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Existence and multiplicity of positive solutions for a new class of singular higher-order fractional differential equations with Riemann–Stieltjes integral boundary value conditions

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Abstract

In this work, the aim is to discuss a new class of singular nonlinear higher-order fractional boundary value problems involving multiple Riemann–Liouville fractional derivatives. The boundary conditions are constituted by Riemann–Stieltjes integral boundary conditions. The existence and multiplicity of positive solutions are derived via employing the Guo–Krasnosel'skii fixed point theorem. In addition, the main results are demonstrated by some examples to show their validity.

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1 Introduction

We consider the existence and multiplicity of positive solutions for the following nonlinear singular higher-order fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t), D_{0+}^{\alpha_1} u(t), D_{0+}^{\alpha_2} u(t), \dots, D_{0+}^{\alpha_{n-2}} u(t)) = 0, & t \in (0, 1), \\ u(0) = D_{0+}^{\gamma_1} u(0) = D_{0+}^{\gamma_2} u(0) = \dots = D_{0+}^{\gamma_{n-2}} u(0) = 0, \\ D_{0+}^{\beta_1} u(1) = \int_0^{\eta} h(s) D_{0+}^{\beta_2} u(s) dA(s) + \int_0^1 a(s) D_{0+}^{\beta_3} u(s) dA(s), \end{cases} \quad (1.1)$$

where $D_{0+}^{\alpha} u$, $D_{0+}^{\alpha_k} u$, $D_{0+}^{\gamma_k} u$ ($k = 1, 2, \dots, n - 2$), $D_{0+}^{\beta_i} u$ ($i = 1, 2, 3$) are the standard Riemann–Liouville derivatives, and $n - 1 < \alpha \leq n$ ($n \geq 3$), $k - 1 < \alpha_k$, $\gamma_k \leq k$ ($k = 1, 2, \dots, n - 2$), $n - j - 1 < \alpha - \gamma_j \leq n - j$ ($j = 1, 2, \dots, n - 2$), $1 < \alpha - \alpha_{n-2} - 1 \leq 2$, $\gamma_{n-2} - \alpha_{n-2} \geq 0$, $\beta_1 \geq \beta_2$, $\beta_1 \geq \beta_3$, $\alpha - \beta_i \geq 1$, $\beta_i - \alpha_{n-2} - 1 \geq 0$ ($i = 1, 2, 3$), $\beta_1 \leq n - 1$, $f : (0, 1) \times \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}_+^1 = [0, +\infty)$ is continuous and $a, h \in C((0, 1), \mathbb{R}_+^1)$, A is a function of bounded variation, $\int_0^{\eta} h(s) D_{0+}^{\beta_2} u(s) dA(s)$, $\int_0^1 a(s) D_{0+}^{\beta_3} u(s) dA(s)$ denote the Riemann–Stieltjes integrals with respect to A .

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Fractional differential equations appear naturally in various fields of science and engineering. This is due to the fact that the differential equations of arbitrary order provide an excellent instrument for the description of memory and hereditary properties of various materials and processes and they have numerous applications in multifarious fields of science and engineering including physics, blood flow phenomena, rheology, diffusive transport akin to diffusion, electrical networks, probability, etc. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials. For some recent work on this branch of differential equations, see [1–4] and the references therein. Fractional-order differential equations have been addressed by several researchers with the sphere of study ranging from the theoretical aspects to the analytic and numerical methods for finding solutions (see [1–40] and the references therein). The existence of positive solutions for fractional differential equation boundary value problems has attracted much attention, and a great deal of results have been developed for differential and integral boundary value problems. For example, in [5], Ma and Yang studied the following higher-order boundary value problems with a sign-changing nonlinear term:

$$\begin{cases} u^{(n)}(t) + \lambda a(t)f(t, u, u', u'', u^{(3)}, \dots, u^{(n-2)}) = 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \leq i \leq n - 3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{cases}$$

where $n \geq 2$, $0 < \lambda, \alpha, \beta, \gamma$ and δ are constants satisfying $\alpha, \gamma > 0$ and $\beta, \delta \geq 0$, $f : [0, 1] \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^1 = (-\infty, +\infty)$ is continuous. Intervals of λ are determined to ensure the existence of a positive solution of the boundary value problem according to the signs of a and f . By using the Schauder fixed point theorem, the authors obtained the existence of a positive solution.

In [6], by means of the fixed point index theory, Zhang et al. investigated the existence of positive solutions for the fractional differential equation with integral boundary conditions:

$$\begin{cases} -D_t^\alpha x(t) = f(t, x(t), D_t^\beta x(t)), & t \in (0, 1), \\ D_t^\beta x(0) = 0, & D_t^\beta x(1) = \int_0^1 D_t^\beta x(s) dA(s), \end{cases}$$

where $0 < \beta \leq 1 < \alpha \leq 2$, $\alpha - \beta > 1$, $D_t^\alpha x, D_t^\beta x$ are the standard Riemann–Liouville derivatives, $\int_0^1 x(s) dA(s)$ denotes a Riemann–Stieltjes integral, A is a function of bounded variation, and dA can be a signed measure, $f : (0, 1) \times (0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}_+^1$ is continuous, $f(t, x, y)$ may be singular at both $t = 0, 1$ and $x = y = 0$.

In [7], via employing the topological degree theory, Zhang et al. investigated the existence of positive nontrivial solutions for the following nonlinear fractional differential equations with integral boundary conditions:

$$\begin{cases} D_{0^+}^\alpha u(t) + h(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & u(1) = \lambda \int_0^\eta u(s) ds, \end{cases}$$

where $3 < \alpha \leq 4$, $0 < \eta \leq 1$, $0 \leq \frac{\lambda\eta^\alpha}{\alpha} < 1$, $D_{0^+}^\alpha u$ is the Riemann–Liouville fractional derivative, $h : (0, 1) \rightarrow [0, \infty)$ is continuous, and $f : [0, 1] \times \mathbb{R}_+^1 \rightarrow \mathbb{R}_+^1$ is also continuous.

In [8], Graef et al. obtained a new upper estimate for the Green’s function associated with a higher-order fractional boundary value problem and applied these properties of $G(t, s)$ and the well-known Schauder fixed point theorem. Criteria for the existence of positive solutions of the following problem are then established:

$$\begin{cases} -D_{0^+}^\nu u(t) = g(t, u), & t \in (0, 1), \\ [D_{0^+}^\alpha u(t)]_{t=1} = 0, & u^{(j)}(0) = 0, \quad j = 0, 1, \dots, n - 2, \end{cases}$$

where $n \geq 3$, $n - 1 < \nu \leq n$, $1 \leq \alpha \leq n - 2$, $D_{0^+}^\nu$ is the standard Riemann–Liouville fractional derivative of order ν , the nonlinearity $g : [0, 1] \times \mathbb{R}_+^1 \rightarrow \mathbb{R}^1$ is a continuous function. For more research results, we refer the reader to [18, 19, 22, 24, 25, 27–38].

Inspired by the works illustrated above, we are committed to establishing the existence and multiplicity of positive solutions for the fractional differential equation boundary value problem (BVP for short) (1.1). The novelty of this article is as follows: Firstly, fractional derivatives are involved in the nonlinear terms and boundary conditions; what is new is that the orders of the fractional derivatives in the nonlinear terms and boundary conditions are different. Moreover, the orders of the fractional derivatives in the boundary conditions can be different, but up to now, there have been few papers dealing with this case where the Riemann–Stieltjes integral boundary conditions contain fractional derivative of different orders. Since fractional derivatives of different orders and high-order fractional derivatives are taken into account in BVP (1.1), it makes the research more complicated. In order to reduce the complexity, we need to use the reduced-order method for fractional differential equation and overcome the difficulties in finding the properties of Green’s function. Secondly, the boundary value conditions involving high-order fractional derivatives of unknown function are more general as they contain multi-point boundary conditions and integral boundary conditions [6, 7, 20, 28–30, 32] as special cases. Thirdly, the given conditions f_0, f_∞ and $(H_3), (H_5)$ are quite different from those in other papers such as [18–40] and are weaker and wider.

The remaining part of article is structured as follows. In Sect. 2, we present some preliminaries and lemmas which are required in later considerations. We also develop and prove some properties of Green’s function. In Sect. 3, we discuss the existence and multiplicity of positive solutions for BVP (1.1). In Sect. 4, two examples are presented to illustrate our fundamental results.

2 Preliminaries and lemmas

In this section, some notations and lemmas, which will be used in the proof of our main results, are stated. They can be found in the literature, see [2, 3, 21].

Definition 2.1 ([2, 3]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \rightarrow \mathbb{R}^1$ is given by

$$I_{0^+}^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([2, 3]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \rightarrow \mathbb{R}^1$ is given by

$$D_{0^+}^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{y(s)}{(t - s)^{\alpha - n + 1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α , provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([3]) *Let $\alpha > 0$. If we assume $u \in C(0, 1) \cap L^1(0, 1)$, then the fractional differential equation*

$$D_{0^+}^\alpha u(t) = 0$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}^1$ ($i = 1, 2, \dots, N$) as the unique solution, where $N = [\alpha] + 1$.

From the definition of the Riemann–Liouville derivative, we can obtain the following lemmas.

Lemma 2.2 ([3]) *Assume that $u \in C(0, 1) \cap L^1(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L^1(0, 1)$. Then*

$$I_{0^+}^\alpha D_{0^+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}^1$ ($i = 1, 2, \dots, N$), where $N = [\alpha] + 1$.

Lemma 2.3 ([3]) *If $x \in L^1(0, 1)$, $\alpha > \beta > 0$, then*

$$I_{0^+}^\alpha I_{0^+}^\beta x(t) = I_{0^+}^{\alpha+\beta} x(t), \quad D_{0^+}^\beta I_{0^+}^\alpha x(t) = I_{0^+}^{\alpha-\beta} x(t), \quad D_{0^+}^\beta I_{0^+}^\beta x(t) = x(t).$$

Lemma 2.4 (Auxiliary lemma) *Let $l_1 := \int_0^\eta h(t)t^{\alpha-\beta_2-1} dA(t) > 0$, $l_2 := \int_0^1 a(t)t^{\alpha-\beta_3-1} dA(t) > 0$. Given $y \in C(0, 1) \cap L^1(0, 1)$ and the following condition is satisfied:*

(H₁)

$$\begin{aligned} \Delta &= \frac{1}{\Gamma(\alpha - \beta_1)} - \frac{1}{\Gamma(\alpha - \beta_2)} l_1 - \frac{1}{\Gamma(\alpha - \beta_3)} l_2 > 0, \\ \frac{\Gamma(\alpha - \beta_1)}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t)t^{\alpha-\beta_2-1} dA(t) + \frac{\Gamma(\alpha - \beta_1)}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t)t^{\alpha-\beta_3-1} dA(t) &< 1. \end{aligned}$$

The unique solution of

$$\begin{cases} D_{0^+}^{\alpha-\alpha_{n-2}} x(t) + y(t) = 0, & 0 < t < 1, \\ D_{0^+}^{\gamma_{n-2}-\alpha_{n-2}} x(0) = 0, \\ D_{0^+}^{\beta_1-\alpha_{n-2}} x(1) = \int_0^\eta h(s)D_{0^+}^{\beta_2-\alpha_{n-2}} x(s) dA(s) + \int_0^1 a(s)D_{0^+}^{\beta_3-\alpha_{n-2}} x(s) dA(s) \end{cases} \tag{2.1}$$

is

$$x(t) = \int_0^1 G(t,s)y(s) ds, \quad t \in [0, 1], \tag{2.2}$$

where

$$\begin{aligned} G(t,s) &= H_1(t,s) + H_2(t,s) + H_3(t,s), \\ H_1(t,s) &= L_1(t,s), \\ H_2(t,s) &= \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-\beta_2)} \Delta^{-1} \int_0^\eta h(t)L_2(t,s) dA(t), \\ H_3(t,s) &= \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-\beta_3)} \Delta^{-1} \int_0^1 a(t)L_3(t,s) dA(t), \end{aligned}$$

in which

$$\begin{aligned} L_1(t,s) &= \frac{1}{\Gamma(\alpha-\alpha_{n-2})} \begin{cases} t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-\beta_1-1} - (t-s)^{\alpha-\alpha_{n-2}-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-\beta_1-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ L_2(t,s) &= \frac{1}{\Gamma(\alpha-\alpha_{n-2})} \begin{cases} t^{\alpha-\beta_2-1}(1-s)^{\alpha-\beta_1-1} - (t-s)^{\alpha-\beta_2-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\beta_2-1}(1-s)^{\alpha-\beta_1-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ L_3(t,s) &= \frac{1}{\Gamma(\alpha-\alpha_{n-2})} \begin{cases} t^{\alpha-\beta_3-1}(1-s)^{\alpha-\beta_1-1} - (t-s)^{\alpha-\beta_3-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\beta_3-1}(1-s)^{\alpha-\beta_1-1}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \tag{2.3}$$

Proof We may apply Lemma 2.2 to reduce (2.1) to an equivalent integral equation

$$x(t) = -I_{0^+}^{\alpha-\alpha_{n-2}}y(t) + d_1t^{\alpha-\alpha_{n-2}-1} + d_2t^{\alpha-\alpha_{n-2}-2}$$

for some $d_i \in \mathbb{R}^1$ ($i = 1, 2$). Consequently, the general solution of (2.1) is

$$x(t) = -\frac{1}{\Gamma(\alpha-\alpha_{n-2})} \int_0^t (t-s)^{\alpha-\alpha_{n-2}-1}y(s) ds + d_1t^{\alpha-\alpha_{n-2}-1} + d_2t^{\alpha-\alpha_{n-2}-2}. \tag{2.4}$$

By (2.4) and Lemma 2.3, we have

$$\begin{aligned} D_{0^+}^{\gamma_{n-2}-\alpha_{n-2}}x(t) &= -D_{0^+}^{\gamma_{n-2}-\alpha_{n-2}}I_{0^+}^{\alpha-\alpha_{n-2}}y(t) + d_1D_{0^+}^{\gamma_{n-2}-\alpha_{n-2}}t^{\alpha-\alpha_{n-2}-1} \\ &\quad + d_2D_{0^+}^{\gamma_{n-2}-\alpha_{n-2}}t^{\alpha-\alpha_{n-2}-2} \\ &= -I_{0^+}^{\alpha-\gamma_{n-2}}y(t) + d_1\frac{\Gamma(\alpha-\alpha_{n-2})}{\Gamma(\alpha-\gamma_{n-2})}t^{\alpha-\gamma_{n-2}-1} \\ &\quad + d_2\frac{\Gamma(\alpha-\alpha_{n-2}-1)}{\Gamma(\alpha-\gamma_{n-2}-1)}t^{\alpha-\gamma_{n-2}-2} \\ &= -\int_0^t \frac{(t-s)^{\alpha-\gamma_{n-2}-1}}{\Gamma(\alpha-\gamma_{n-2})}y(s) ds + d_1\frac{\Gamma(\alpha-\alpha_{n-2})}{\Gamma(\alpha-\gamma_{n-2})}t^{\alpha-\gamma_{n-2}-1} \\ &\quad + d_2\frac{\Gamma(\alpha-\alpha_{n-2}-1)}{\Gamma(\alpha-\gamma_{n-2}-1)}t^{\alpha-\gamma_{n-2}-2}. \end{aligned} \tag{2.5}$$

Considering the fact that $D_{0^+}^{\gamma_{n-2}-\alpha_{n-2}}x(0) = 0$, one gets that $d_2 = 0$ by (2.5). Then we obtain

$$x(t) = -\frac{1}{\Gamma(\alpha - \alpha_{n-2})} \int_0^t (t-s)^{\alpha-\alpha_{n-2}-1} y(s) ds + d_1 t^{\alpha-\alpha_{n-2}-1}. \tag{2.6}$$

By (2.6) and Lemma 2.3, we have

$$\begin{aligned} D_{0^+}^{\beta_i-\alpha_{n-2}}x(t) &= -D_{0^+}^{\beta_i-\alpha_{n-2}}I_{0^+}^{\alpha-\alpha_{n-2}}y(t) + d_1 D_{0^+}^{\beta_i-\alpha_{n-2}}t^{\alpha-\alpha_{n-2}-1} \\ &= -I_{0^+}^{\alpha-\beta_i}y(t) + d_1 \frac{\Gamma(\alpha - \alpha_{n-2})}{\Gamma(\alpha - \beta_i)} t^{\alpha-\beta_i-1} \\ &= -\int_0^t \frac{(t-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha - \beta_i)} y(s) ds + d_1 \frac{\Gamma(\alpha - \alpha_{n-2})}{\Gamma(\alpha - \beta_i)} t^{\alpha-\beta_i-1}. \end{aligned} \tag{2.7}$$

So, from (2.7), we have

$$\begin{aligned} D_{0^+}^{\beta_1-\alpha_{n-2}}x(1) &= -\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha - \beta_1)} y(s) ds + d_1 \frac{\Gamma(\alpha - \alpha_{n-2})}{\Gamma(\alpha - \beta_1)}, \\ D_{0^+}^{\beta_2-\alpha_{n-2}}x(t) &= -\int_0^t \frac{(t-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha - \beta_2)} y(s) ds + d_1 \frac{\Gamma(\alpha - \alpha_{n-2})}{\Gamma(\alpha - \beta_2)} t^{\alpha-\beta_2-1}, \\ D_{0^+}^{\beta_3-\alpha_{n-2}}x(t) &= -\int_0^t \frac{(t-s)^{\alpha-\beta_3-1}}{\Gamma(\alpha - \beta_3)} y(s) ds + d_1 \frac{\Gamma(\alpha - \alpha_{n-2})}{\Gamma(\alpha - \beta_3)} t^{\alpha-\beta_3-1}. \end{aligned} \tag{2.8}$$

On the other hand, by

$$D_{0^+}^{\beta_1-\alpha_{n-2}}x(1) = \int_0^\eta h(s) D_{0^+}^{\beta_2-\alpha_{n-2}}x(s) dA(s) + \int_0^1 a(s) D_{0^+}^{\beta_3-\alpha_{n-2}}x(s) dA(s)$$

combining with (2.8), we obtain

$$\begin{aligned} d_1 &= [\Gamma(\alpha - \alpha_{n-2})\Delta]^{-1} \left(\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha - \beta_1)} y(s) ds - \int_0^\eta h(t) \int_0^t \frac{(t-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha - \beta_2)} y(s) ds dA(t) \right. \\ &\quad \left. - \int_0^1 a(t) \int_0^t \frac{(t-s)^{\alpha-\beta_3-1}}{\Gamma(\alpha - \beta_3)} y(s) ds dA(t) \right). \end{aligned}$$

So, substituting d_1 into (2.6), one has the unique solution of problem (2.1)

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha - \alpha_{n-2})} \int_0^t (t-s)^{\alpha-\alpha_{n-2}-1} y(s) ds \\ &\quad + t^{\alpha-\alpha_{n-2}-1} [\Gamma(\alpha - \alpha_{n-2})\Delta]^{-1} \left(\int_0^1 \frac{(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha - \beta_1)} y(s) ds \right. \\ &\quad \left. - \int_0^\eta h(t) \int_0^t \frac{(t-s)^{\alpha-\beta_2-1}}{\Gamma(\alpha - \beta_2)} y(s) ds dA(t) - \int_0^1 a(t) \int_0^t \frac{(t-s)^{\alpha-\beta_3-1}}{\Gamma(\alpha - \beta_3)} y(s) ds dA(t) \right) \\ &= -\frac{1}{\Gamma(\alpha - \alpha_{n-2})} \int_0^t (t-s)^{\alpha-\alpha_{n-2}-1} y(s) ds + \left[\frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \alpha_{n-2})} \right. \\ &\quad \left. + (\Gamma(\alpha - \alpha_{n-2})\Delta)^{-1} \left(\frac{l_1 t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_2)} + \frac{l_2 t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_3)} \right) \right] \int_0^1 (1-s)^{\alpha-\beta_1-1} y(s) ds \\ &\quad - [\Gamma(\alpha - \alpha_{n-2})\Delta]^{-1} \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t) \int_0^t (t-s)^{\alpha-\beta_2-1} y(s) ds dA(t) \end{aligned}$$

$$\begin{aligned}
 & - [\Gamma(\alpha - \alpha_{n-2})\Delta]^{-1} \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t) \int_0^t (t-s)^{\alpha-\beta_3-1} y(s) ds dA(t) \\
 = & - \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \int_0^t (t-s)^{\alpha-\alpha_{n-2}-1} y(s) ds \\
 & + \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \int_0^1 t^{\alpha-\alpha_{n-2}-1} (1-s)^{\alpha-\beta_1-1} y(s) ds \\
 & + [\Gamma(\alpha - \alpha_{n-2})\Delta]^{-1} \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t) \int_0^1 t^{\alpha-\beta_2-1} (1-s)^{\alpha-\beta_1-1} y(s) ds dA(t) \\
 & + [\Gamma(\alpha - \alpha_{n-2})\Delta]^{-1} \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t) \int_0^1 t^{\alpha-\beta_3-1} (1-s)^{\alpha-\beta_1-1} y(s) ds dA(t) \\
 & - [\Gamma(\alpha - \alpha_{n-2})\Delta]^{-1} \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t) \int_0^t (t-s)^{\alpha-\beta_2-1} y(s) ds dA(t) \\
 & - [\Gamma(\alpha - \alpha_{n-2})\Delta]^{-1} \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t) \int_0^t (t-s)^{\alpha-\beta_3-1} y(s) ds dA(t) \\
 = & \int_0^1 \left[L_1(t,s) + \Delta^{-1} \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t) L_2(t,s) dA(t) \right. \\
 & \left. + \Delta^{-1} \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t) L_3(t,s) dA(t) \right] y(s) ds \\
 = & \int_0^1 G(t,s) y(s) ds.
 \end{aligned}$$

The proof is complete. □

Using similar arguments as those used in the proof of Lemma 3 from [19], we obtain the following properties of the functions L_i ($i = 1, 2, 3$).

Lemma 2.5 *If condition (H_1) in Lemma 2.4 is satisfied, the function L_i ($i = 1, 2, 3$) defined by (2.3) has the following properties:*

- (1) $L_1(t,s) \leq \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-\alpha_{n-2})}$ for all $t,s \in [0,1]$;
- (2) $L_1(t,s) \leq \tilde{h}_1(s)$ for all $t,s \in [0,1]$, where

$$\tilde{h}_1(s) = \frac{1}{\Gamma(\alpha - \alpha_{n-2})} (1-s)^{\alpha-\beta_1-1} [1 - (1-s)^{\beta_1-\alpha_{n-2}}], \quad s \in [0,1];$$

- (3) $L_1(t,s) \geq t^{\alpha-\alpha_{n-2}-1} \tilde{h}_1(s)$ for all $t,s \in [0,1]$;
- (4) $L_2(t,s) \leq \frac{t^{\alpha-\beta_2-1}}{\Gamma(\alpha-\alpha_{n-2})}$ for all $t,s \in [0,1]$;
- (5) $L_2(t,s) \leq \frac{t^{\alpha-\beta_2-1}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\alpha_{n-2})}$ for all $t,s \in [0,1]$;
- (6) $L_2(t,s) \geq t^{\alpha-\beta_2-1} \tilde{h}_2(s)$ for all $t,s \in [0,1]$, where

$$\tilde{h}_2(s) = \frac{1}{\Gamma(\alpha - \alpha_{n-2})} (1-s)^{\alpha-\beta_1-1} [1 - (1-s)^{\beta_1-\beta_2}], \quad s \in [0,1];$$

- (7) $L_3(t,s) \leq \frac{t^{\alpha-\beta_3-1}}{\Gamma(\alpha-\alpha_{n-2})}$ for all $t,s \in [0,1]$;
- (8) $L_3(t,s) \leq \frac{t^{\alpha-\beta_3-1}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\alpha_{n-2})}$ for all $t,s \in [0,1]$;

(9) $L_3(t, s) \geq t^{\alpha-\beta_3-1} \tilde{h}_3(s)$ for all $t, s \in [0, 1]$, where

$$\tilde{h}_3(s) = \frac{1}{\Gamma(\alpha - \alpha_{n-2})} (1 - s)^{\alpha-\beta_1-1} [1 - (1 - s)^{\beta_1-\beta_3}], \quad s \in [0, 1];$$

(10) The functions L_i ($i = 1, 2, 3$) are continuous on $[0, 1] \times [0, 1]$, $L_i(t, s) \geq 0$ for any $t, s \in [0, 1]$, $L_i(t, s) > 0$ for any $t, s \in (0, 1)$ ($i = 1, 2, 3$).

Proof (1) From (2.3), we have

$$L_1(t, s) \leq \frac{t^{\alpha-\alpha_{n-2}-1} (1 - s)^{\alpha-\beta_1-1}}{\Gamma(\alpha - \alpha_{n-2})} \leq \frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha - \alpha_{n-2})}, \quad \forall t, s \in [0, 1].$$

(2) The function L_1 is nondecreasing in the first variable. In fact, from (2.3), if $s \leq t$, we obtain

$$\begin{aligned} \frac{\partial L_1(t, s)}{\partial t} &= \frac{\alpha - \alpha_{n-2} - 1}{\Gamma(\alpha - \alpha_{n-2})} [t^{\alpha-\alpha_{n-2}-2} (1 - s)^{\alpha-\beta_1-1} - (t - s)^{\alpha-\alpha_{n-2}-2}] \\ &\geq \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} [t^{\alpha-\alpha_{n-2}-2} (1 - s)^{\alpha-\alpha_{n-2}-2} - (t - s)^{\alpha-\alpha_{n-2}-2}] \\ &= \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} [(t - ts)^{\alpha-\alpha_{n-2}-2} - (t - s)^{\alpha-\alpha_{n-2}-2}] \\ &\geq 0. \end{aligned}$$

Then $L_1(t, s) \leq L_1(1, s)$ for every $(t, s) \in [0, 1] \times [0, 1]$ with $s \leq t$. For $s \geq t$, we acquire

$$\frac{\partial L_1(t, s)}{\partial t} = \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} [t^{\alpha-\alpha_{n-2}-2} (1 - s)^{\alpha-\beta_1-1}] \geq 0.$$

Hence, $L_1(t, s) \leq L_1(s, s)$ for every $(t, s) \in [0, 1] \times [0, 1]$ with $s \geq t$. Therefore, we deduce that $L_1(t, s) \leq \tilde{h}_1(s)$ for every $(t, s) \in [0, 1] \times [0, 1]$, where $\tilde{h}_1(s) = \frac{1}{\Gamma(\alpha-\alpha_{n-2})} (1 - s)^{\alpha-\beta_1-1} [1 - (1 - s)^{\beta_1-\alpha_{n-2}}]$, $s \in [0, 1]$.

(3) From (2.3), if $s \leq t$, we obtain

$$\begin{aligned} L_1(t, s) &= \frac{1}{\Gamma(\alpha - \alpha_{n-2})} [t^{\alpha-\alpha_{n-2}-1} (1 - s)^{\alpha-\beta_1-1} - (t - s)^{\alpha-\alpha_{n-2}-1}] \\ &\geq \frac{1}{\Gamma(\alpha - \alpha_{n-2})} [t^{\alpha-\alpha_{n-2}-1} (1 - s)^{\alpha-\beta_1-1} - (t - ts)^{\alpha-\alpha_{n-2}-1}] \\ &= \frac{1}{\Gamma(\alpha - \alpha_{n-2})} t^{\alpha-\alpha_{n-2}-1} [(1 - s)^{\alpha-\beta_1-1} - (1 - s)^{\alpha-\alpha_{n-2}-1}] \\ &= \frac{1}{\Gamma(\alpha - \alpha_{n-2})} t^{\alpha-\alpha_{n-2}-1} (1 - s)^{\alpha-\beta_1-1} [1 - (1 - s)^{\beta_1-\alpha_{n-2}}] \\ &= t^{\alpha-\alpha_{n-2}-1} \tilde{h}_1(s), \end{aligned}$$

if $s \leq t$, we obtain

$$L_1(t, s) = \frac{1}{\Gamma(\alpha - \alpha_{n-2})} t^{\alpha-\alpha_{n-2}-1} (1 - s)^{\alpha-\beta_1-1} \geq t^{\alpha-\alpha_{n-2}-1} \tilde{h}_1(s),$$

where $\tilde{h}_1(s) = \frac{1}{\Gamma(\alpha - \alpha_{n-2})}(1 - s)^{\alpha - \beta_1 - 1}[1 - (1 - s)^{\beta_1 - \alpha_{n-2}}]$, $s \in [0, 1]$. Therefore, we conclude $L_1(t, s) \geq t^{\alpha - \alpha_{n-2} - 1}\tilde{h}_1(s)$ for all $t, s \in [0, 1]$.

For the proof of properties (4)–(9) is similar to the proof of properties (1)–(3), we omit it here.

(10) This property is evident, it follows from the definitions of L_i ($i = 1, 2, 3$) and from properties (3), (6), and (9). The proof is complete. \square

Now, from the definitions of the Green’s functions G and the properties of functions L_i ($i = 1, 2, 3$), we obtain the following lemma.

Lemma 2.6 *If condition (H_1) in Lemma 2.4 is satisfied, the functions H_i ($i = 1, 2, 3$) defined by (2.3) satisfy the following:*

- (1) $H_1(t, s) \leq J_1(s)$ for any $(t, s) \in [0, 1] \times [0, 1]$, where

$$J_1(s) = \tilde{h}_1(s) = \frac{1}{\Gamma(\alpha - \alpha_{n-2})}(1 - s)^{\alpha - \beta_1 - 1}[1 - (1 - s)^{\beta_1 - \alpha_{n-2}}], \quad s \in [0, 1];$$

- (2) $H_1(t, s) \geq t^{\alpha - \alpha_{n-2} - 1}J_1(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$;
- (3) $H_2(t, s) \leq J_2(s)$ for any $(t, s) \in [0, 1] \times [0, 1]$, where

$$J_2(s) = \frac{\Delta^{-1}}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t)L_2(t, s) dA(t);$$

- (4) $H_2(t, s) = t^{\alpha - \alpha_{n-2} - 1}J_2(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$;
- (5) $H_3(t, s) \leq J_3(s)$ for any $(t, s) \in [0, 1] \times [0, 1]$, where

$$J_3(s) = \frac{\Delta^{-1}}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t)L_3(t, s) dA(t);$$

- (6) $H_3(t, s) = t^{\alpha - \alpha_{n-2} - 1}J_3(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$;
- (7) Functions H_i ($i = 1, 2, 3$) are continuous on $[0, 1] \times [0, 1]$, $H_i(t, s) \geq 0$ for any $t, s \in [0, 1]$, $H_i(t, s) > 0$ for all $t, s \in (0, 1)$ ($i = 1, 2, 3$).

Proof (1) It follows from $H_1(t, s) = L_1(t, s)$, $J_1(s) = \tilde{h}_1(s)$, $L_1(t, s) \leq \tilde{h}_1(s)$ for all $t, s \in [0, 1]$ that $H_1(t, s) \leq J_1(s)$ for any $(t, s) \in [0, 1] \times [0, 1]$.

(2) From property (3) in Lemma 2.5, we have $H_1(t, s) \geq t^{\alpha - \alpha_{n-2} - 1}J_1(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$.

(3)–(6) follow from the definitions of H_i ($i = 2, 3$).

(7) This property follows from the definitions of L_i , H_i ($i = 2, 3$), and properties (2), (4), (6), and condition (H_1) in Lemma 2.4, property (10) in Lemma 2.5. The proof is complete. \square

Let $u(t) = I_{0^+}^{\alpha_{n-2}}x(t)$, $x(t) \in C[0, 1]$, then $D_{0^+}^{\alpha_{n-2}}u(t) = x(t)$, problem (1.1) can turn into the following modified problem of BVP (2.9):

$$\begin{cases} D_{0^+}^{\alpha - \alpha_{n-2}}x(t) + f(t, I_{0^+}^{\alpha_{n-2}}x(t), I_{0^+}^{\alpha_{n-2} - \alpha_1}x(t), \dots, x(t)) = 0, & t \in (0, 1), \\ D_{0^+}^{\gamma_{n-2} - \alpha_{n-2}}x(0) = 0, \\ D_{0^+}^{\beta_1 - \alpha_{n-2}}x(1) = \int_0^\eta h(s)D_{0^+}^{\beta_2 - \alpha_{n-2}}x(s) dA(s) + \int_0^1 a(s)D_{0^+}^{\beta_3 - \alpha_{n-2}}x(s) dA(s). \end{cases} \tag{2.9}$$

Obviously, the solution of BVP (2.9) is

$$x(t) = \int_0^1 G(t, s)f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds. \tag{2.10}$$

Lemma 2.7 *If $x \in C([0, 1], \mathbb{R}_+^1)$ is a positive solution of BVP (2.9), let $u(t) = I_{0^+}^{\alpha_{n-2}}x(t)$, then $u(t) = I_{0^+}^{\alpha_{n-2}}x(t)$ is a positive solution of BVP (1.1).*

Proof We assume that $x \in C([0, 1])$ is a positive solution of BVP (2.9). Let $u(t) = I_{0^+}^{\alpha_{n-2}}x(t)$, we have

$$D_{0^+}^{\alpha_{n-2}}u(t) = x(t), \quad D_{0^+}^{\alpha_i}u(t) = D_{0^+}^{\alpha_i}I_{0^+}^{\alpha_{n-2}}x(t) = I_{0^+}^{\alpha_{n-2}-\alpha_i}x(t) \quad (i = 1, 2, \dots, n - 3),$$

$$D_{0^+}^{\alpha}u(t) = \frac{d^n}{dt^n}I_{0^+}^{n-\alpha}I_{0^+}^{\alpha_{n-2}}x(t) = \frac{d^n}{dt^n}I_{0^+}^{n-\alpha+\alpha_{n-2}}x(t) = D_{0^+}^{\alpha-\alpha_{n-2}}x(t),$$

which implies that

$$D_{0^+}^{\alpha}u(t) + f(t, u(t), D_{0^+}^{\alpha_1}u(t), D_{0^+}^{\alpha_2}u(t), \dots, D_{0^+}^{\alpha_{n-2}}u(t))$$

$$= D_{0^+}^{\alpha-\alpha_{n-2}}x(t) + f(t, I_{0^+}^{\alpha_{n-2}}x(t), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(t), \dots, x(t))$$

$$= 0,$$

$$D_{0^+}^{\gamma_i}u(t) = D_{0^+}^{\gamma_i}I_{0^+}^{\alpha_{n-2}}x(t) = I_{0^+}^{\alpha_{n-2}-\gamma_i}x(t) \quad (i = 0, 1, 2, \dots, n - 3),$$

$$D_{0^+}^{\gamma_{n-2}}u(t) = D_{0^+}^{\gamma_{n-2}}I_{0^+}^{\alpha_{n-2}}x(t) = D_{0^+}^{\gamma_{n-2}-\alpha_{n-2}}x(t).$$

From $u(t) = I_{0^+}^{\alpha_{n-2}}x(t)$ and the above expression, we have

$$u(0) = D_{0^+}^{\gamma_1}u(0) = D_{0^+}^{\gamma_2}u(0) = \dots = D_{0^+}^{\gamma_{n-2}}u(0) = 0,$$

$$D_{0^+}^{\beta_1}u(1) = \int_0^\eta h(s)D_{0^+}^{\beta_2}u(s) dA(s) + \int_0^1 a(s)D_{0^+}^{\beta_3}u(s) dA(s).$$

Hence, we demonstrate that $u(t) = I_{0^+}^{\alpha_{n-2}}x(t)$ is a positive solution of BVP (1.1). The proof is complete. □

Lemma 2.8 *If condition (H_1) in Lemma 2.4 is satisfied and $y \in C(0, 1) \cap L^1(0, 1)$ with $y(t) > 0$ for all $t \in (0, 1)$, then the solution x of problem (2.1) satisfies the inequality $x(t) > 0$ for all $t \in (0, 1)$. Moreover, we have the inequalities $x(t) \geq t^{\alpha-\alpha_{n-2}-1}x(t')$ for all $t, t' \in [0, 1]$.*

Proof In view of Lemma 2.6, we acquire

$$x(t) = \int_0^1 G(t, s)y(s) ds = \int_0^1 [H_1(t, s) + H_2(t, s) + H_3(t, s)]y(s) ds$$

$$\geq \int_0^1 [t^{\alpha-\alpha_{n-2}-1}J_1(s) + t^{\alpha-\alpha_{n-2}-1}J_2(s) + t^{\alpha-\alpha_{n-2}-1}J_3(s)]y(s) ds$$

$$\geq \int_0^1 [t^{\alpha-\alpha_{n-2}-1}H_1(t', s) + t^{\alpha-\alpha_{n-2}-1}H_2(t', s) + t^{\alpha-\alpha_{n-2}-1}H_3(t', s)]y(s) ds$$

$$\begin{aligned}
 &= t^{\alpha-\alpha_{n-2}-1} \int_0^1 G(t',s)y(s) ds \\
 &= t^{\alpha-\alpha_{n-2}-1} x(t'), \quad \forall t, t' \in [0, 1].
 \end{aligned}$$

The proof is complete. □

Remark 2.1 Under the assumptions of Lemma 2.8, for $c \in (0, 1/2)$, the solutions of problem (2.1) satisfy the inequality $\min_{t \in [c, 1]} x(t) \geq c^{\alpha-\alpha_{n-2}-1} \|x\|$.

Let $E = C[0, 1]$. Clearly, $(E, \|\cdot\|)$ is a Banach space with supremum norm $\|x\| = \sup_{t \in [0, 1]} |x(t)|$. Let $P = \{x \in E : x(t) \geq 0, 0 \leq t \leq 1\}$. It is easy to see that P is a normal cone of E . We define an operator $A : E \rightarrow E$ by

$$Ax(t) = \int_0^1 G(t,s)f(s, I_0^{\alpha_{n-2}}x(s), I_0^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds, \quad t \in [0, 1].$$

Obviously, if x is a fixed point of operator A , then x is a solution of problem (2.9). We present the basic assumptions that we shall use in the sequel.

(H₂) The function $f \in C((0, 1) \times \mathbb{R}_+^{n-1}, \mathbb{R}_+^1)$ and there exist the functions $p \in C((0, 1), \mathbb{R}_+^1)$ and $q \in C([0, 1] \times \mathbb{R}_+^{n-1}, \mathbb{R}_+^1)$ with $p \not\equiv 0$ and $\int_0^1 (1-s)^{\alpha-\beta_1-1} p(s) ds < +\infty$ such that

$$\begin{aligned}
 f(t, x_0, x_1, \dots, x_{n-2}) &\leq p(t)q(t, x_0, x_1, \dots, x_{n-2}), \\
 \forall t \in (0, 1), x_i &\in \mathbb{R}_+^1, i = 0, 1, \dots, n-2.
 \end{aligned}$$

Lemma 2.9 *If conditions (H₁) and (H₂) hold, then $A : P \rightarrow P$ is completely continuous.*

Proof We denote by $Q_i = \int_0^1 J_i(s)p(s) ds$ ($i = 1, 2, 3$), where J_i ($i = 1, 2, 3$) are defined in Lemma 2.6. Using (H₂) and Lemma 2.5, we deduce that $Q_i > 0$ ($i = 1, 2, 3$) and

$$\begin{aligned}
 Q_1 &= \int_0^1 J_1(s)p(s) ds \leq \frac{1}{\Gamma(\alpha-\alpha_{n-2})} \int_0^1 (1-s)^{\alpha-\beta_1-1} p(s) ds < +\infty, \\
 Q_2 &= \int_0^1 J_2(s)p(s) ds \leq \frac{\Delta^{-1}}{\Gamma(\alpha-\beta_2)} \int_0^1 \int_0^\eta h(t)L_2(t,s) dA(t)p(s) ds \\
 &\leq \frac{\Delta^{-1}}{\Gamma(\alpha-\beta_2)\Gamma(\alpha-\alpha_{n-2})} \int_0^1 (1-s)^{\alpha-\beta_1-1} \int_0^\eta h(t)t^{\alpha-\beta_2-1} dA(t)p(s) ds \\
 &\leq \frac{\Delta^{-1} \int_0^\eta h(t)t^{\alpha-\beta_2-1} dA(t)}{\Gamma(\alpha-\beta_2)\Gamma(\alpha-\alpha_{n-2})} \int_0^1 (1-s)^{\alpha-\beta_1-1} p(s) ds < +\infty, \\
 Q_3 &= \int_0^1 J_3(s)p(s) ds \leq \frac{\Delta^{-1}}{\Gamma(\alpha-\beta_3)} \int_0^1 \int_0^1 a(t)L_3(t,s) dA(t)p(s) ds \\
 &\leq \frac{\Delta^{-1}}{\Gamma(\alpha-\beta_3)\Gamma(\alpha-\alpha_{n-2})} \int_0^1 (1-s)^{\alpha-\beta_1-1} \int_0^1 a(t)t^{\alpha-\beta_3-1} dA(t)p(s) ds \\
 &\leq \frac{\Delta^{-1} \int_0^1 a(t)t^{\alpha-\beta_3-1} dA(t)}{\Gamma(\alpha-\beta_3)\Gamma(\alpha-\alpha_{n-2})} \int_0^1 (1-s)^{\alpha-\beta_1-1} p(s) ds < +\infty.
 \end{aligned}$$

By Lemma 2.6, we also conclude that $A : P \rightarrow P$. We prove that A maps bounded sets into relatively compact sets. Suppose that $D \subset P$ is an arbitrary bounded set of E , then there

exists $M_1 > 0$ such that $\|x\| \leq M_1$ for all $x \in D$. By the continuity of q , we obtain

$$M_2 = \sup_{\substack{t \in [0,1], x_i \in [0,\rho] \\ i=0,1,\dots,n-2}} q(t, x_0, x_1, \dots, x_{n-2}) > 0,$$

where $\rho = \sum_{i=0}^{n-2} \frac{M_1}{\Gamma(\alpha_{n-2-\alpha_i+1)}$, $\alpha_0 = 0$. By using Lemma 2.6, for any $x \in D$ and $t \in [0, 1]$, we obtain

$$\begin{aligned} Ax(t) &= \int_0^1 G(t,s)f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\leq \int_0^1 G(t,s)p(s)q(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\leq \int_0^1 (J_1(s) + J_2(s) + J_3(s))p(s)q(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\leq M_2(Q_1 + Q_2 + Q_3). \end{aligned}$$

Thus, $A(D)$ is bounded in E . In what follows, we prove that $A(D)$ is equicontinuous. By using Lemma 2.4, for any $x \in D$ and $t \in [0, 1]$, we have

$$\begin{aligned} Ax(t) &= \int_0^1 G(t,s)f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &= \int_0^t \frac{t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-\beta_1-1} - (t-s)^{\alpha-\alpha_{n-2}-1}}{\Gamma(\alpha-\alpha_{n-2})} f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\quad + \int_t^1 \frac{t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\alpha_{n-2})} f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\quad + \int_0^1 t^{\alpha-\alpha_{n-2}-1} J_2(s) f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\quad + \int_0^1 t^{\alpha-\alpha_{n-2}-1} J_3(s) f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds. \end{aligned} \tag{2.11}$$

Thus, for any $t \in (0, 1)$, we conclude

$$\begin{aligned} (Ax)'(t) &= \int_0^t \frac{(\alpha-\alpha_{n-2}-1)t^{\alpha-\alpha_{n-2}-2}(1-s)^{\alpha-\beta_1-1} - (\alpha-\alpha_{n-2}-1)(t-s)^{\alpha-\alpha_{n-2}-2}}{\Gamma(\alpha-\alpha_{n-2})} \\ &\quad \times f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\quad + \int_t^1 \frac{(\alpha-\alpha_{n-2}-1)t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\alpha_{n-2})} \\ &\quad \times f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\quad + (\alpha-\alpha_{n-2}-1) \int_0^1 t^{\alpha-\alpha_{n-2}-2} J_2(s) f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\quad + (\alpha-\alpha_{n-2}-1) \int_0^1 t^{\alpha-\alpha_{n-2}-2} J_3(s) f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\leq \frac{1}{\Gamma(\alpha-\alpha_{n-2}-1)} \int_0^t t^{\alpha-\alpha_{n-2}-2}(1-s)^{\alpha-\beta_1-1} + (t-s)^{\alpha-\alpha_{n-2}-2} p(s) \\ &\quad \times q(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_t^1 t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} p(s) \\
 & \times q(s, I_0^{\alpha_{n-2}} x(s), I_0^{\alpha_{n-2} - \alpha_1} x(s), \dots, x(s)) ds \\
 & + (\alpha - \alpha_{n-2} - 1) \int_0^1 t^{\alpha - \alpha_{n-2} - 2} J_2(s) p(s) q(s, I_0^{\alpha_{n-2}} x(s), I_0^{\alpha_{n-2} - \alpha_1} x(s), \dots, x(s)) ds \\
 & + (\alpha - \alpha_{n-2} - 1) \int_0^1 t^{\alpha - \alpha_{n-2} - 2} J_3(s) p(s) q(s, I_0^{\alpha_{n-2}} x(s), I_0^{\alpha_{n-2} - \alpha_1} x(s), \dots, x(s)) ds \\
 \leq & M_2 \left[\frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_0^t [t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} + (t-s)^{\alpha - \alpha_{n-2} - 2}] p(s) ds \right. \\
 & + \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_t^1 t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} p(s) ds \\
 & + (\alpha - \alpha_{n-2} - 1) \int_0^1 t^{\alpha - \alpha_{n-2} - 2} J_2(s) p(s) ds \\
 & \left. + (\alpha - \alpha_{n-2} - 1) \int_0^1 t^{\alpha - \alpha_{n-2} - 2} J_3(s) p(s) ds \right]. \tag{2.12}
 \end{aligned}$$

We denote

$$\begin{aligned}
 g(t) &= \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_0^t [t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} + (t-s)^{\alpha - \alpha_{n-2} - 2}] p(s) ds \\
 & + \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_t^1 t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} p(s) ds, \quad t \in [0, 1], \\
 \mu(t) &= g(t) + (\alpha - \alpha_{n-2} - 1) \left[\int_0^1 t^{\alpha - \alpha_{n-2} - 2} J_2(s) p(s) ds \right. \\
 & \left. + \int_0^1 t^{\alpha - \alpha_{n-2} - 2} J_3(s) p(s) ds \right], \quad t \in [0, 1].
 \end{aligned}$$

For the integral of the function g , by exchanging the order of integration, we obtain

$$\begin{aligned}
 \int_0^1 g(t) dt &= \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_0^1 \int_0^t [t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} + (t-s)^{\alpha - \alpha_{n-2} - 2}] p(s) ds dt \\
 & + \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_0^1 \int_t^1 t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} p(s) ds dt \\
 &= \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_0^1 \int_s^1 [t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} + (t-s)^{\alpha - \alpha_{n-2} - 2}] dt p(s) ds \\
 & + \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_0^1 \int_0^s t^{\alpha - \alpha_{n-2} - 2} (1-s)^{\alpha - \beta_1 - 1} dt p(s) ds \\
 &= \frac{1}{\Gamma(\alpha - \alpha_{n-2} - 1)} \int_0^1 \left[\frac{(1-s)^{\alpha - \beta_1 - 1}}{\alpha - \alpha_{n-2} - 1} - \frac{s^{\alpha - \alpha_{n-2} - 1} (1-s)^{\alpha - \beta_1 - 1}}{\alpha - \alpha_{n-2} - 1} \right. \\
 & \left. + \frac{(1-s)^{\alpha - \alpha_{n-2} - 1}}{\alpha - \alpha_{n-2} - 1} + \frac{s^{\alpha - \alpha_{n-2} - 1} (1-s)^{\alpha - \beta_1 - 1}}{\alpha - \alpha_{n-2} - 1} \right] p(s) ds \\
 &= \frac{1}{\Gamma(\alpha - \alpha_{n-2})} \int_0^1 (1-s)^{\alpha - \beta_1 - 1} (1 + (1-s)^{\beta_1 - \alpha_{n-2}}) p(s) ds \\
 &\leq \frac{2}{\Gamma(\alpha - \alpha_{n-2})} \int_0^1 (1-s)^{\alpha - \beta_1 - 1} p(s) ds < +\infty.
 \end{aligned}$$

For the integral of the function μ , we have

$$\begin{aligned} \int_0^1 \mu(t) dt &= \int_0^1 g(t) dt + \int_0^1 (\alpha - \alpha_{n-2} - 1) \left[\int_0^1 t^{\alpha - \alpha_{n-2} - 2} J_2(s) p(s) ds \right. \\ &\quad \left. + \int_0^1 t^{\alpha - \alpha_{n-2} - 2} J_3(s) p(s) ds \right] dt \\ &\leq \frac{2}{\Gamma(\alpha - \alpha_{n-2})} \int_0^1 (1 - s)^{\alpha - \beta_1 - 1} p(s) ds \\ &\quad + (\alpha - \alpha_{n-2} - 1) \left[\int_0^1 J_2(s) p(s) ds + \int_0^1 J_3(s) p(s) ds \right] \\ &\leq \frac{2}{\Gamma(\alpha - \alpha_{n-2})} \int_0^1 (1 - s)^{\alpha - \beta_1 - 1} p(s) ds \\ &\quad + (\alpha - \alpha_{n-2} - 1) \left[\frac{\Delta^{-1} \int_0^1 h(t) t^{\alpha - \beta_2 - 1} dA(t)}{\Gamma(\alpha - \beta_2) \Gamma(\alpha - \alpha_{n-2})} \int_0^1 (1 - s)^{\alpha - \beta_1 - 1} p(s) ds \right. \\ &\quad \left. + \frac{\Delta^{-1} \int_0^1 a(t) t^{\alpha - \beta_3 - 1} dA(t)}{\Gamma(\alpha - \beta_3) \Gamma(\alpha - \alpha_{n-2})} \int_0^1 (1 - s)^{\alpha - \beta_1 - 1} p(s) ds \right] < +\infty. \end{aligned} \tag{2.13}$$

We deduce that $\mu \in L^1(0, 1)$. Thus, for any $t_1, t_2 \in [0, 1]$ with $t_1 \leq t_2$ and $x \in D$, by (2.12) and (2.13), we conclude

$$|Ax(t_1) - Ax(t_2)| = \left| \int_{t_1}^{t_2} (Ax)'(t) dt \right| \leq M_2 \int_{t_1}^{t_2} \mu(t) dt. \tag{2.14}$$

From (2.13), (2.14) and the absolute continuity of the integral function, we obtain that $A(D)$ is equicontinuous. By the Ascoli–Arzela theorem, we conclude that $A(D)$ is a relatively compact set of E . Therefore A is a compact operator. Besides, we can show that A is continuous on P (see the proof of Lemma 1.4.1 in [21]). Hence $A : P \rightarrow P$ is completely continuous. \square

Lemma 2.10 ([16]) *Let K be a cone of the real Banach space E , $\Omega_1, \Omega_2 \subset E$ be bounded open sets of E , $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$ is a completely continuous mapping such that one of the following two conditions is satisfied:*

- (i) $\|Au\| \leq \|u\|, \forall u \in K \cap \partial\Omega_1; \|Au\| \geq \|u\|, \forall u \in K \cap \partial\Omega_2;$
- (ii) $\|Au\| \geq \|u\|, \forall u \in K \cap \partial\Omega_1; \|Au\| \leq \|u\|, \forall u \in K \cap \partial\Omega_2.$

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

For $c \in (0, \frac{1}{2})$, we define the cone

$$P_0 = \left\{ x \in P, \min_{t \in [c, 1]} x(t) \geq c^{\alpha - \alpha_{n-2} - 1} \|x\| \right\}.$$

Under assumptions $(H_1), (H_2)$ and Remark 2.1, we have $A(P) \subset P_0$, and so $A|_{P_0} : P_0 \rightarrow P_0$ (denoted again by A) is also a completely continuous operator.

3 Existence of positive solutions for BVP (1.1)

We first offer some fixed numbers $k_i, \mu_i, \varsigma_i, \vartheta_i \geq 0$ ($i = 0, 1, 2, \dots, n - 2$) with $\sum_{i=0}^{n-2} k_i > 0, \sum_{i=0}^{n-2} \mu_i > 0, \sum_{i=0}^{n-2} \varsigma_i > 0, \sum_{i=0}^{n-2} \vartheta_i > 0$. Now, we list our assumptions:

(H₃) There exists $a \geq 1$ such that

$$q_0 = \limsup_{\sum_{i=0}^{n-2} k_i x_i \rightarrow 0} \max_{t \in [0,1]} \frac{q(t, x_0, x_1, \dots, x_{n-2})}{(k_0 x_0 + k_1 x_1 + \dots + k_{n-2} x_{n-2})^a} < b_1,$$

where $b_1 = \frac{1}{(Q_1+Q_2+Q_3)\varrho_1^a} > 0$, $\varrho_1 = \sum_{i=0}^{n-2} k_i \Gamma^{-1}(\alpha_{n-2} - \alpha_i + 1)$, $Q_i = \int_0^1 J_i(s)p(s) ds$ ($i = 1, 2, 3$), in which J_i are defined in Lemma 2.6.

(H₄) There exists $c \in (0, \frac{1}{2})$ such that

$$f_\infty = \liminf_{\sum_{i=0}^{n-2} \mu_i x_i \rightarrow \infty} \min_{t \in [c, 1-c]} \frac{f(t, x_0, x_1, \dots, x_{n-2})}{\mu_0 x_0 + \mu_1 x_1 + \dots + \mu_{n-2} x_{n-2}} > b_2,$$

where $b_2 = \frac{2}{c^{2(\alpha_{n-2}-1)m_1\varrho_2c_1}}$, $\varrho_2 = \sum_{i=0}^{n-2} \mu_i \Gamma^{-1}(\alpha_{n-2} - \alpha_i + 1)$, $c_1 = \min_{i=0,1,2,\dots,n-2} \{c^{\alpha_{n-2}-\alpha_i}\}$, $m_1 = \int_c^{1-c} J_1(s) ds$, in which J_1 is defined in Lemma 2.6.

(H₅)

$$q_\infty = \limsup_{\sum_{i=0}^{n-2} \varsigma_i x_i \rightarrow \infty} \max_{t \in [0,1]} \frac{q(t, x_0, x_1, \dots, x_{n-2})}{\varsigma_0 x_0 + \varsigma_1 x_1 + \dots + \varsigma_{n-2} x_{n-2}} < b_3,$$

where $b_3 = \frac{1}{2\varrho_3(Q_1+Q_2+Q_3)}$, $\varrho_3 = \sum_{i=0}^{n-2} \varsigma_i \Gamma^{-1}(\alpha_{n-2} - \alpha_i + 1)$, $Q_i = \int_0^1 J_i(s)p(s) ds$ ($i = 1, 2, 3$), in which J_i are defined in Lemma 2.6.

(H₆) There exist $c \in (0, \frac{1}{2})$, $\hat{a} \in (0, 1]$ such that

$$f_0 = \liminf_{\sum_{i=0}^{n-2} \vartheta_i x_i \rightarrow 0} \min_{t \in [c, 1-c]} \frac{f(t, x_0, x_1, \dots, x_{n-2})}{(\vartheta_0 x_0 + \vartheta_1 x_1 + \dots + \vartheta_{n-2} x_{n-2})^{\hat{a}}} > b_4,$$

where $b_4 = \frac{1}{c^{(1+\hat{a})(\alpha_{n-2}-1)(\varrho_4 c_1)^{\hat{a}} m_1}}$, $\varrho_4 = \sum_{i=0}^{n-2} \vartheta_i \Gamma^{-1}(\alpha_{n-2} - \alpha_i + 1)$, $m_1 = \int_c^{1-c} J_1(s) ds$, $c_1 = \min_{i=0,1,2,\dots,n-2} \{c^{\alpha_{n-2}-\alpha_i}\}$, in which J_1 is defined in Lemma 2.6.

(H₇) $M_q > 0$ satisfies

$$M_q \left[\max_{t \in [0,1]} \int_0^1 G(t,s)p(s) ds \right] < 1,$$

where $M_q = \max\{q(t, x_0, x_1, \dots, x_{n-2}) : 0 \leq t \leq 1, x_i \in [0, \tilde{\rho}], i = 0, 1, \dots, n - 2\}$, $\tilde{\rho} = \sum_{i=0}^{n-2} \frac{1}{\Gamma(\alpha_{n-2}-\alpha_i+1)}$, $\alpha_0 = 0$.

Theorem 3.1 Assume that (H₁), (H₂), (H₃), and (H₄) hold, then fractional BVP (1.1) has at least one positive solution.

Proof From (H₃), for fixed $b_1 > \epsilon_1 > 0$, there exists $r \in (0, 1)$ such that

$$q(t, x_0, x_1, \dots, x_{n-2}) \leq (b_1 - \epsilon_1)(k_0 x_0 + k_1 x_1 + \dots + k_{n-2} x_{n-2})^a \tag{3.1}$$

for all $t \in [0, 1]$, $x_i \geq 0$, $\sum_{i=0}^{n-2} k_i x_i \leq r$. Taking $r_1 < \min\{\frac{r}{\varrho_1}, r\}$. By Definition 2.1, for any $t \in [0, 1]$, we can obtain that

$$\begin{aligned} 0 < I_{0^+}^{\alpha_{n-2}-\alpha_i} x(t) &= \frac{1}{\Gamma(\alpha_{n-2}-\alpha_i)} \int_0^t (t-s)^{\alpha_{n-2}-\alpha_i-1} x(s) ds \\ &\leq \frac{t^{\alpha_{n-2}-\alpha_i} \|x\|}{\Gamma(\alpha_{n-2}-\alpha_i+1)} \leq \frac{\|x\|}{\Gamma(\alpha_{n-2}-\alpha_i+1)}, \quad i = 0, 1, 2, \dots, n-2, \end{aligned} \tag{3.2}$$

where $\alpha_0 = 0$. For any $x \in \partial B_{r_1} \cap P_0$ and $t \in [0, 1]$, by (3.2), we know that

$$k_0 I_{0^+}^{\alpha_{n-2}} x(s) + k_1 I_{0^+}^{\alpha_{n-2}-\alpha_1} x(s) + \dots + k_{n-2} x(s) \leq \varrho_1 r_1 < r.$$

Thus, by (3.1), Lemma 2.6, and (3.2), for any $x \in \partial B_{r_1} \cap P_0$ and $t \in [0, 1]$, we obtain

$$\begin{aligned} Ax(t) &= \int_0^1 G(t,s) f(s, I_{0^+}^{\alpha_{n-2}} x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1} x(s), \dots, x(s)) ds \\ &\leq \int_0^1 [J_1(s) + J_2(s) + J_3(s)] p(s) q(s, I_{0^+}^{\alpha_{n-2}} x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1} x(s), \dots, x(s)) ds \\ &\leq (b_1 - \epsilon_1) \int_0^1 [J_1(s) + J_2(s) + J_3(s)] p(s) (k_0 I_{0^+}^{\alpha_{n-2}} x(s) + k_1 I_{0^+}^{\alpha_{n-2}-\alpha_1} x(s) + \dots \\ &\quad + k_{n-2} x(s))^a ds \\ &\leq (b_1 - \epsilon_1) \int_0^1 [J_1(s) + J_2(s) + J_3(s)] p(s) (\varrho_1 \|x\|)^a ds \\ &\leq (Q_1 + Q_2 + Q_3) b_1 \varrho_1^a \|x\|. \end{aligned}$$

Therefore

$$\|Ax\| \leq \|x\|, \quad \forall x \in \partial B_{r_1} \cap P_0. \tag{3.3}$$

From (H_4) , for fixed $\epsilon_2 > 0$, there exists $c_2 > 0$ such that

$$f(t, x_0, x_1, \dots, x_{n-2}) \geq (b_2 + \epsilon_2)(\mu_0 x_0 + \mu_1 x_1 + \dots + \mu_{n-2} x_{n-2}) - c_2 \tag{3.4}$$

for any $t \in [c, 1 - c]$, $x_i \geq 0$ ($i = 0, 1, 2, \dots, n - 2$). By Definition 2.1, for any $t \in [0, 1]$, we can obtain that

$$\begin{aligned} I_{0^+}^{\alpha_{n-2}-\alpha_i} x(t) &= \frac{1}{\Gamma(\alpha_{n-2}-\alpha_i)} \int_0^t (t-s)^{\alpha_{n-2}-\alpha_i-1} x(s) ds \\ &\geq \frac{t^{\alpha_{n-2}-\alpha_i} \min_{t \in [0,1]} x(s)}{\Gamma(\alpha_{n-2}-\alpha_i+1)}, \quad i = 0, 1, 2, \dots, n-2, \end{aligned} \tag{3.5}$$

where $\alpha_0 = 0$. Then, by using (3.4) and (3.5), for any $x \in P_0$ and $t \in [c, 1]$, we have

$$\begin{aligned} Ax(t) &\geq \int_c^{1-c} G(t,s)f(s, I_0^{\alpha_{n-2}}x(s), I_0^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\geq c^{\alpha-\alpha_{n-2}-1}(b_2 + \epsilon_2) \int_c^{1-c} J_1(s)\varrho_2c_1 \min(x(s)) ds - c^{\alpha-\alpha_{n-2}-1}c_2 \int_c^{1-c} J_1(s) ds \\ &\geq c^{\alpha-\alpha_{n-2}-1}b_2m_1\varrho_2c_1 \min_{s \in [c, 1-c]} x(s) - c^{\alpha-\alpha_{n-2}-1}c_2m_1 \\ &\geq c^{\alpha-\alpha_{n-2}-1}b_2m_1\varrho_2c_1 \min_{s \in [c, 1]} x(s) - c^{\alpha-\alpha_{n-2}-1}c_2m_1 \\ &\geq c^{\alpha-\alpha_{n-2}-1}b_2m_1\varrho_2c_1c^{\alpha-\alpha_{n-2}-1}\|x\| - \tilde{c}_2 \\ &\geq 2\|x\| - \tilde{c}_2, \end{aligned}$$

where $\tilde{c}_2 = c^{\alpha-\alpha_{n-2}-1}c_2m_1$. We can choose $R \geq \max\{\tilde{c}_2, 1\}$, then we deduce

$$\|Ax\| \geq \|x\|, \quad \forall x \in \partial B_R \cap P_0. \tag{3.6}$$

By (3.3), (3.6), and the Guo–Krasnosel'skii fixed point theorem, we conclude that A has a fixed point $x \in (\bar{B}_R \setminus B_{r_1}) \cap P_0$, that is, $r_1 \leq \|x\| \leq R$. By Lemma 2.8, we obtain that $x(t) > 0$ for all $t \in (0, 1)$. By Lemma 2.7, we obtain that BVP (1.1) has at least one positive solution. Therefore, the proof of Theorem 3.1 is completed. \square

Theorem 3.2 *Assume that (H_1) , (H_2) , (H_5) , and (H_6) hold, then fractional BVP (1.1) has at least one positive solution.*

Proof From (H_5) , for fixed $b_3 > \epsilon_3 > 0$, there exists $c_3 > 0$ such that

$$\begin{aligned} q(t, x_0, x_1, \dots, x_{n-2}) &\leq (b_3 - \epsilon_3)(\varsigma_0x_0 + \varsigma_1x_1 + \dots + \varsigma_{n-2}x_{n-2}) + c_3, \quad \forall t \in [0, 1], x_i \geq 0. \end{aligned} \tag{3.7}$$

By using of (3.7) and (H_2) , for any $x \in P_0$, $t \in [0, 1]$, we conclude

$$\begin{aligned} |Ax(t)| &\leq \int_0^1 [J_1(s) + J_2(s) + J_3(s)]p(s)q(s, I_0^{\alpha_{n-2}}x(s), I_0^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\leq \int_0^1 [J_1(s) + J_2(s) + J_3(s)]p(s)[(b_3 - \epsilon_3)(\varsigma_0I_0^{\alpha_{n-2}}x(s) + \varsigma_1I_0^{\alpha_{n-2}-\alpha_1}x(s) + \dots \\ &\quad + \varsigma_{n-2}x(s)) + c_3] ds \\ &\leq b_3(Q_1 + Q_2 + Q_3)\varrho_3\|x\| + (Q_1 + Q_2 + Q_3)c_3 \\ &\leq 2^{-1}\|x\| + \tilde{c}_3, \end{aligned}$$

where $\tilde{c}_3 = (Q_1 + Q_2 + Q_3)c_3$. We can choose large $\tilde{R} > \max\{2\tilde{c}_3, 1\}$ such that

$$\|Ax\| \leq \|x\|, \quad \forall x \in \partial B_{\tilde{R}} \cap P_0. \tag{3.8}$$

From (H_6) , for small enough $\epsilon_4 > 0$, there exists $\tilde{r} \in (0, 1]$ such that

$$f(t, x_0, x_1, \dots, x_{n-2}) \geq (b_4 + \epsilon_4)(\vartheta_0 x_0 + \vartheta_1 x_1 + \dots + \vartheta_{n-2} x_{n-2})^{\hat{a}} \tag{3.9}$$

for all $t \in [c, 1 - c]$, $x_i \geq 0$, $\sum_{i=0}^{n-2} \vartheta_i x_i \leq \tilde{r}$. Taking $\tilde{r}_1 < \min\{\frac{\tilde{r}}{\varrho_4}, \tilde{r}\}$. For any $x \in \partial B_{\tilde{r}_1} \cap P_0$ and $t \in [0, 1]$, from (3.2), we know that

$$\vartheta_0 I_{0^+}^{\alpha_{n-2}} x(s) + \vartheta_1 I_{0^+}^{\alpha_{n-2}-\alpha_1} x(s) + \dots + \vartheta_{n-2} x(s) \leq \varrho_4 \tilde{r}_1 < \tilde{r}.$$

Thus, by using of (3.9), for any $x \in \partial B_{\tilde{r}_1} \cap P_0$ and $t \in [c, 1 - c]$, we have

$$\begin{aligned} Ax(t) &\geq \int_c^{1-c} G(t,s) f(s, I_{0^+}^{\alpha_{n-2}} x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1} x(s), \dots, x(s)) ds \\ &\geq c^{\alpha-\alpha_{n-2}-1} \int_c^{1-c} J_1(s) (b_4 + \epsilon_4) (\vartheta_0 I_{0^+}^{\alpha_{n-2}} x(s) + \vartheta_1 I_{0^+}^{\alpha_{n-2}-\alpha_1} x(s) + \dots \\ &\quad + \vartheta_{n-2} x(s))^{\hat{a}} ds \\ &\geq c^{\alpha-\alpha_{n-2}-1} b_4 \int_c^{1-c} J_1(s) [\varrho_4 c_1 \min(x(s))]^{\hat{a}} ds \\ &= c^{\alpha-\alpha_{n-2}-1} b_4 (\varrho_4 c_1)^{\hat{a}} m_1 \left[\min_{s \in [c, 1-c]} x(s) \right]^{\hat{a}} \\ &\geq c^{\alpha-\alpha_{n-2}-1} b_4 (\varrho_4 c_1)^{\hat{a}} m_1 \left[\min_{s \in [c, 1]} x(s) \right]^{\hat{a}} \\ &\geq c^{\alpha-\alpha_{n-2}-1} b_4 (\varrho_4 c_1)^{\hat{a}} m_1 (c^{\alpha-\alpha_{n-2}-1})^{\hat{a}} \|x\| \\ &\geq \|x\|. \end{aligned}$$

Therefore

$$\|Ax\| \geq \|x\|, \quad \forall x \in \partial B_{\tilde{r}_1} \cap P_0. \tag{3.10}$$

By (3.8), (3.10) and the Guo-Krasnosel'skii fixed point theorem, we deduce that A has at least one point $x \in (\overline{B_{\tilde{r}}} \setminus B_{\tilde{r}_1}) \cap P_0$, that is, $\tilde{r}_1 \leq \|x\| \leq \tilde{r}$. By Lemma 2.7, we obtain that BVP (1.1) has at least one positive solution. Therefore, the proof of Theorem 3.2 is completed. \square

Theorem 3.3 *Assume that (H_1) , (H_2) , (H_4) , (H_6) , and (H_7) are satisfied, then BVP (1.1) has at least two positive solutions.*

Proof Firstly, when (H_1) , (H_2) , (H_4) hold, by the proof of Theorem 3.1, we know that there exists $R > 1$ such that

$$\|Ax\| \geq \|x\|, \quad \forall x \in \partial B_R \cap P_0. \tag{3.11}$$

Again, when (H_1) , (H_2) , (H_6) hold, by the proof of Theorem 3.2, we know that there exists $\tilde{r}_1 < 1$ such that

$$\|Ax\| \geq \|x\|, \quad \forall x \in \partial B_{\tilde{r}_1} \cap P_0. \tag{3.12}$$

On the other hand, let $\Omega = \{x \in P : \|x\| = 1\}$. By (H_7) , for any $x \in \partial\Omega \cap P_0$ and $t \in [0, 1]$, we have

$$\begin{aligned} Ax(t) &= \int_0^1 G(t,s)f(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\leq \int_0^1 G(t,s)p(s)q(s, I_{0^+}^{\alpha_{n-2}}x(s), I_{0^+}^{\alpha_{n-2}-\alpha_1}x(s), \dots, x(s)) ds \\ &\leq M_q \max_{t \in [0,1]} \int_0^1 G(t,s)p(s) ds \\ &< 1 = \|x\|. \end{aligned}$$

Consequently,

$$\|Ax\| < \|x\|, \quad \forall x \in \partial\Omega \cap P_0. \tag{3.13}$$

Therefore, from (3.12) and (3.13) and Lemma 2.9, it follows that BVP (2.10) has one positive solution x_1^* with $\tilde{r}_1 \leq \|x_1^*\| < 1$. In the same way, from (3.11) and (3.13) and Lemma 2.10, it follows that BVP (2.9) has another positive solution x_2^* with $1 \leq \|x_2^*\| \leq R$. By Lemma 2.7, we obtain that BVP (1.1) has at least two positive solutions. Therefore, the proof of Theorem 3.3 is completed. \square

4 Examples

Example 4.1 We consider the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{7}{2}}u(t) + \frac{(u(t)+D_{0^+}^{\frac{1}{2}}u(t)+D_{0^+}^{\frac{11}{10}}u(t))^{a_0}}{(1-t)^{\tau_1}} = 0, & t \in (0, 1), \\ u(0) = D_{0^+}^{\frac{3}{4}}u(0) = D_{0^+}^{\frac{8}{5}}u(0) = 0, \\ D_{0^+}^{\frac{5}{2}}u(1) = \int_0^{\frac{2}{3}} D_{0^+}^{\frac{49}{20}}u(s) dA(s) + \int_0^1 D_{0^+}^{\frac{12}{5}}u(s) dA(s), \end{cases} \tag{4.1}$$

where $\alpha = \frac{7}{2}, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{11}{10}, \gamma_1 = \frac{3}{4}, \gamma_2 = \frac{8}{5}, \beta_1 = \frac{5}{2}, \beta_2 = \frac{49}{20}, \beta_3 = \frac{12}{5}, h(s) = a(s) = 1, s \in [0, 1], \eta = \frac{2}{3}$, and

$$\begin{aligned} A(t) &= \begin{cases} 0, & t \in [0, \frac{1}{2}), \\ \frac{1}{100}, & t \in [\frac{1}{2}, 1], \end{cases} \\ f(t, x, y, z) &= \frac{(x + y + z)^{a_0}}{(1-t)^{\tau_1}}, \quad t \in (0, 1), x, y, z \geq 0, \end{aligned}$$

with $a_0 > 1$ and $\tau_1 \in (0, 1)$. Let $u(t) = I_{0^+}^{\frac{11}{10}}x(t)$, the equation can be changed to the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{12}{5}}x(t) + f(t, I_{0^+}^{\frac{11}{10}}x(t), I_{0^+}^{\frac{3}{5}}x(t), x(t)) = 0, & t \in (0, 1), \\ D_{0^+}^{\frac{1}{2}}x(0) = 0, \quad D_{0^+}^{\frac{7}{5}}x(1) = \int_0^{\frac{2}{3}} D_{0^+}^{\frac{27}{20}}x(s) dA(s) + \int_0^1 D_{0^+}^{\frac{13}{10}}x(s) dA(s). \end{cases} \tag{4.2}$$

Here, $f(t, x, y, z) = p(t)q(t, x, y, z)$, where $p(t) = \frac{1}{(1-t)^{\tau_1}}$ for all $t \in (0, 1)$ and $q(t, x, y, z) = (x + y + z)^{a_0}$ for all $t \in (0, 1), x, y, z \geq 0$, we have $\int_0^1 p(s) ds < +\infty$. By direct calculation, $l_1 =$

$$\int_0^{\frac{2}{3}} t^{\frac{1}{20}} dA(t) \approx 0.00965936 > 0, l_2 = \int_0^1 t^{\frac{1}{10}} dA(t) \approx 0.0093303299 > 0,$$

$$\begin{aligned} \Delta &= \frac{1}{\Gamma(\alpha - \beta_1)} - \frac{1}{\Gamma(\alpha - \beta_2)} l_1 - \frac{1}{\Gamma(\alpha - \beta_3)} l_2 \\ &> 1 - \frac{1}{\Gamma(\frac{21}{20})} l_1 - \frac{1}{\Gamma(\frac{11}{10})} l_2 \\ &\approx 0.980270282 > 0, \end{aligned}$$

$$\begin{aligned} &\frac{\Gamma(\alpha - \beta_1)}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t)t^{\alpha-\beta_2-1} dA(t) + \frac{\Gamma(\alpha - \beta_1)}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t)t^{\alpha-\beta_3-1} dA(t) \\ &= \frac{\Gamma(1)}{\Gamma(\frac{21}{20})} \int_0^{\frac{2}{3}} t^{\frac{1}{20}} dA(t) + \frac{\Gamma(1)}{\Gamma(\frac{11}{10})} \int_0^1 t^{\frac{1}{10}} dA(t) \\ &\approx 0.019729718 \\ &< 1. \end{aligned}$$

In (H_3) for $a = 1, k_0 = 3, k_1 = 4, k_2 = 5$, we obtain $q_0 = 0$. In (H_4) , for $c \in (0, \frac{1}{2}), \mu_0 = \mu_1 = \mu_2 = \frac{1}{2}$, we have $f_\infty = +\infty$. Then, by Theorem 3.1, we deduce that BVP (1.1) has at least one positive solution.

Example 4.2 We consider the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{7}{2}} u(t) + \frac{e^{u(t)+D_{0^+}^{\frac{1}{2}} u(t)+D_{0^+}^{\frac{11}{10}} u(t)}}{(1-t)^{\tau_1}} = 0, & t \in (0, 1), \\ u(0) = D_{0^+}^{\frac{3}{4}} u(0) = D_{0^+}^{\frac{8}{5}} u(0) = 0, \\ D_{0^+}^{\frac{5}{2}} u(1) = \int_0^{\frac{2}{3}} D_{0^+}^{\frac{49}{20}} u(s) dA(s) + \int_0^1 D_{0^+}^{\frac{12}{5}} u(s) dA(s), \end{cases} \tag{4.3}$$

where $\alpha = \frac{7}{2}, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{11}{10}, \gamma_1 = \frac{3}{4}, \gamma_2 = \frac{8}{5}, \beta_1 = \frac{5}{2}, \beta_2 = \frac{49}{20}, \beta_3 = \frac{12}{5}, h(s) = a(s) = 1, s \in [0, 1], \eta = \frac{2}{3}$, and

$$\begin{aligned} A(t) &= \begin{cases} 0, & t \in [0, \frac{1}{2}), \\ \frac{1}{100}, & t \in [\frac{1}{2}, 1], \end{cases} \\ f(t, x, y, z) &= \frac{e^{x+y+z}}{(1-t)^{\tau_1}}, \quad t \in (0, 1), x, y, z \geq 0, \end{aligned}$$

with $\tau_1 \in (0, 1)$. Let $u(t) = I_{0^+}^{\frac{11}{10}} x(t)$, the equation can be changed to the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{12}{5}} x(t) + f(t, I_{0^+}^{\frac{11}{10}} x(t), I_{0^+}^{\frac{3}{5}} x(t), x(t)) = 0, & t \in (0, 1), \\ D_{0^+}^{\frac{1}{2}} x(0) = 0, \quad D_{0^+}^{\frac{7}{5}} x(1) = \int_0^{\frac{2}{3}} D_{0^+}^{\frac{27}{20}} x(s) dA(s) + \int_0^1 D_{0^+}^{\frac{13}{10}} x(s) dA(s). \end{cases} \tag{4.4}$$

Here, $f(t, x, y, z) = p(t)q(t, x, y, z)$, where $p(t) = \frac{1}{(1-t)^{\tau_1}}$ for all $t \in (0, 1)$ and $q(t, x, y, z) = e^{x+y+z}$ for all $t \in (0, 1), x, y, z \geq 0$, we have $\int_0^1 p(s) ds < +\infty$. By direct calculation, $l_1 =$

$$\int_0^{\frac{2}{3}} t^{\frac{1}{20}} dA(t) \approx 0.00965936 > 0, l_2 = \int_0^1 t^{\frac{1}{10}} dA(t) \approx 0.0093303299 > 0,$$

$$\Delta = \frac{1}{\Gamma(\alpha - \beta_1)} - \frac{1}{\Gamma(\alpha - \beta_2)} l_1 - \frac{1}{\Gamma(\alpha - \beta_3)} l_2 > 1 - \frac{1}{\Gamma(\frac{21}{20})} l_1 - \frac{1}{\Gamma(\frac{11}{10})} l_2 \approx 0.980270282 > 0,$$

$$\begin{aligned} & \frac{\Gamma(\alpha - \beta_1)}{\Gamma(\alpha - \beta_2)} \int_0^\eta h(t)t^{\alpha-\beta_2-1} dA(t) + \frac{\Gamma(\alpha - \beta_1)}{\Gamma(\alpha - \beta_3)} \int_0^1 a(t)t^{\alpha-\beta_3-1} dA(t) \\ &= \frac{\Gamma(1)}{\Gamma(\frac{21}{20})} \int_0^{\frac{2}{3}} t^{\frac{1}{20}} dA(t) + \frac{\Gamma(1)}{\Gamma(\frac{11}{10})} \int_0^1 t^{\frac{1}{10}} dA(t) \approx 0.019729718 < 1, \end{aligned}$$

$$L_1(t, s) = \frac{1}{\Gamma(\frac{12}{5})} \begin{cases} t^{\frac{7}{5}} - (t-s)^{\frac{7}{5}}, & 0 \leq s \leq t \leq 1, \\ t^{\frac{7}{5}}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$L_2(t, s) = \frac{1}{\Gamma(\frac{12}{5})} \begin{cases} t^{\frac{1}{20}} - (t-s)^{\frac{1}{20}}, & 0 \leq s \leq t \leq 1, \\ t^{\frac{1}{20}}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$L_3(t, s) = \frac{1}{\Gamma(\frac{12}{5})} \begin{cases} t^{\frac{1}{10}} - (t-s)^{\frac{1}{10}}, & 0 \leq s \leq t \leq 1, \\ t^{\frac{1}{10}}, & 0 \leq t \leq s \leq 1. \end{cases}$$

$$H_2(t, s) = \frac{t^{\frac{7}{5}}}{\Gamma(\frac{21}{20})} \Delta^{-1} \int_0^{\frac{2}{3}} L_2(t, s) dA(t), \quad H_3(t, s) = \frac{t^{\frac{7}{5}}}{\Gamma(\frac{11}{10})} \Delta^{-1} \int_0^1 L_3(t, s) dA(t),$$

$$H_1(t, s) = L_1(t, s), \quad G(t, s) = H_1(t, s) + H_2(t, s) + H_3(t, s).$$

In (H_4) , for $\mu_0 = \mu_1 = \mu_2 = 5$, we obtain

$$f_\infty = \liminf_{\sum_{i=0}^2 \mu_i x_i \rightarrow \infty} \min_{t \in [c, 1-c]} \frac{f(t, x_0, x_1, x_2)}{\mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2} = \frac{e^{x_0+x_1+x_2}}{5(1-t)^{\tau_1}(x_0+x_1+x_2)} = \infty.$$

In (H_6) , for $c \in (0, \frac{1}{2})$, $\vartheta_0 = \vartheta_1 = \vartheta_2 = 4$, $\hat{a} = 1$, we obtain

$$f_0 = \liminf_{\sum_{i=0}^2 \vartheta_i x_i \rightarrow 0} \min_{t \in [c, 1-c]} \frac{f(t, x_0, x_1, x_2)}{(\vartheta_0 x_0 + \vartheta_1 x_1 + \vartheta_2 x_2)^{\hat{a}}} = \frac{e^{x_0+x_1+x_2}}{4(1-t)^{\tau_1}(x_0+x_1+x_2)} = \infty.$$

Choose $0 < M_q < \frac{1}{\max_{t \in [0,1]} \int_0^1 G(t,s) \frac{1}{(1-s)^{\tau_1}} ds}$, we see that all the conditions of Theorem 3.3 are satisfied. Thus, by Theorem 3.3, we deduce that BVP (1.1) has at least two positive solutions.

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