac{1}{\Gamma(\gamma)}\int_1^{t_1}\bigg[\bigg(\ln\frac{t_2}{s}\bigg)^{\gamma-1}-\bigg(\ln\frac{t_1}{s}\bigg)^{\gamma-1}\bigg]\big|\Theta\big(s,\varrho(s)\big)\big|\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}\bigg(\ln\frac{t_2}{s}\bigg)^{\gamma-1}\big|\Theta\big(s,\varrho(s)\big)\big|\frac{ds}{s}\\
&\quad+\frac{(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]+2[\ln t_2-\ln t_1]}{2\Gamma(\gamma-1)}\int_1^e\bigg(\ln\frac{e}{s}\bigg)^{\gamma-2}\big|\Theta\big(s,\varrho(s)\big)\big|\frac{ds}{s}\\
&\quad+\frac{2[\ln t_2-\ln t_1]+(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma-2)}\int_1^e\bigg(\ln\frac{e}{s}\bigg)^{\gamma-3}\big|\Theta\big(s,\varrho(s)\big)\big|\frac{ds}{s}\\
&\quad+\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma+\mu)}\int_1^e\bigg(\ln\frac{e}{s}\bigg)^{\gamma+\mu-1}\big|\Theta\big(s,\varrho(s)\big)\big|\frac{ds}{s}\\
&\leq\frac{1}{\Gamma(\gamma)}\int_1^{t_1}\bigg[\bigg(\ln\frac{t_2}{s}\bigg)^{\gamma-1}-\bigg(\ln\frac{t_1}{s}\bigg)^{\gamma-1}\bigg]\psi^*\xi\big(\|\varrho\|\big)\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}\bigg(\ln\frac{t_2}{s}\bigg)^{\gamma-1}\psi^*\xi\big(\|\varrho\|\big)\frac{ds}{s}\\
&\quad+\frac{(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]+2[\ln t_2-\ln t_1]}{2\Gamma(\gamma-1)}\int_1^e\bigg(\ln\frac{e}{s}\bigg)^{\gamma-2}\psi^*\xi\big(\|\varrho\|\big)\frac{ds}{s}\\
&\quad+\frac{2[\ln t_2-\ln t_1]+(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma-2)}\int_1^e\bigg(\ln\frac{e}{s}\bigg)^{\gamma-3}\psi^*\xi\big(\|\varrho\|\big)\frac{ds}{s}\\
&\quad+\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma+\mu)}\int_1^e\bigg(\ln\frac{e}{s}\bigg)^{\gamma+\mu-1}\psi^*\xi\big(\|\varrho\|\big)\frac{ds}{s}.
\end{aligned}$$

Hence, the RHS of the above inequalities tends to 0 free of $\varrho\in\overline{\mathcal{V}}_\rho(0)$ as $t_1\to t_2$. Thus, $|(\Phi_2\varrho)(t_2)-(\Phi_2\varrho)(t_1)|\to 0$ as $t_1\to t_2$ and so Φ_2 is equicontinuous. Therefore by utilizing the Arzela–Ascoli theorem, we find that Φ_2 is completely continuous on $\overline{\mathcal{V}}_\rho(0)$.

In the next step, by considering hypothesis $(\mathcal{C}3)$, we may write

$$\begin{aligned}
\hat{\Delta}&=\big\|\Phi_2\big(\overline{\mathcal{V}}_\rho(0)\big)\big\|_{\mathcal{X}}=\sup_{t\in[1,e]}\big\{\big|(\Phi_2\varrho)(t)\big|:\varrho\in\overline{\mathcal{V}}_\rho(0)\big\}\\
&=\psi^*\xi\big(\|\varrho\|\big)\bigg[\frac{1}{\Gamma(\gamma+1)}+\frac{(2+\mu)^2+4}{2\Gamma(\gamma)}+\frac{4+(2+\mu)^2}{2\Gamma(\gamma-1)}+\frac{\Gamma(3+\mu)}{2\Gamma(\gamma+\mu+1)}\bigg]\\
&=\psi^*\xi\big(\|\varrho\|\big)\tilde{M}.
\end{aligned}$$

Setting $l^*=\theta^*$, we get $\hat{\Delta}l^*<1$. Thus, one of conditions $(a1)$ or $(a2)$ in Theorem 1 holds. Let $\alpha_0\in(0,1)$. We claim that k satisfies the equation $\varrho=\alpha_0\Phi_1\varrho\Phi_2\varrho$. Hence, $\|\varrho\|=\rho$ and

$$\big|\varrho(t)\big|$$

$$
\begin{aligned}
&= \alpha_0 \left|(\Phi_1 \varrho)(t)\right| \left|(\Phi_2 \varrho)(t)\right| = \alpha_0 \left|\Lambda\left(t, \varrho(t)\right)\right| \\
&\quad \times \left| \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} \Theta\left(s, \varrho(s)\right) \frac{\mathrm{d}s}{s} \right. \\
&\quad + \frac{(2+\mu)^2 (\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma-2} \Theta\left(s, \varrho(s)\right) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{2(1+\ln t) - (2+\mu)^2 (\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma-3} \Theta\left(s, \varrho(s)\right) \frac{\mathrm{d}s}{s} \\
&\quad \left. - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma+\mu-1} \Theta\left(s, \varrho(s)\right) \frac{\mathrm{d}s}{s} \right| \\
&\leq \left(\left|\Lambda\left(t, \varrho(t)\right) - \Lambda(t,0)\right| + \left|\Lambda(t,0)\right| \right) \\
&\quad \times \left(\frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} \left|\Theta\left(s, \varrho(s)\right)\right| \frac{\mathrm{d}s}{s} \right. \\
&\quad + \frac{(2+\mu)^2 (\ln t)^2 + 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma-2} \left|\Theta\left(s, \varrho(s)\right)\right| \frac{\mathrm{d}s}{s} \\
&\quad + \frac{2(1+\ln t) + (2+\mu)^2 (\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma-3} \left|\Theta\left(s, \varrho(s)\right)\right| \frac{\mathrm{d}s}{s} \\
&\quad \left. + \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln \frac{e}{s}\right)^{\gamma+\mu-1} \left|\Theta\left(s, \varrho(s)\right)\right| \frac{\mathrm{d}s}{s} \right) \\
&\leq \left(\theta(t)\left|\varrho(t)\right| + \Lambda^*\right) \tilde{M} \psi(t) \xi\left(\|\varrho\|\right) \\
&\leq \left(\theta^* \|\varrho\| + \Lambda^*\right) \tilde{M} \psi^* \xi\left(\|\varrho\|\right).
\end{aligned}
$$

This yields $\rho \leq \frac{\Lambda^* \tilde{M} \psi^* \xi(\|\varrho\|)}{1-\theta^* \tilde{M} \psi^* \xi(\|\varrho\|)}$, which is impossible due to inequality (9). Hence, condition ($a2$) in Theorem 1 is not valid. Thus, condition ($a1$) in Theorem 1 holds and so hybrid BVP (1)–(2) has a solution. □

In what follows, we are going to provide another essential result for the fractional hybrid inclusion problem (3)–(4). Existence results herein are carried out in the light of the assumptions of Theorem 3.

Definition 6 We say that the function $\varrho \in AC_{\mathbb{R}}([1,e])$ is a solution for the hybrid inclusion BVP (3)–(4) whenever there exists an integrable function $\vartheta \in \mathcal{L}^1_{\mathbb{R}}([1,e])$ with $\vartheta(t) \in \Psi(t, \varrho(t))$ for almost all $t \in [1,e]$ satisfying

$$
\begin{aligned}
&\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=1} = {}^{CH}\mathcal{D}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=1}, \\
&{}^{CH}\mathcal{D}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=e} = {}^{CH}\mathcal{D}^2_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=e}, \\
&{}^{H}\mathcal{I}^{\mu}_{1^+}\left(\frac{\varrho(t)}{\Lambda(t,\varrho(t))}\right)\bigg|_{t=e} = \frac{1}{\Gamma(\mu)} \int_1^e \left(\ln \frac{e}{s}\right)^{\mu-1} \left(\frac{\varrho(s)}{\Lambda(s,\varrho(s))}\right) \frac{\mathrm{d}s}{s} = 0
\end{aligned}
$$

and

$$
\varrho(t) = \Lambda\left(t, \varrho(t)\right) \left[\frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} \vartheta(s) \frac{\mathrm{d}s}{s} \right.
$$

$$+\frac{(2+\mu)^2(\ln t)^2-2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-2}\vartheta(s)\frac{ds}{s}$$
$$+\frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}\vartheta(s)\frac{ds}{s}$$
$$-\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\vartheta(s)\frac{ds}{s}\Bigg]$$

for any $t \in [1,e]$.

Here, we can formulate the desired theorem on the existence of a solution function of the above form.

Theorem 7 *Assume that the following statements are valid.*

($\mathcal{C}4$) *There is a bounded real-valued function $\theta : [1,e] \to \mathbb{R}^+$ such that, for all $\varrho_1, \varrho_2 \in \mathcal{X}$ and $t \in [1,e]$, we have $|\Lambda(t,\varrho_1(t)) - \Lambda(t,\varrho_2(t))| \le \theta(t)|\varrho_1(t) - \varrho_2(t)|$.*

($\mathcal{C}5$) *The convex and compact-valued multifunction $\Psi : [1,e] \times \mathbb{R} \to \mathfrak{P}_{cmp,cvx}(\mathbb{R})$ is $\mathcal{L}^1$-Caratheodory.*

($\mathcal{C}6$) *There is a map $q(t) \in \mathcal{L}^1([1,e],\mathbb{R}^+)$ such that*

$$\|\Psi(t,\varrho)\| = \sup\{|\vartheta| : \vartheta \in \Psi(t,\varrho(t))\} \le q(t)$$

for any $\varrho \in \mathcal{X}$ and almost all $t \in [1,e]$. Here, $\|q\|_{\mathcal{L}^1} = \int_1^e |q(s)|\,ds$.

($\mathcal{C}7$) *There is a number $\tilde{\rho} \in \mathbb{R}^+$ so that*

$$\tilde{\rho} > \frac{\Lambda^*\tilde{M}\|q\|_{\mathcal{L}^1}}{1-\theta^*\tilde{M}\|q\|_{\mathcal{L}^1}}, \tag{11}$$

where $\Lambda^* = \sup_{t\in[1,e]}|\Lambda(t,0)|$, $\theta^* = \sup_{t\in[1,e]}|\theta(t)|$, and $\tilde{M}$ is illustrated by (10).

In this case, the hybrid inclusion BVP (3)–(4) has a solution whenever

$$\theta^*\tilde{M}\|q\|_{\mathcal{L}^1} < \frac{1}{2}.$$

Proof To transform the hybrid inclusion BVP (3)–(4) into a corresponding fixed point problem, we formulate the set-valued map $\mathcal{G} : \mathcal{X} \to \mathfrak{P}(\mathcal{X})$ by

$$\mathcal{G}(\varrho) = \{g \in \mathcal{X} : g(t) = \kappa_1(t) \textit{forallt} \in [1,e]\},$$

where

$$\kappa_1(t) = \begin{cases} \Lambda(t,\varrho(t))[\frac{1}{\Gamma(\gamma)}\int_1^t(\ln\frac{t}{s})^{\gamma-1}\vartheta(s)\frac{ds}{s} \\ \quad + \frac{(2+\mu)^2(\ln t)^2-2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\vartheta(s)\frac{ds}{s} \\ \quad + \frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e(\ln\frac{e}{s})^{\gamma-3}\vartheta(s)\frac{ds}{s} \\ \quad - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e(\ln\frac{e}{s})^{\gamma+\mu-1}\vartheta(s)\frac{ds}{s}], \quad \vartheta \in (\mathcal{SEL})_{\Psi,\varrho}. \end{cases}$$

It is evident that each fixed point of $\mathcal{G}$ is a solution for the hybrid inclusion BVP (3)–(4). Now, we split the operator $\mathcal{G}$ into two parts as $\Phi_1 : \mathcal{X} \to \mathcal{X}$ and $\Phi_2 : \mathcal{X} \to \mathfrak{P}(\mathcal{X})$ given by

$(\Phi_1\varrho)(t) = \Lambda(t,\varrho(t))$ and $(\Phi_2\varrho)(t) = \{\zeta \in \mathcal{X} : \zeta(t) = \kappa_2(t)\}$, where

$$\kappa_2(t) = \begin{cases} \frac{1}{\Gamma(\gamma)}\int_1^t (\ln\frac{t}{s})^{\gamma-1}\vartheta(s)\frac{ds}{s} \\ \quad + \frac{(2+\mu)^2(\ln t)^2-2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e (\ln\frac{e}{s})^{\gamma-2}\vartheta(s)\frac{ds}{s} \\ \quad + \frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e (\ln\frac{e}{s})^{\gamma-3}\vartheta(s)\frac{ds}{s} \\ \quad - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e (\ln\frac{e}{s})^{\gamma+\mu-1}\vartheta(s)\frac{ds}{s}, \quad \vartheta \in (\mathcal{SEL})_{\Psi,\varrho} \end{cases}$$

for all $t \in [1,e]$, respectively. Thus, we have $\mathcal{G}(\varrho) = \Phi_1\varrho\Phi_2\varrho$. In this moment, we must show that Φ_1 and Φ_2 satisfy all the hypotheses of Theorem 3. In view of assumption $(\mathcal{C}4)$ and by a similar deduction in Theorem 5, one can easily verify that Φ_1 is Lipschitzian. Now, we check that Φ_2 is convex-valued. Let $\varrho_1, \varrho_2 \in \Phi_2\varrho$. Select $\vartheta_1, \vartheta_2 \in (\mathcal{SEL})_{\Psi,\varrho}$ provided that

$$\begin{aligned} \varrho_j(t) &= \frac{1}{\Gamma(\gamma)}\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}\vartheta_j(s)\frac{ds}{s} \\ &\quad + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2}\vartheta_j(s)\frac{ds}{s} \\ &\quad + \frac{2(1+\ln t) - (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3}\vartheta_j(s)\frac{ds}{s} \\ &\quad - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\vartheta_j(s)\frac{ds}{s}, \quad (j=1,2) \end{aligned}$$

for almost all $t \in [1,e]$. Let $\lambda \in (0,1)$. Then one can write

$$\begin{aligned} &\lambda\varrho_1(t) + (1-\lambda)\varrho_2(t) \\ &\quad = \frac{1}{\Gamma(\gamma)}\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}\left[\lambda\vartheta_1(s) + (1-\lambda)\vartheta_2(s)\right]\frac{ds}{s} \\ &\qquad + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2}\left[\lambda\vartheta_1(s) + (1-\lambda)\vartheta_2(s)\right]\frac{ds}{s} \\ &\qquad + \frac{2(1+\ln t) - (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3}\left[\lambda\vartheta_1(s) + (1-\lambda)\vartheta_2(s)\right]\frac{ds}{s} \\ &\qquad - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\left[\lambda\vartheta_1(s) + (1-\lambda)\vartheta_2(s)\right]\frac{ds}{s} \end{aligned}$$

for almost all $t \in [1,e]$. As Ψ has convex values, so $(\mathcal{SEL})_{\Psi,\varrho}$ is convex-valued. This yields $\lambda\vartheta_1(t) + (1-\lambda)\vartheta_2(t) \in (\mathcal{SEL})_{\Psi,\varrho}$ for any $t \in [1,e]$, and so $\Phi_2\varrho$ is a convex set for each $\varrho \in \mathcal{X}$.

To confirm the complete continuity of Φ_2, we need to verify the equicontinuity and uniform boundedness of $\Phi_2(\mathcal{X})$. For this reason, we first check that Φ_2 maps all bounded sets into bounded subsets of $\mathcal{X}$. For a number $\rho^* \in \mathbb{R}^+$, construct the bounded ball $\mathcal{V}_{\rho^*} = \{\varrho \in \mathcal{X} : \|\varrho\|_{\mathcal{X}} \le \rho^*\}$. For each $\varrho \in \mathcal{V}_{\rho^*}$ and $\zeta \in \Phi_2\varrho$, there is a function $\vartheta \in (\mathcal{SEL})_{\Psi,\varrho}$ provided that

$$\begin{aligned} \zeta(t) &= \frac{1}{\Gamma(\gamma)}\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}\vartheta(s)\frac{ds}{s} \\ &\quad + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2}\vartheta(s)\frac{ds}{s} \end{aligned}$$

$$+\frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}\vartheta(s)\frac{ds}{s}$$
$$-\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\vartheta(s)\frac{ds}{s}$$

for each $t\in[1,e]$. Then we get

$$\begin{aligned}
\left|\zeta(t)\right| &\le \frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\left|\vartheta(s)\right|\frac{ds}{s}\\
&\quad+\frac{(2+\mu)^2(\ln t)^2+2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-2}\left|\vartheta(s)\right|\frac{ds}{s}\\
&\quad+\frac{2(1+\ln t)+(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}\left|\vartheta(s)\right|\frac{ds}{s}\\
&\quad+\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\left|\vartheta(s)\right|\frac{ds}{s}\\
&\le \frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}q(s)\frac{ds}{s}\\
&\quad+\frac{(2+\mu)^2(\ln t)^2+2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-2}q(s)\frac{ds}{s}\\
&\quad+\frac{2(1+\ln t)+(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}q(s)\frac{ds}{s}\\
&\quad+\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}q(s)\frac{ds}{s}\\
&\le\left[\frac{1}{\Gamma(\gamma+1)}+\frac{(2+\mu)^2+4}{2\Gamma(\gamma)}+\frac{4+(2+\mu)^2}{2\Gamma(\gamma-1)}+\frac{\Gamma(3+\mu)}{2\Gamma(\gamma+\mu+1)}\right]\|q\|_{\mathcal{L}^1}\\
&=\tilde{M}\|q\|_{\mathcal{L}^1},
\end{aligned}$$

where $\tilde{M}$ is illustrated by (10). Thus, $\|\zeta\|\le\tilde{M}\|q\|_{\mathcal{L}^1}$, and this implies that $\Phi_2(\mathcal{X})$ is uniformly bounded. In the sequel, we establish that the operator Φ_2 maps bounded sets into equicontinuous sets. Let $\varrho\in V_{\rho^*}$ and $\zeta\in\Phi_2\varrho$. We select $\vartheta\in(\mathcal{SEL})_{\Psi,\varrho}$ so that

$$\begin{aligned}
\zeta(t)&=\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\vartheta(s)\frac{ds}{s}\\
&\quad+\frac{(2+\mu)^2(\ln t)^2-2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-2}\vartheta(s)\frac{ds}{s}\\
&\quad+\frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma-3}\vartheta(s)\frac{ds}{s}\\
&\quad-\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\vartheta(s)\frac{ds}{s}
\end{aligned}$$

for each $t\in[1,e]$. Let $t_1,t_2\in[1,e]$ so that $t_1<t_2$. Then we write

$$\left|\zeta(t_2)-\zeta(t_1)\right|$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\gamma)}\int_1^{t_1}\left[\left(\ln\frac{t_2}{s}\right)^{\gamma-1}-\left(\ln\frac{t_1}{s}\right)^{\gamma-1}\right]\left|\vartheta(s)\right|\frac{\mathrm{d}s}{s}\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\gamma-1}\left|\vartheta(s)\right|\frac{\mathrm{d}s}{s}\\
&\quad+\frac{(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]+2[\ln t_2-\ln t_1]}{2\Gamma(\gamma-1)}\int_1^{e}\left(\ln\frac{e}{s}\right)^{\gamma-2}\left|\vartheta(s)\right|\frac{\mathrm{d}s}{s}\\
&\quad+\frac{2[\ln t_2-\ln t_1]+(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma-2)}\int_1^{e}\left(\ln\frac{e}{s}\right)^{\gamma-3}\left|\vartheta(s)\right|\frac{\mathrm{d}s}{s}\\
&\quad+\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma+\mu)}\int_1^{e}\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\left|\vartheta(s)\right|\frac{\mathrm{d}s}{s}\\
&\leq \frac{1}{\Gamma(\gamma)}\int_1^{t_1}\left[\left(\ln\frac{t_2}{s}\right)^{\gamma-1}-\left(\ln\frac{t_1}{s}\right)^{\gamma-1}\right]q(s)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\gamma-1}q(s)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]+2[\ln t_2-\ln t_1]}{2\Gamma(\gamma-1)}\int_1^{e}\left(\ln\frac{e}{s}\right)^{\gamma-2}q(s)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{2[\ln t_2-\ln t_1]+(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma-2)}\int_1^{e}\left(\ln\frac{e}{s}\right)^{\gamma-3}q(s)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma+\mu)}\int_1^{e}\left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}q(s)\frac{\mathrm{d}s}{s}\\
&\leq \|q\|_{\mathcal{L}^1}\left[\frac{2}{\Gamma(\gamma+1)}\left(\ln\left(\frac{t_2}{t_1}\right)\right)^{\gamma}+\frac{1}{\Gamma(\gamma+1)}\left|(\ln t_2)^{\gamma}-(\ln t_1)^{\gamma}\right|\right.\\
&\quad+\frac{(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]+2[\ln t_2-\ln t_1]}{2\Gamma(\gamma)}\\
&\quad+\frac{2[\ln t_2-\ln t_1]+(2+\mu)^2[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma-1)}\\
&\quad\left.+\frac{(2+\mu)(1+\mu)\Gamma(1+\mu)[(\ln t_2)^2-(\ln t_1)^2]}{2\Gamma(\gamma+\mu+1)}\right].
\end{aligned}
$$

Notice that the RHS of inequalities converges to 0 free of $\varrho \in \mathcal{V}_{\rho^*}$ letting $t_1 \to t_2$. With due attention to the Arzela–Ascoli theorem, we realize that $\Phi_2 : C_{\mathbb{R}}([1,e]) \to \mathfrak{P}(C_{\mathbb{R}}([1,e]))$ is completely continuous. Now, we intend to show that Φ_2 has a closed graph and this confirms the upper semi-continuity of Φ_2. To reach this goal, assume that $\varrho_n \in \mathcal{V}_{\rho^*}$ and $\zeta_n \in \Phi_2\varrho_n$ so that $\varrho_n \to \varrho^*$ and $\zeta_n \to \zeta^*$. We claim that $\zeta^* \in \Phi_2\varrho^*$. For each $n \geq 1$ and $\zeta_n \in \Phi_2\varrho_n$, we select $\vartheta_n \in (\mathcal{SEL})_{\Psi,\varrho_n}$ provided that

$$
\begin{aligned}
\zeta_n(t) &= \frac{1}{\Gamma(\gamma)}\int_1^{t}\left(\ln\frac{t}{s}\right)^{\gamma-1}\vartheta_n(s)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{(2+\mu)^2(\ln t)^2-2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^{e}\left(\ln\frac{e}{s}\right)^{\gamma-2}\vartheta_n(s)\frac{\mathrm{d}s}{s}\\
&\quad+\frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^{e}\left(\ln\frac{e}{s}\right)^{\gamma-3}\vartheta_n(s)\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$- \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \vartheta_n(s)\frac{ds}{s}$$

for any $t \in [1, e]$. It is enough to show that there is $\vartheta^* \in (\mathcal{SEL})_{\Psi,\varrho^*}$ so that

$$\begin{aligned}\zeta^*(t) = {} & \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} \vartheta^*(s)\frac{ds}{s} \\ & + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2} \vartheta^*(s)\frac{ds}{s} \\ & + \frac{2(1+\ln t) - (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} \vartheta^*(s)\frac{ds}{s} \\ & - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \vartheta^*(s)\frac{ds}{s}\end{aligned}$$

for any $t \in [1, e]$. Define the continuous linear operator $\Xi : \mathcal{L}^1_{\mathbb{R}}([1,e]) \to \mathcal{X} = C_{\mathbb{R}}([1,e])$ by

$$\begin{aligned}\Xi(\vartheta)(t) = \varrho(t) = {} & \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} \vartheta(s)\frac{ds}{s} \\ & + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2} \vartheta(s)\frac{ds}{s} \\ & + \frac{2(1+\ln t) - (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} \vartheta(s)\frac{ds}{s} \\ & - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \vartheta(s)\frac{ds}{s}\end{aligned}$$

for each $t \in [1, e]$. Hence,

$$\begin{aligned}\left\|\zeta_n(t) - \zeta^*(t)\right\| = {} & \bigg\| \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} \big(\vartheta_n(s) - \vartheta^*(s)\big)\frac{ds}{s} \\ & + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2} \big(\vartheta_n(s) - \vartheta^*(s)\big)\frac{ds}{s} \\ & + \frac{2(1+\ln t) - (2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} \big(\vartheta_n(s) - \vartheta^*(s)\big)\frac{ds}{s} \\ & - \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \big(\vartheta_n(s) - \vartheta^*(s)\big)\frac{ds}{s} \bigg\| \\ \to {} & 0\end{aligned}$$

letting $n \to \infty$. By applying Theorem 2, we deduce that $\Xi \circ (\mathcal{SEL})_\Psi$ has a closed graph. Since $\zeta_n \in \Xi((\mathcal{SEL})_{\Psi,\varrho_n})$ and $\varrho_n \to \varrho^*$, so there is $\vartheta^* \in (\mathcal{SEL})_{\Psi,\varrho^*}$ such that

$$\begin{aligned}\zeta^*(t) = {} & \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} \vartheta^*(s)\frac{ds}{s} \\ & + \frac{(2+\mu)^2(\ln t)^2 - 2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2} \vartheta^*(s)\frac{ds}{s}\end{aligned}$$

$$+ \frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} \vartheta^*(s)\frac{\mathrm{d}s}{s}$$
$$- \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \vartheta^*(s)\frac{\mathrm{d}s}{s}$$

for any $t \in [1,e]$. Therefore, $\zeta^* \in \Phi_2\varrho^*$ and so Φ_2 has a closed graph. This concludes that Φ_2 is upper semi-continuous. Since the operator Φ_2 has compact values, thus Φ_2 is compact and upper semi-continuous. In view of hypothesis $(\mathcal{C}6)$, we have

$$\begin{aligned}
\hat{\Delta} &= \left\|\Phi_2(\mathcal{X})\right\| = \sup_{t\in[1,e]} \left\{|\Phi_2\varrho| : \varrho \in \mathcal{X}\right\} \\
&= \left[\frac{1}{\Gamma(\gamma+1)} + \frac{(2+\mu)^2+4}{2\Gamma(\gamma)} + \frac{4+(2+\mu)^2}{2\Gamma(\gamma-1)} + \frac{\Gamma(3+\mu)}{2\Gamma(\gamma+\mu+1)}\right]\|q\|_{\mathcal{L}^1} \\
&= \tilde{M}\|q\|_{\mathcal{L}^1}.
\end{aligned}$$

Setting $l^* = \theta^*$, we get $\hat{\Delta} l^* < \frac{1}{2}$. Now, by applying Theorem 3 for Φ_2, we find that one of conditions $(a'1)$ or $(a'2)$ is valid. We claim that condition $(a'2)$ is invalid. By considering Theorem 3 and hypothesis $(\mathcal{C}7)$, assume that ϱ is an arbitrary element of $\mathcal{O}^*$ with $\|\varrho\| = \tilde{\rho}$. Then $\alpha_0\varrho(t) \in \Phi_1\varrho(t)\Phi_2\varrho(t)$ for each $\alpha_0 > 1$. Select $\vartheta \in (\mathcal{SEL})_{\Psi,\varrho}$. Then, for each $\alpha_0 > 1$, we have

$$\begin{aligned}
\varrho(t) = \frac{1}{\alpha_0}\Lambda\big(t,\varrho(t)\big)\Bigg[&\frac{1}{\Gamma(\gamma)}\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}\vartheta(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{(2+\mu)^2(\ln t)^2-2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2}\vartheta(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{2(1+\ln t)-(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3}\vartheta(s)\frac{\mathrm{d}s}{s} \\
&- \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\vartheta(s)\frac{\mathrm{d}s}{s}\Bigg]
\end{aligned}$$

for any $t \in [1,e]$. Thus, one can write

$$\begin{aligned}
\big|\varrho(t)\big| = \frac{1}{\alpha_0}\big|\Lambda\big(t,\varrho(t)\big)\big|\Bigg[&\frac{1}{\Gamma(\gamma)}\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}\big|\vartheta(s)\big|\frac{\mathrm{d}s}{s} \\
&+ \frac{(2+\mu)^2(\ln t)^2+2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2}\big|\vartheta(s)\big|\frac{\mathrm{d}s}{s} \\
&+ \frac{2(1+\ln t)+(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3}\big|\vartheta(s)\big|\frac{\mathrm{d}s}{s} \\
&+ \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1}\big|\vartheta(s)\big|\frac{\mathrm{d}s}{s}\Bigg] \\
= \Big[\big|\Lambda\big(t,k(t)\big)-\Lambda(t,0)\big| + \big|\Lambda(t,0)\big|\Big]\Bigg[&\frac{1}{\Gamma(\gamma)}\int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1}\big|\vartheta(s)\big|\frac{\mathrm{d}s}{s} \\
&+ \frac{(2+\mu)^2(\ln t)^2+2(1+\ln t)}{2\Gamma(\gamma-1)}\int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2}\big|\vartheta(s)\big|\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2(1+\ln t)+(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} \left|\vartheta(s)\right| \frac{\mathrm{d}s}{s} \\
&+ \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} \left|\vartheta(s)\right| \frac{\mathrm{d}s}{s}\Bigg] \\
\le{}& \left[\theta^*\|k\| + \Lambda^*\right]\Bigg[\frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln\frac{t}{s}\right)^{\gamma-1} q(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{(2+\mu)^2(\ln t)^2+2(1+\ln t)}{2\Gamma(\gamma-1)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-2} q(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{2(1+\ln t)+(2+\mu)^2(\ln t)^2}{2\Gamma(\gamma-2)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma-3} q(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{(2+\mu)(1+\mu)\Gamma(1+\mu)(\ln t)^2}{2\Gamma(\gamma+\mu)} \int_1^e \left(\ln\frac{e}{s}\right)^{\gamma+\mu-1} q(s)\frac{\mathrm{d}s}{s}\Bigg] \\
\le{}& \left[\theta^*\tilde{\rho} + \Lambda^*\right]\tilde{M}\|q\|_{\mathcal{L}^1}
\end{aligned}
$$

for any $t \in [1,e]$. Hence, $\tilde{\rho} \le \frac{\Lambda^*\tilde{M}\|q\|_{\mathcal{L}^1}}{1-\theta^*\tilde{M}\|q\|_{\mathcal{L}^1}}$. According to condition (11), we conclude that condition $(a'2)$ of Theorem 3 is not valid. Thus, $\varrho \in \Phi_1\varrho\Phi_2\varrho$. So, it is verified that $\mathcal{G}$ has a fixed point, and thus the hybrid inclusion BVP (3)–(4) has a solution. □

4 Examples

To demonstrate the consistency and applicability of the obtained results, two illustrative numerical examples are provided herein.

Example 1 Corresponding to the proposed hybrid BVP (1)–(2), we formulate the hybrid fractional Caputo–Hadamard differential equation

$$
{}^{CH}\mathcal{D}_{1^+}^{2.08}\left(\frac{\varrho(t)}{\frac{0.57(t+1)}{3}\frac{|\varrho(t)|}{3+|\varrho(t)|}+0.112}\right) = \frac{(t+1)^2\cos(\varrho(t))}{981} \quad \bigl(t\in[1,e]\bigr) \tag{12}
$$

endowed with the hybrid Hadamard integral boundary conditions

$$
\begin{cases}
(\frac{\varrho(t)}{\frac{0.57(t+1)}{3}\frac{|\varrho(t)|}{3+|\varrho(t)|}+0.112})|_{t=1} = {}^{CH}\mathcal{D}_{1^+}(\frac{\varrho(t)}{\frac{0.57(t+1)}{3}\frac{|\varrho(t)|}{3+|\varrho(t)|}+0.112})|_{t=1}, \\
{}^{CH}\mathcal{D}_{1^+}(\frac{\varrho(t)}{\frac{0.57(t+1)}{3}\frac{|\varrho(t)|}{3+|\varrho(t)|}+0.112})|_{t=e} = {}^{CH}\mathcal{D}_{1^+}^2(\frac{\varrho(t)}{\frac{0.57(t+1)}{3}\frac{|\varrho(t)|}{3+|\varrho(t)|}+0.112})|_{t=e}, \\
{}^{H}\mathcal{I}_{1^+}^{0.92}(\frac{\varrho(t)}{\frac{0.57(t+1)}{3}\frac{|\varrho(t)|}{3+|\varrho(t)|}+0.112})|_{t=e} \\
\quad = \frac{1}{\Gamma(0.92)}\int_1^e(\ln\frac{e}{s})^{0.92-1}(\frac{\varrho(s)}{\frac{0.57(s+1)}{3}\frac{|\varrho(s)|}{3+|\varrho(s)|}+0.112})\frac{\mathrm{d}s}{s} = 0,
\end{cases} \tag{13}
$$

so that $\gamma = 2.08$ and $\mu = 0.92$. Define the nonzero real-valued continuous map Λ on $[1,e]\times\mathbb{R}$ as follows: $\Lambda(t,\varrho(t)) = \frac{0.57(t+1)}{3}\frac{|\varrho(t)|}{3+|\varrho(t)|} + 0.112$ with $\Lambda^* = \sup_{t\in[1,e]}|\Lambda(t,0)| = 0.112$. Furthermore, define the continuous map $\Theta : [1,e]\times\mathbb{R}\to\mathbb{R}^+$ by $\Theta(t,\varrho(t)) = \frac{(t+1)^2\cos(\varrho(t))}{981}$. Now, put $\theta(t) = \frac{0.57(t+1)}{3}$ and $\psi(t) = \frac{(t+1)^2}{981}$. Then we get $\theta^* = \sup_{t\in[1,e]}|\theta(t)| \simeq 0.7049$, $\psi^* = \sup_{t\in[1,e]}|\psi(t)| = \frac{(e+1)^2}{981} \simeq 0.01403$, $\xi(\|\varrho\|) = 1$, and $\tilde{M} \simeq 13.4852$. Choose $\rho > 0.024449$. On the other hand, notice that $\theta^*\tilde{M}\psi^*\xi(\|\varrho\|) \simeq 0.1333 < 1$. Now, by utilizing Theorem 5, the hybrid fractional Caputo–Hadamard BVP (12)–(13) has a solution.

Example 2 Corresponding to the proposed hybrid BVP (3)–(4), we formulate the hybrid Caputo–Hadamard inclusion BVP:

$$ {}^{CH}\mathcal{D}_{1^+}^{2.35}\left(\frac{\varrho(t)}{\frac{t|\sin\varrho(t)|}{1200(1+|\sin\varrho(t)|)}+0.007}\right)\in\left[\frac{|\varrho(t)|}{1+|\varrho(t)|}+0.8,\frac{3|3\varrho(t)|}{8(1+|3\varrho(t)|)}+1.625\right] \tag{14}$$

furnished with the hybrid Hadamard integral boundary conditions

$$\begin{cases} (\frac{\varrho(t)}{\frac{t|\sin\varrho(t)|}{1200(1+|\sin\varrho(t)|)}+0.007})|_{t=1}={}^{CH}\mathcal{D}_{1^+}(\frac{\varrho(t)}{\frac{t|\sin\varrho(t)|}{1200(1+|\sin\varrho(t)|)}+0.007})|_{t=1}, \\ {}^{CH}\mathcal{D}_{1^+}(\frac{\varrho(t)}{\frac{t|\sin\varrho(t)|}{1200(1+|\sin\varrho(t)|)}+0.007})|_{t=e}={}^{CH}\mathcal{D}_{1^+}^{2}(\frac{\varrho(t)}{\frac{t|\sin\varrho(t)|}{1200(1+|\sin\varrho(t)|)}+0.007})|_{t=e}, \\ {}^{H}\mathcal{I}_{1^+}^{0.78}(\frac{\varrho(t)}{\frac{t|\sin\varrho(t)|}{1200(1+|\sin\varrho(t)|)}+0.007})|_{t=e} \\ \quad=\frac{1}{\Gamma(0.78)}\int_1^e(\ln\frac{e}{s})^{0.78-1}(\frac{\varrho(s)}{\frac{s|\sin\varrho(s)|}{1200(1+|\sin\varrho(s)|)}+0.007})\frac{\mathrm{d}s}{s}=0 \end{cases} \tag{15}$$

so that $t\in[1,e]$, $\gamma=2.35$, and $\mu=0.78$. Consider the nonzero real-valued continuous map Λ on $[1,e]\times\mathbb{R}$ given by $\Lambda(t,\varrho(t))=\frac{t|\sin\varrho(t)|}{1200(1+|\sin\varrho(t)|)}+0.007$ with $\Lambda^*=\sup_{t\in[1,e]}|\Lambda(t,0)|=0.007$. Define the set-valued map $\Psi:[1,e]\times\mathbb{R}\to\mathfrak{P}(\mathbb{R})$ by

$$\Psi\big(t,\varrho(t)\big)=\left[\frac{|\varrho(t)|}{1+|\varrho(t)|}+0.8,\frac{3|3\varrho(t)|}{8(1+|3\varrho(t)|)}+1.625\right].$$

If $\theta(t)=\frac{t}{1200}$, then $\theta^*=\sup_{t\in[1,e]}|\theta(t)|=\frac{e}{1200}\simeq 0.002258$. Since

$$|\zeta|\leq\max\left[\frac{|\varrho(t)|}{1+|\varrho(t)|}+0.8,\frac{3|3\varrho(t)|}{8(1+|3\varrho(t)|)}+1.625\right]\leq 2$$

for all $\zeta\in\Psi(t,\varrho(t))$, we get $\|\Psi(t,\varrho(t))\|=\sup\{|\vartheta|:\vartheta\in\Psi(t,\varrho(t))\}\leq 2$. Put $q(t)=2$ for any $t\in[1,e]$. Then $\|q\|_{\mathcal{L}^1}=\int_1^e|q(s)|\,\mathrm{d}s=2(e-1)\simeq 3.42$. Hence, we obtain $\tilde{M}\simeq 12.1327$. Now, select $\tilde{\rho}>0$ with $\tilde{\rho}>0.32035$. Then $\theta^*\tilde{M}\|q\|_{\mathcal{L}^1}\simeq 0.09336<\frac{1}{2}$. Now, by applying Theorem 7, the hybrid inclusion BVP (14)–(15) has a solution.

5 Conclusion

It is known that the most natural phenomena are modeled by different types of fractional differential equations and inclusions. This diversity in investigating complicate fractional differential equations and inclusions increases our ability for exact modelings of more phenomena. This is useful in making modern software which helps us to allow for more cost-free testing and less material consumption. In this work, we investigate the existence of solutions for a hybrid fractional Caputo–Hadamard differential equation and its related inclusion problem with hybrid Hadamard integral boundary value conditions. In this way, we use some Dhage's fixed point results in our proofs. Eventually, we give two numerical examples to support the applicability of our findings.

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