Advances in Difference Equations
a SpringerOpen Journal

Reconstruction of nonlinear integral inequalities associated with time scales calculus

Zareen A. Khan[1*]

[*]Correspondence:
dr.zareenkhan@ymail.com
[1]Department of Mathematics,
College of Science, Princess Nourah
bint Abdulrahman University,
Riyadh, Saudi Arabia

Abstract

In this paper, we build up some generalizations of nonlinear integral inequalities and recreate the results of some Pachpatte's inequalities on time scales. We not just settle new estimated bounds of a particular class of nonlinear retarded dynamic inequalities, but additionally determine and unify continuous analogs alongside a subjective time scale $\mathbb{T}$. We demonstrate applications of the treated inequalities to reflect the benefits of our work. The key effects will be proven by using the analysis procedure and the standard time-scale comparison theorem technique.

MSC: 35A23; 26E70; 34N05

Keywords: Integral Pachpatte's inequalities; Time scales; Dynamic equations

1 Introduction

A dynamic system containing discrete and continuous times is an important tool for modeling real-world problems. It is fair to check if a structure can be given that helps us to integrate all dynamic systems simultaneously to gain some perspective and a superior comprehension of the contrasts between discrete and continuous domains. To counter this, a concept was composed by Hilger [1]. The primary target of dynamic equations on time scales is that they construct a connection between continuous and discrete situations. A while later, this perception was evolved by many researchers [2–4].

Over the most recent couple of years, great efforts have been made to unify and expand integral inequalities on time scales [5–12]. These essential inequalities are promoted in numerous classifications for the boundedness, uniqueness, and the solutions of various dynamic equations [13–16].

Linear and nonlinear versions of Pachpatte's inequalities on time scale have been a matter of conversation for quite a while. These inequalities were advanced by means of several authors [17–22]. Bohner has planned an assortment of dynamic inequalities, which are basically founded on the inequality of Gronwall. Originally, Bohner et al. [23] unify the continuous-type Gronwall inequality as follows

$$x(l) \leq b(l) + \int_{l_0}^{l} j(l_1)x(l_1)\Delta l_1, \quad l \in \mathbb{T},$$

Springer

where j is a right-dense continuous function, $x \geq 0$ is a regressive right-dense continuous function, and $\mathbb{T}$ is a time scale. Bohner et al. [24] further suggested the integral inequality on time scales

$$x(l) \leq a(l) + p(l) \int_{l_0}^{l} \left[b(l_1)x(l_1) + q(l_1) \right] \Delta l_1.$$

After that in 2010, Li [25] considered the nonlinear integral inequality of one independent variable associated with time scales

$$x^{\gamma}(l) \leq a(l) + c(l) \int_{l_0}^{l} \left[f(l_1)x(\rho(l_1)) + n(l_1) \right] \Delta l_1$$

for $l \in l_0$ with initial conditions $x(l) = \Omega(l)$, $l \in [\beta, l_0] \cap \mathbb{T}$, $\Upsilon(\rho(l)) \leq (a(l))^{\frac{1}{\gamma}}$ for $l \in l_0$, $\rho(l) \leq l_0$, where $\gamma \geq 1$ is a constant, $\rho(l) \leq l$, $-\infty < \beta = \inf\{\rho(l), l \in \mathbb{T}_0\} \leq l_0$, and $\Omega(l) \in C_{rd}([\beta, l_0] \cap \mathbb{T}, \mathbb{R}_+)$. Meanwhile, Pachpatte [26] stepped forward to discover the extension of the integral inequality of the form

$$x(l) \leq a(l) + \int_{l_0}^{l} [f(l_1) \left[x(l_1) + \int_{l_0}^{l_1} m(l_1, h_1)x(h_1)\Delta h_1 \right] \Delta l_1$$

such that $m(l_1, h_1) \geq 0$, $m^{\Delta}(l_1, h_1) \geq 0$ for $l, h_1 \in \mathbb{T}$ and $h_1 \leq l$. Later, Meng et al. [27] inquired the expansion of the nonlinear integral inequality on time scales as follows:

$$x(l) \leq x_0 + \int_{l_0}^{l} f(l_1) \left[x(l_1) + \int_{l_0}^{l_1} j(h_1)x(h_1)\Delta h_1 \right] \Delta l_1 + \int_{l_0}^{\alpha} s(l_1)x(l_1)\Delta l_1$$

with $\alpha > l_0$. Recently, in 2017, Haidong [28] proved the retarded Volterra–Fredholm integral inequality on time scales

$$x(l) \leq a(l) + b(l) \int_{\rho(l_0)}^{\rho(l)} \left[f_1(l_1)x(l_1) + f_2(l_1) \int_{\rho(l_0)}^{l_1} j(h_1)x(h_1)\Delta h_1 \right] \Delta l_1$$

$$+ \lambda b(T) \int_{\rho(l_0)}^{\rho(T)} \left[f_1(l_1)x(l_1) + f_2(l_1) \int_{\rho(l_0)}^{l_1} j(h_1)x(h_1)\Delta h_1 \right] \Delta l_1,$$

where $\lambda \geq 0$. To delineate the hypothetical theorems, it has been demonstrated that the acquired inequalities can be utilized as significant apparatuses in the investigation of specific properties of dynamic equations on time scales.

Moreover, Nasser et al. [29] introduced some new generalizations and rectifications of many known results of Pachpatte kind, consolidating two nonlinear integral terms on time scales. These acquired consequences played a crucial role in reading a few lessons of integral and integro-differential equations.

Often, the previously noted inequalities are not practical directly in the evaluation of certain retarded differential and integral equations. Therefore it is alluring to discover a few new estimates in which the nonretarded term l is changed to the retarded argument $\rho(l)$ in specific circumstances. To overcome this hollow, primarily based on the expertise of the research mentioned, in this text, we are able to seek the nonlinear dynamic inequalities constructed up for the solution of the integral inequalities and unifying some known results in the literature.

At the point when we want to examine certain properties of a differential equation, these types of inequalities have many applications (see[30–32]). Around the completion of this paper, we discuss several applications to investigate the uniqueness and global existence of solutions of nonlinear delay dynamic integral equations.

The remaining portions of the document are structured as follows. In Sect. 2, we describe major realities and fundamental lemmas that are key devices for our primary results. Theoretical conversations on nonlinear dynamic Pachpatte's inequalities on general time scales with some finishing remarks are committed in Sect. 3. The final section accomplishes the applications of the abstract results.

2 Preliminaries on time scales

A time scale $\mathbb{T}$ is a nonempty closed subset of the real line $\mathbb{R}$. For $l \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $\sigma(l) = \inf\{n \in \mathbb{T} : n > l\}$, the backward jump operator $\varsigma : \mathbb{T} \to \mathbb{R}$ by $\varsigma(l) = \sup\{n \in \mathbb{T} : n < l\}$ and the graininess function $\psi : \mathbb{T} \to [0, \infty)$ by $\psi(l) = \sigma(l) - l$. An element $l \in \mathbb{T}$ is said to be right-dense if $\sigma(l) = l$ and right-scattered if $\sigma(l) > l$, left-dense if $\varsigma(l) = l$ and left-scattered if $\varsigma(l) < l$. The set $\mathbb{T}^k$ is defined to be $\mathbb{T}$ if it has a left-scattered maximum g, then $\mathbb{T}^k = \mathbb{T} - \{g\}$ otherwise, $\mathbb{T}^k = \mathbb{T}$. $\Re$ is the set of all regressive and rd-continuous functions, and $\Re^+ = \{y \in \Re : 1 + \psi(l)y(l) > 0, l \in \mathbb{T}\}$.

On time scales, the reader is supposed to be acquainted with the skills and basic ideas about the analytics given by Bohner [3]. Next, we give some basic lemmas on time scales which will be required in the evidence of the exhibited paper.

Lemma 2.1 ([20]) *If j, h are delta differentiable at l, then jh is also delta differentiable at l, and*

$$(jh)^\Delta(l) = j^\Delta(l)h(l) + j\big(\sigma(l)\big)h^\Delta(l).$$

Lemma 2.2 ([19]) *Let $l_0 \in \mathbb{T}^k$, and let $j : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ be continuous at (l, l), where $l > l_0$ and $l \in \mathbb{T}^k$. Assume that $j^\Delta(l, \cdot)$ is rd-continuous on $[l_0, \sigma(l)]_\mathbb{T}$. Suppose that, for every $\epsilon > 0$, there exists a neighborhood Ω of l, independent of $\eta \in [l_0, \sigma(l)]_\mathbb{T}$, such that*

$$\big|\big[j\big(\sigma(l), \eta\big) - j(l_1, \eta)\big] - j^\Delta(l, \eta)\big[\sigma(l) - l_1\big]\big| \le \epsilon\big|\sigma(l) - l_1\big|, \quad l_1 \in \Omega,$$

where j^Δ be the derivative of j with respect to the first variable. Then $x(l) = \int_{l_0}^l j(l, \eta)\Delta\eta$ yields

$$x^\Delta(l) = \int_{l_0}^l j^\Delta(l, \eta)\Delta\eta + j\big(\sigma(l), l\big).$$

Lemma 2.3 ([23]) *Chain Rule* 1: *Let $j : \mathbb{R} \to \mathbb{R}$ be differentiable and suppose that $h : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $j \circ h : \mathbb{T} \to \mathbb{R}$ is delta differentiable, and*

$$(j \circ h)^\Delta(l) = \left\{\int_0^1 \big[j'\big(h(l)\big) + y\psi(l)h^\Delta(l)\big] \, dy\right\}h^\Delta(l).$$

Chain Rule 2: *Assume that $j : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\mathbb{T}^* = j(\mathbb{T})$ is a time scale. Let $v : \mathbb{T}^* \to \mathbb{R}$ and $j^\Delta(l), v^\Delta(j(l))$ exist for $l \in \mathbb{T}^k$. Then*

$$(j \circ v)^\Delta = \big(j^\Delta \circ v\big)v^\Delta.$$

Lemma 2.4 ([32]) *Let $j \in C_{rd}$ and $l \in \mathbb{T}^k$. Then*

$$\int_l^{\sigma(l)} j(\tau)\Delta\tau = \psi(l)j(l) = j(l)\big(\sigma(l) - l\big).$$

Lemma 2.5 ([19]) *If $j \in \Re$ and $l \in \mathbb{T}$, then the exponential function $e_j(l, l_0)$ is the unique solution of the initial value problem*

$$\begin{cases} x^\Delta(l) = j(l)x(l), \\ x(l_0) = 1. \end{cases}$$

3 Results and discussion

Without compromising nonspecific statements, throughout in this task, we denote $\mathbb{R}_+ = [0, \infty)$ and $l_0 \in \mathbb{T}$, $l_0 \geq 0$, $\mathbb{T}_0 = [l_0, \infty) \cap \mathbb{T}$.

To demonstrate our elementary results, we first rundown the accompanying suppositions:

(P1) The functions $j(l, l_1), j^\Delta(l, l_1), h(l, l_1), h^\Delta(l, l_1), m(l, l_1), m^\Delta(l, l_1) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+)$.

(P2) $x \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$.

(P3) $\varrho_i \in C_{rd}(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$, are continuous nondecreasing functions with $\varrho_i(l) > 0$ for $l > 0$.

(P4) The function $\rho \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ is strictly increasing.

(P5) $b \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$.

We now present the principle lemma and theorems.

Lemma 3.1 *Let $a \in C_{rd}$, $l \in \mathbb{T}_k$, and let $\rho(l) \in C_{rd}$ be a strictly increasing function for $l \in \mathbb{T}$. Then*

$$\int_{\sigma(l)}^{\rho(\sigma(l))} a\big(\sigma(l), \lambda\big)\Delta\lambda = a\big(\sigma(l), \rho(l)\big)\big(\rho\big(\sigma(l)\big) - \rho(l)\big).$$

Proof If A is the antiderivative of a and $A^{\Delta\lambda}(\sigma(l), \lambda) = a(\sigma(l), \lambda)$, then

$$\begin{aligned} \int_{\sigma(l)}^{\rho(\sigma(l))} a\big(\sigma(l), \lambda\big)\Delta\lambda &= A\big(\sigma(l), \rho\big(\rho(l)\big)\big) - A\big(\sigma(l), \rho(l)\big) \\ &= \frac{A(\sigma(l), \rho(\rho(l))) - A(\sigma(l), \rho(l))}{\rho(\sigma(l)) - \rho(l)}\big(\rho\big(\sigma(l)\big) - \rho(l)\big) \\ &= A^{\Delta\lambda}\big(\sigma(l), \lambda\big)|_{\lambda=\rho(l)}\big(\rho\big(\sigma(l)\big) - \rho(l)\big) \\ &= a\big(\sigma(l), \rho(l)\big)\big(\rho\big(\sigma(l)\big) - \rho(l)\big). \qquad\square \end{aligned}$$

Theorem 3.2 *Suppose that suppositions* (P1)–(P5) *with $\varrho_1^\Delta(l) = \varrho_2(l)$ and the inequality*

$$\begin{aligned} \varrho_1\big(x(l)\big) \leq{}& b(l) + \int_{l_0}^{\rho(l)} j(l, l_1)\varrho_2\big(x(l_1)\big) \\ &\times \left[x(l_1) + \int_{l_0}^{\rho(l_1)} h(l_1, q_1)\varrho_1\big(x(q_1)\big)\Delta q_1 \int_{l_0}^{\rho(l_1)} m(l_1, q_1)\varrho_1\big(x(q_1)\big)\Delta q_1 \right]^\xi \Delta l_1, \\ & l \in \mathbb{T}_0, \end{aligned} \tag{1}$$

are satisfied. Then

$$x(l) \le \varrho_1^{-1}\big(b(l)\big) + \int_{l_0}^{\rho(l)} j(l, l_1) \bigg\{ \Lambda^{-1}\bigg(\Lambda\bigg[\varrho_1^{\xi-1}\big(b(l)\big)$$

$$+ (1 - \xi) \int_{l_0}^{\rho(l_1)} j(l_1, q_1) \Delta q_1 \bigg]^{\frac{1}{1-\xi}} + \int_{l_0}^{\rho(l_1)} \big(h(l_1, q_1) m(l_1, q_1)\big) \Delta q_1 \bigg) \bigg\}^{\xi} \Delta l_1, \qquad (2)$$

where $\xi \ne 1$,

$$\Lambda(v) = \int_{v_0}^{v} \frac{\Delta p}{\varrho_1^2(p)}, \quad v \ge v_0 > 0, \Lambda(+\infty) = +\infty, \qquad (3)$$

Λ^{-1} is the inverse function of Λ, and L_1 is the largest number for all $l < L_1$ with

$$\Lambda\bigg[\varrho_1^{\xi-1}\big(b(l)\big) + (1 - \xi) \int_{l_0}^{\rho(l)} j(l, l_1) \Delta l_1 \bigg]^{\frac{1}{1-\xi}}$$

$$+ \int_{l_0}^{\rho(l)} \big(h(l, l_1) m(l, l_1)\big) \Delta l_1 \in \mathrm{Dom}\big(\Lambda^{-1}\big). \qquad (4)$$

Proof Fix an arbitrary $l^* \in \mathbb{T}_0$ for $l \in [l_0, l^*] \cap \mathbb{T}$ and denote by $\varrho_1(J(l))$ the function on the right side of (1), which is nonnegative and nondecreasing. Therefore

$$\varrho_1\big(J(l)\big)$$

$$= b\big(l^*\big) + \int_{l_0}^{\rho(l)} j(l, l_1) \varrho_2\big(x(l_1)\big)$$

$$\times \bigg[x(l_1) + \int_{l_0}^{\rho(l_1)} h(l_1, q_1) \varrho_1\big(x(q_1)\big) \Delta q_1 \int_{l_0}^{\rho(l_1)} m(l_1, q_1) \varrho_1\big(x(q_1)\big) \Delta q_1 \bigg]^{\xi} \Delta l_1 \qquad (5)$$

and

$$J(l_0) = \varrho_1^{-1}\big(b\big(l^*\big)\big), \qquad (6)$$

so that by (1)

$$x(l) \le J(l), \quad l \in \mathbb{T}_0. \qquad (7)$$

Equation (5) by Lemma 2.2 and delta derivative with respect to l imply that

$$\varrho_1^{\Delta}\big(J(l)\big) J^{\Delta}(l) = \bigg\{ \int_{l_0}^{\rho(l)} j^{\Delta}(l, l_1) \varrho_2\big(x(l_1)\big) \Delta l_1 + j\big(\sigma(l), \rho(l)\big) \rho^{\Delta}(l) \varrho_2\big(x(\rho(l))\big) \bigg\}$$

$$\times \bigg[x(l) + \int_{l_0}^{\rho(l)} h(l, q_1) \varrho_1\big(x(q_1)\big) \Delta q_1 \int_{l_0}^{\rho(l)} m(l, q_1) \varrho_1\big(x(q_1)\big) \Delta q_1 \bigg]^{\xi}$$

$$\le \bigg\{ \int_{l_0}^{\rho(l)} j(l, l_1) \Delta l_1 \bigg\}^{\Delta} \varrho_2\big(J(l)\big)$$

$$\times \bigg[J(l) + \int_{l_0}^{\rho(l)} h(l, q_1) \varrho_1\big(J(q_1)\big) \Delta q_1 \int_{l_0}^{\rho(l)} m(l, q_1) \varrho_1\big(J(q_1)\big) \Delta q_1 \bigg]^{\xi},$$

since $\varrho_1^\Delta(J(l)) = \varrho_2(J(l))$. This inequality becomes

$$
\begin{aligned}
J^\Delta(l) &\leq \left\{ \int_{l_0}^{\rho(l)} j(l, l_1) \Delta l_1 \right\}^\Delta \\
&\quad \times \left[J(l) + \int_{l_0}^{\rho(l)} h(l, q_1) \varrho_1\big(J(q_1)\big) \Delta q_1 \int_{l_0}^{\rho(l)} m(l, q_1) \varrho_1\big(J(q_1)\big) \Delta q_1 \right]^\xi \\
&\leq \left\{ \int_{l_0}^{\rho(l)} j(l, l_1) \Delta l_1 \right\}^\Delta W^\xi(l),
\end{aligned}
\tag{8}
$$

where

$$
W(l) = J(l) + \int_{l_0}^{\rho(l)} h(l, q_1) \varrho_1\big(J(q_1)\big) \Delta q_1 \int_{l_0}^{\rho(l)} m(l, q_1) \varrho_1\big(J(q_1)\big) \Delta q_1.
\tag{9}
$$

Delta differentiating (9) and utilizing $J(l) \leq W(l)$ and (8), we derive that

$$
\begin{aligned}
W^\Delta(l) &= J^\Delta(l) + \left[\int_{l_0}^{\rho(l)} h(l, q_1) \varrho_1\big(J(q_1)\big) \Delta q_1 \int_{l_0}^{\rho(l)} m(l, q_1) \varrho_1\big(J(q_1)\big) \Delta q_1 \right]^\Delta \\
&\leq \left\{ \int_{l_0}^{\rho(l)} j(l, l_1) \Delta l_1 \right\}^\Delta W^\xi(l) \\
&\quad + \left[\int_{l_0}^{\rho(l)} h(l, q_1) \varrho_1\big(W(q_1)\big) \Delta q_1 \int_{l_0}^{\rho(l)} m(l, q_1) \varrho_1\big(W(q_1)\big) \Delta q_1 \right]^\Delta.
\end{aligned}
\tag{10}
$$

Consider

$$
\begin{aligned}
&\left[\int_{l_0}^{\rho(l)} h(l, q_1) \varrho_1\big(W(q_1)\big) \Delta q_1 \int_{l_0}^{\rho(l)} m(l, q_1) \varrho_1\big(W(q_1)\big) \Delta q_1 \right]^\Delta \\
&= \left\{ \int_{l_0}^{\rho(l)} h^\Delta(l, q_1) \varrho_1\big(W(q_1)\big) \Delta q_1 + h\big(\sigma(l), \rho(l)\big) \rho^\Delta(l) \varrho_1\big(W\big(\rho(l)\big)\big) \right\} \\
&\quad \times \left(\int_{l_0}^{\rho(l)} m(l, q_1) \varrho_1\big(W(q_1)\big) \Delta q_1 \right) \\
&\quad + \left\{ \int_{l_0}^{\rho(l)} m^\Delta(l, q_1) \varrho_1\big(W(q_1)\big) \Delta q_1 + m\big(\sigma(l), \rho(l)\big) \rho^\Delta(l) \varrho_1\big(W\big(\rho(l)\big)\big) \right\} \\
&\quad \times \left(\int_{l_0}^{\rho(\sigma(l))} h\big(\sigma(l), q_1\big) \varrho_1\big(J(q_1)\big) \Delta q_1 \right) \\
&\leq \varrho_1^2\big(W(q_1)\big) \left\{ \int_{l_0}^{\rho(l)} h^\Delta(l, q_1) \Delta q_1 + h\big(\sigma(l), \rho(l)\big) \rho^\Delta(l) \right\} \left(\int_{l_0}^{\rho(l)} m(l, q_1) \Delta q_1 \right) \\
&\quad + \varrho_1\big(W(q_1)\big) \left\{ \int_{l_0}^{\rho(l)} m^\Delta(l, q_1) \Delta q_1 + m\big(\sigma(l), \rho(l)\big) \rho^\Delta(l) \right\} \\
&\quad \times \left(\int_{l_0}^{\rho(\sigma(l))} h\big(\sigma(l), q_1\big) \varrho_1\big(W(q_1)\big) \Delta q_1 \right).
\end{aligned}
\tag{11}
$$

It is easy to observe from Lemma 3.1 that

$$
\int_{l_0}^{\rho(\sigma(l))} h\big(\sigma(l), q_1\big) \varrho_1\big(W(q_1)\big) \Delta q_1
$$

$$= \int_{l_0}^{\rho(l)} h\big(\sigma(l),q_1\big)\varrho_1\big(W(q_1)\big)\Delta q_1 + \int_{\rho(l)}^{\rho(\sigma(l))} h\big(\sigma(l),q_1\big)\varrho_1\big(W(q_1)\big)\Delta q_1$$

$$\leq \varrho_1\big(W(l)\big)\int_{l_0}^{\rho(l)} h\big(\sigma(l),q_1\big)\Delta q_1 + h\big(\sigma(l),\rho(l)\big)\varrho_1\big(\rho(l)\big)\big(\rho\big(\sigma(l)\big)-\rho(l)\big)$$

$$\leq \varrho_1\big(W(l)\big)\bigg[\int_{l_0}^{\rho(l)} h\big(\sigma(l),q_1\big)\Delta q_1 + \int_{l_0}^{\rho(\sigma(l))} h\big(\sigma(l),q_1\big)\Delta q_1\bigg]$$

$$\leq \varrho_1\big(W(l)\big)\int_{l_0}^{\rho(\sigma(l))} h\big(\sigma(l),q_1\big)\Delta q_1. \tag{12}$$

By substituting (12) into (11) we have

$$\bigg[\int_{l_0}^{\rho(l)} h(l,q_1)\varrho_1\big(W(q_1)\big)\Delta q_1 \int_{l_0}^{\rho(l)} m(l,q_1)\varrho_1\big(W(q_1)\big)\Delta q_1\bigg]^{\Delta}$$

$$\leq \varrho_1^2\big(W(l)\big)\bigg\{\int_{l_0}^{\rho(l)} \big(h(l,q_1)m(l,q_1)\big)\Delta q_1\bigg\}^{\Delta}. \tag{13}$$

From (10) and (13) we obtain

$$W^{\Delta}(l) \leq \bigg\{\int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1\bigg\}^{\Delta} W^{\xi}(l) + \varrho_1^2\big(W(l)\big)\bigg\{\int_{l_0}^{\rho(l)} \big(h(l,q_1)m(l,q_1)\big)\Delta q_1\bigg\}^{\Delta}$$

or, equivalently,

$$\frac{W^{\Delta}(l)}{W^{\xi}(l)} \leq \bigg\{\int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1\bigg\}^{\Delta} + \frac{\varrho_1^2(W(l))}{W^{\xi}(l)}\bigg\{\int_{l_0}^{\rho(l)} \big(h(l,q_1)m(l,q_1)\big)\Delta q_1\bigg\}^{\Delta}. \tag{14}$$

Integrating both sides of (14) from l_0 to l and using $W(l_0)=\varrho_1^{-1}(b(l^*))$ and $W(l)>0$ yield the estimate

$$W^{1-\xi}(l) \leq \varrho_1^{\xi-1}\big(b\big(l^*\big)\big) + (1-\xi)\int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1$$

$$+ (1-\xi)\int_{l_0}^{l} \frac{\varrho_1^2(W(l_1))}{W^{\xi}(l_1)}\bigg\{\int_{l_0}^{\rho(l)} \big(h(l_1,q_1)m(l_1,q_1)\big)\Delta q_1\bigg\}^{\Delta}\Delta l_1, \quad \forall l \in \mathbb{T}_0;$$

also,

$$W^{1-\xi}(l) \leq \varrho_1^{\xi-1}\big(b\big(l^*\big)\big) + (1-\xi)\int_{l_0}^{\rho(l^*)} j(l,l_1)\Delta l_1$$

$$+ (1-\xi)\int_{l_0}^{l} \frac{\varrho_1^2(W(l_1))}{W^{\xi}(l_1)}\bigg\{\int_{l_0}^{\rho(l)} \big(h(l_1,q_1)m(l_1,q_1)\big)\Delta q_1\bigg\}^{\Delta}\Delta g. \tag{15}$$

Define the function $R^{1-\xi}(l)$ as the right-hand side of (15). Since $R(l)$ is nondecreasing, we have

$$W(l) \leq R(l), \quad \forall l < L. \tag{16}$$

From (16) by delta differentiating $R^{1-\xi}(l)$ with respect to l we get that

$$R^{1-\xi}(l)R^{\Delta}(l) = \frac{\varrho_1^2(W(l))}{W^{\xi}(l)} \left\{ \int_{l_0}^{\rho(l)} \big(h(l,q_1)m(l,q_1)\big)\Delta q_1 \right\}^{\Delta},$$

which leads to

$$\frac{R^{\Delta}(l)}{\varrho_1^2(R(l))} \leq \left\{ \int_{l_0}^{\rho(l)} \big(h(l,q_1)m(l,q_1)\big)\Delta q_1 \right\}^{\Delta}, \quad \forall l < L. \tag{17}$$

In comparison, for $l \in [l_0, \mathbb{T}] \cap \mathbb{T}$, if $\sigma(l) > l$, then

$$\big[\Lambda(R(l))\big]^{\Delta} = \frac{\Lambda(R(\sigma(l))) - \Lambda(R(l))}{\sigma(l) - l} = \frac{1}{\sigma(l) - l} \int_{R(l)}^{R(\sigma(l))} \frac{1}{\varrho_1^2(v)} \Delta v$$

$$\leq \frac{R(\sigma(l)) - R(l)}{\sigma(l) - l} \frac{1}{\varrho_1^2(R(l))} = \frac{R^{\Delta}(l)}{\varrho_1^2(R(l))}. \tag{18}$$

If $\sigma(l) = l$, then we have

$$\big[\Lambda(R(l))\big]^{\Delta} = \lim_{g \to l} \frac{\Lambda(R(l)) - \Lambda(R(l_1))}{l - l_1} = \lim_{g \to l} \frac{1}{l - g} \int_{R(l_1)}^{R(l)} \frac{1}{\varrho_1^2(v)} \Delta v$$

$$= \lim_{l_1 \to l} \frac{R(l) - R(l_1)}{l - l_1} \frac{1}{\varrho_1^2(\mu)} = \frac{R^{\Delta}(l)}{\varrho_1^2(R(l))}, \tag{19}$$

where μ lies between $R(l_1)$ and $R(l)$. Together (18) and (19) produce

$$\big[\Lambda(R(l))\big]^{\Delta} \leq \frac{R^{\Delta}(l)}{\varrho_1^2(R(l))}. \tag{20}$$

Inequalities (17) and (20) turn out into

$$\big[\Lambda(R(l))\big]^{\Delta} \leq \left\{ \int_{l_0}^{\rho(l)} \big(h(l,q_1)m(l,q_1)\big)\Delta q_1 \right\}^{\Delta}.$$

From the definition of Λ in (3) by integration (20) from l_0 to l we get

$$\Lambda(R(l)) - \Lambda(R(l_0)) \leq \int_{l_0}^{\rho(l)} \big(h(l,q_1)m(l,q_1)\big)\Delta q_1.$$

Since Λ is increasing and $R(l_0) = [\varrho_1^{\xi-1}(b(l^*)) + (1-\xi)\int_{l_0}^{\rho(l^*)} j(l,l_1)\Delta l_1]^{\frac{1}{1-\xi}}$, the last inequality takes the form

$$R(l) \leq \Lambda^{-1}\Bigg(\Lambda\Bigg[\varrho_1^{\xi-1}\big(b(l^*)\big) + (1-\xi)\int_{l_0}^{\rho(l^*)} j(l,l_1)\Delta l_1\Bigg]^{\frac{1}{1-\xi}}$$

$$+ \int_{l_0}^{\rho(l)} \big(h(l,q_1)m(l,q_1)\big)\Delta q_1 \Bigg). \tag{21}$$

The conclusion in (2) can be achieved by the arbitrariness of l^*, inserting (21) into (16) and (8) simultaneously, integrating the resulting inequality, and taking the benefit of (6) and (7). Explanations are discarded.   □

Essential comments on Theorem 3.2 are listed underneath.

Remark 3.3 By taking $\varrho_1(x(l)) = u(t)$, $\rho(l) \leq t$, $b(l) = a(t)$, $\varrho_2 = 1$, $h = 0$, $\xi = 1$, $j(l, l_1) = k(t, s)$ and $x(l) = u(t)$ Theorem 3.2 changes into Corollary 3.9 of [24].

Remark 3.4 It is very amazing to realize that, as a distinctive case, Theorem 3.2 diminishes into [7, Theorem 3.2] by setting $\varrho_1(x(l)) = u(t)$, $\rho(l) \leq t$, $h = 0$, $\xi = 1$, $b(l) = c$, $c \geq 0$, $j(l, l_1) = f(t)p(t)$, $\varrho_2(x(l)) = 1$, and $x(l) = u(t) + f(t)q(t)$.

Theorem 3.5 *Suppose that the relation*

$$\varrho_1\big(x(l)\big) \leq b(l) + \int_{l_0}^{\rho(l)} \Big[j(l, l_1)\varrho_2\big(x(l_1)\big)\varrho_1\big(x(l_1)\big) + u(l_1)\varrho_2\big(x(l_1)\big)\Big]\Delta l_1$$

$$+ \int_{l_0}^{\rho(l)} h(l, l_1)\varrho_2\big(x(l_1)\big)\Delta l_1 \int_{l_0}^{\rho(l)} m(l, l_1)\varrho_1\big(x(l_1)\big)\Delta l_1, \quad l \in \mathbb{T}_0, \tag{22}$$

with $u \in C_{rd}(\mathbb{T}, \mathbb{R}_+)$ and conditions (P1)–(P5) *are fulfilled. Then*

$$x(l) \leq \varrho_1^{-1}\bigg[\Upsilon^{-1}\bigg\{\Theta^{-1}\bigg(\Theta\bigg[\Upsilon\big(b(l)\big) + \int_{l_0}^{\rho(l)} u(l_1)\Delta l_1$$

$$+ \int_{l_0}^{\rho(l)} \big(j(l, l_1) + h(l, l_1)m(l, l_1)\big)\Delta l_1\bigg]\bigg)\bigg\}\bigg], \tag{23}$$

where

$$\Upsilon(v) = \int_{v_0}^{v} \frac{\Delta r}{\varrho_2(\varrho_1^{-1}(r))}, \quad v \geq v_0 > 0, \Upsilon(+\infty) = +\infty, \tag{24}$$

$$\Theta(s) = \int_{s_0}^{s} \frac{\Delta k}{\Upsilon^{-1}(k)}, \quad s \geq s_0 > 0, \Theta(+\infty) = +\infty, \tag{25}$$

Υ^{-1}, Θ^{-1} *are the inverses of Υ, Θ, and L_1 is the largest number for all $l < L_1$ with*

$$\Theta\bigg[\Upsilon\big(b(l)\big) + \int_{l_0}^{\rho(l)} u(l_1)\Delta l_1 + \int_{l_0}^{\rho(l)} \big(j(l, l_1) + h(l, l_1)m(l, l_1)\big)\Delta l_1\bigg] \in \mathrm{Dom}\big(\Theta^{-1}\big). \tag{26}$$

Proof Fixing $l^* \in \mathbb{T}_0$ for $l \in [l_0, l^*] \cap \mathbb{T}$ and denoting the nondecreasing function

$$J_1(l) = b\big(l^*\big) + \int_{l_0}^{\rho(l)} \Big[j(l, l_1)\varrho_2\big(x(l_1)\big)\varrho_1\big(x(l_1)\big) + u(l_1)\varrho_2\big(x(l_1)\big)\Big]\Delta l_1$$

$$+ \int_{l_0}^{\rho(l)} h(l, l_1)\varrho_2\big(x(l_1)\big)\Delta l_1 \int_{l_0}^{\rho(l)} m(l, l_1)\varrho_1\big(x(l_1)\big)\Delta l_1, \tag{27}$$

from (22) and (27), we obtain

$$x(l) \leq \varrho_1^{-1}\big(J_1(l)\big), \quad l \in \mathbb{T}_0. \tag{28}$$

Delta differentiating (27) and applying the same analysis from (11)–(13), Lemmas 2.1 and 2.2, and (28), we notice that

$$
\begin{aligned}
J_1^\Delta(l) &= u(l)\varrho_2\big(x(l)\big)\rho^\Delta(l) \\
&\quad + \left\{ \int_{l_0}^{\rho(l)} j^\Delta(l,l_1)\varrho_2\big(x(l_1)\big)\varrho_1\big(x(l_1)\big)\Delta l_1 + j\big(\sigma(l),\rho(l)\big)\rho^\Delta(l)\varrho_2\big(x(l)\big)\varrho_1\big(x(l)\big) \right\} \\
&\quad + \left[\int_{l_0}^{\rho(l)} h(l,l_1)\varrho_2\big(x(l_1)\big)\Delta l_1 \int_{l_0}^{\rho(l)} m(l,l_1)\varrho_1\big(x(l_1)\big)\Delta l_1 \right]^\Delta, \\
&\leq u(l)\varrho_2\big(\varrho_1^{-1}\big(J_1(l)\big)\big)\rho^\Delta(l) \\
&\quad + \varrho_2\big(\varrho_1^{-1}\big(J_1(l)\big)\big)\left[\left\{ \int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1 \right\}^\Delta + \left\{ \int_{l_0}^{\rho(l)} \big(h(l,l_1)m(l,l_1)\big)\Delta l_1 \right\}^\Delta \right] J_1(l),
\end{aligned}
$$

which can be transformed into

$$
\frac{J_1^\Delta(l)}{\varrho_2(\varrho_1^{-1}(J_1(l)))} \leq u(l)\rho^\Delta(l) + \left\{ \int_{l_0}^{\rho(l)} \big(j(l,l_1)+h(l,l_1)m(l,l_1)\big)\Delta l_1 \right\}^\Delta J_1(l). \tag{29}
$$

Integrating (29) from l_0 to l and using $\Upsilon(l_0) = b(l^*)$ and $J_1(l) > 0$, from (24) we acquire

$$
\begin{aligned}
J_1(l) &\leq \Upsilon^{-1}\Bigg[\Upsilon\big(b(l^*)\big) + \int_{l_0}^{l} u(l_1)\rho^\Delta(l_1)\Delta l_1 \\
&\quad + \int_{l_0}^{l} J_1(l_1)\left\{ \int_{l_0}^{\rho(l_1)} \big(j(l_1,q_1)+h(l_1,q_1)m(l_1,q_1)\big)\Delta q_1 \right\}^\Delta \Delta l_1 \Bigg] \\
&\leq \Upsilon^{-1}\Bigg[\Upsilon\big(b(l^*)\big) + \int_{l_0}^{\rho(l^*)} u(l_1)\Delta l_1 \\
&\quad + \int_{l_0}^{l} J_1(l_1)\left\{ \int_{l_0}^{\rho(l_1)} \big(j(l_1,q_1)+h(l_1,q_1)m(l_1,q_1)\big)\Delta q_1 \right\}^\Delta \Delta l_1 \Bigg] \\
&\leq \Upsilon^{-1}\big(Z(l)\big), \tag{30}
\end{aligned}
$$

where

$$
\begin{aligned}
Z(l) &= \Upsilon\big(b(l^*)\big) + \int_{l_0}^{\rho(l^*)} u(l_1)\Delta l_1 \\
&\quad + \int_{l_0}^{l} J_1(l_1)\left\{ \int_{l_0}^{\rho(l_1)} \big(j(l_1,q_1)+h(l_1,q_1)m(l_1,q_1)\big)\Delta q_1 \right\}^\Delta \Delta l_1, \tag{31}
\end{aligned}
$$

and

$$
Z(l_0) = \Upsilon\big(b(l^*)\big) + \int_{l_0}^{\rho(l^*)} u(l_1)\Delta l_1. \tag{32}
$$

From the definition of $Z(l)$ with (30) we have

$$
Z^\Delta(l) = \left\{ \int_{l_0}^{\rho(l)} \big(j(l,l_1)+h(l,l_1)m(l,l_1)\big)\Delta l_1 \right\}^\Delta J_1(l),
$$

$$\frac{Z^\Delta(l)}{\Upsilon^{-1}(Z(l))} \leq \left\{ \int_{l_0}^{\rho(l)} \left(j(l,l_1) + h(l,l_1)m(l,l_1) \right) \Delta l_1 \right\}^\Delta.$$

The desired bound in (23) can be carried out by integrating over $[l_0, l]$, using (25) and (32), setting $l = l^*$, and simultaneously putting the resultant inequality into (30) and (28). The proof is completed. $\qquad\square$

Remark 3.6 If $h = 0$, $\rho(l) \leq t$, $b(l) = u_0$, which is a constant, $j(l,l_1) = f(t)$, $\varrho_1(x(l)) = u(t)$, $\varrho_2(x(l)) = W(u(t))$, and $u(l_1) = h(t)$, then Theorem 3.5 becomes [7, Theorem 3.4] by Pachpatte with $g(t) = 1$.

Remark 3.7 If $h = 0$, $b(l) = a(t)$, $\varrho_1(x(l)) = \varrho_1(x(l)) = u(t)$, $j(l,l_1) = b(t)$, $u(l_1) = 0$, and $\rho(l) \leq t$, then from Theorem 3.5 we are able to get Theorem 3.6 in [24].

Theorem 3.8 *Under* (P1)–(P5), *assume that*

$$\varrho_1\big(x(l)\big) \leq b(l) + \int_{l_o}^{\rho(l)} j(l,l_1)\varrho_1\big(x(l_1)\big)\Delta l_1$$
$$+ \int_{l_o}^{\rho(l)} h(l,l_1)\varrho_2\big(x(l_1)\big)\Delta l_1 \int_{l_o}^{\rho(l)} m(l,l_1)\varrho_2\big(x(l_1)\big)\Delta l_1, \quad l \in \mathbb{T}_0. \qquad (33)$$

Then

$$x(l) \leq \varrho_1^{-1}\left[\Delta^{-1}\left\{ \Pi^{-1}\left(\Pi\left[\Delta\big(b(l)\big) + \int_{l_0}^{\rho(l)} \big(h(l,l_1)m(l,l_1)\big)\Delta l_1 \right] \right. \right. \right.$$
$$\left. \left. \left. + \int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1 \right) \right\} \right], \qquad (34)$$

where

$$\Delta(v) = \int_{v_0}^{v} \frac{\Delta r}{\varrho_2^2(\varrho_1^{-1}(r))}, \quad v \geq v_0 > 0, \Delta(+\infty) = +\infty, \qquad (35)$$

$$\Pi(s) = \int_{s_0}^{s} \frac{\varrho_2^2(\varrho_1^{-1}(\Delta^{-1}(k)))}{\Delta^{-1}(k)}\Delta k, \quad s \geq s_0 > 0, \Pi(+\infty) = +\infty, \qquad (36)$$

Δ^{-1}, Π^{-1} *are the inverses of* Δ, Π, *respectively, and* $\Pi[\Delta(b(l)) + \int_{l_0}^{\rho(l)}(h(l,l_1)m(l,l_1))\Delta l_1] + \int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1$ *is in the domain of* Π.

Proof Let $l^* \in \mathbb{T}_0$ for $l \in [l_0, l^*] \cap \mathbb{T}$ be fixed. Define the nondecreasing function

$$J_2(l) = b\big(l^*\big) + \int_{l_o}^{\rho(l)} j(l,l_1)\varrho_1\big(x(l_1)\big)\Delta l_1$$
$$+ \int_{l_o}^{\rho(l)} h(l,l_1)\varrho_2\big(x(l_1)\big)\Delta l_1 \int_{l_o}^{\rho(l)} m(l,l_1)\varrho_2\big(x(l_1)\big)\Delta l_1. \qquad (37)$$

From (33) and (37) we get

$$x(l) \leq \varrho_1^{-1}\big(J_2(l)\big). \qquad (38)$$

By delta differentiating (37), using Lemma 2.1, (38), and similar steps from (11)0-(13), we obtain

$$J_2^\Delta(l) \le \left\{ \int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1 \right\}^\Delta J_2(l) + \left\{ \int_{l_0}^{\rho(l)} \big(h(l,l_1)m(l,l_1)\big)\Delta l_1 \right\}^\Delta \varrho_2^2\big(\varrho_1^{-1}\big(J_2(l)\big)\big)$$

or

$$\frac{J_2^\Delta(l)}{\varrho_2^2(\varrho_1^{-1}(J_2(l)))} \le \left\{ \int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1 \right\}^\Delta \frac{J_2(l)}{\varrho_2^2(\varrho_1^{-1}(J_2(l)))} + \left\{ \int_{l_0}^{\rho(l)} \big(h(l,l_1)m(l,l_1)\big)\Delta l_1 \right\}^\Delta.$$

Integrating over $[l_0, l]$, from (35) we have

$$\Delta J_2(l) = \Delta J_2(l_0) + \int_{l_0}^{\rho(l)} \big(h(l,l_1)m(l,l_1)\big)\Delta l_1$$
$$+ \int_{l_0}^{l} \frac{J_2(l_1)}{\varrho_2^2(\varrho_1^{-1}(J_2(l_1)))} \left\{ \int_{l_0}^{\rho(l_1)} j(l_1,q_1)\Delta q_1 \right\}^\Delta \Delta l_1. \tag{39}$$

Equation (39) with $J_2(l_0) = b(l^*)$ gives

$$J_2(l) \le \Delta^{-1}\Bigg[\Delta\big(b\big(l^*\big)\big) + \int_{l_0}^{\rho(l)} \big(h(l,l_1)m(l,l_1)\big)\Delta l_1$$
$$+ \int_{l_0}^{l} \frac{J_2(l_1)}{\varrho_2^2(\varrho_1^{-1}(J_2(l_1)))} \left\{ \int_{l_0}^{\rho(l_1)} j(l_1,q_1)\Delta q_1 \right\}^\Delta \Delta l_1 \Bigg]$$
$$\le \Delta^{-1}\big(Z_1(l)\big), \tag{40}$$

where

$$Z_1(l_0) = \Delta\big(b\big(l^*\big)\big) + \int_{l_0}^{\rho(l^*)} \big(h(l,l_1)m(l,l_1)\big)\Delta l_1. \tag{41}$$

Differentiation of $Z_1(l)$ with respect to l and (40) imply that

$$\frac{\varrho_2^2(\varrho_1^{-1}(\Delta^{-1}(Z_1(l))))Z_1^\Delta(l)}{\Delta^{-1}(Z_1(l))} \le \left\{ \int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1 \right\}^\Delta,$$

integration the prior inequality from l_0 to l, use (36), (41) and $l^* \in \mathbb{T}_0$ is chosen. The resultant inequality, (38), (40) yield the required bound in (34). Details are omitted. $\qquad\square$

Theorem 3.9 *Under the assumptions* (P1), (P2), *and* (P4), *suppose that*

$$x^\xi(l) \le b^{\frac{\xi}{\xi-\varpi}} + \frac{\xi}{\xi-\varpi} \int_{l_0}^{\rho(l)} j(l,l_1)x^\varpi(l_1)$$
$$\times \left[x^{\xi-\varpi}(l_1) + \int_{l_0}^{\rho(l_1)} h(l_1,q_1)x^\varpi(q_1)\Delta q_1 \int_{l_0}^{\rho(l_1)} m(l_1,q_1)x^{\xi-\varpi}(q_1)\Delta q_1 \right]\Delta l_1,$$
$$l \in \mathbb{T}_0, \tag{42}$$

where ξ, b, ϖ *are constants, $b \geq 0$, and $\xi > \varpi > 0$. Then*

$$x(l) \leq \left[b\left(1 + \int_{l_0}^{\rho(l)} j(l,l_1) e_{G_2}(l_1,l_0) \Delta l_1 \right) \right]^{\frac{1}{\xi-\varpi}}, \tag{43}$$

where

$$G_2(l) = \left\{ \int_{l_0}^{\rho(l)} \left(j(l,l_1) + h(l,l_1)m(l,l_1) \right) \Delta l_1 \right\}^{\Delta}. \tag{44}$$

Proof Defining

$$N(l) = b^{\frac{\xi}{\xi-\varpi}} + \frac{\xi}{\xi-\varpi} \int_{l_0}^{\rho(l)} j(l,l_1)x^{\varpi}(l_1)$$

$$\times \left[x^{\xi-\varpi}(l_1) + \int_{l_0}^{\rho(l_1)} h(l_1,q_1)x^{\varpi}(q_1)\Delta q_1 \right.$$

$$\times \left. \int_{l_0}^{\rho(l_1)} m(l_1,q_1)x^{\xi-\varpi}(q_1)\Delta q_1 \right] \Delta l_1, \tag{45}$$

(42) can be restated as

$$x^{\xi}(l) \leq N(l) \quad \Rightarrow \quad x(l) \leq N^{\frac{1}{\xi}}(l), l \in \mathbb{T}_0. \tag{46}$$

Obviously, $N(l)$ is nondecreasing. The definition of $N(l)$ in (45) with Lemma 2.2 and (46) yields

$$N^{\Delta}(l) \leq \frac{\xi}{\xi-\varpi} \left\{ \int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1 \right\}^{\Delta} N^{\frac{\varpi}{\xi}}(l)$$

$$\times \left[N^{\frac{\xi-\varpi}{\xi}}(l) + \int_{l_0}^{\rho(l_1)} h(l_1,q_1)N^{\frac{\varpi}{\xi}}(q_1)\Delta q_1 \int_{l_0}^{\rho(l_1)} m(l_1,q_1)N^{\frac{\xi-\varpi}{\xi}}(q_1)\Delta q_1 \right],$$

or

$$N^{-\frac{\varpi}{\xi}}(l)N^{\Delta}(l) \leq \frac{\xi}{\xi-\varpi} \left\{ \int_{l_0}^{\rho(l)} j(l,l_1)\Delta l_1 \right\}^{\Delta} N_1(l), \tag{47}$$

where

$$N_1(l) = N^{\frac{\xi-\varpi}{\xi}}(l) + \int_{l_0}^{\rho(l)} h(l,l_1)N^{\frac{\varpi}{\xi}}(l_1)\Delta l_1 \int_{l_0}^{\rho(l)} m(l,l_1)N^{\frac{\xi-\varpi}{\xi}}(l_1)\Delta l_1. \tag{48}$$

Taking the delta derivative of (48) and using (47), (11)–(13), $N(l) \leq N_1(l)$, and $N_1(l_0) = b$, we infer that

$$N_1^{\Delta}(l) = \frac{\xi-\varpi}{\xi}N^{-\frac{\varpi}{\xi}}(l)N^{\Delta}(l) + \left[\int_{l_0}^{\rho(l)} h(l,l_1)\Delta l_1 \int_{l_0}^{\rho(l)} m(l,l_1)\Delta l_1 \right]^{\Delta} N(l)$$

$$\leq \left[\int_{l_0}^{\rho(l)} \left[j(l,l_1) + h(l,l_1)m(l,l_1) \right] \Delta l_1 \right]^{\Delta} N_1(l)$$

$$\leq G_2(l)N_1(l),$$

which yields

$$N_1(l) \leq b e_{G_2}(l, l_0), \tag{49}$$

with G_2 given in (44). From (47) and (49) we claim

$$N^{-\frac{\varpi}{\xi}}(l)N^{\Delta}(l) \leq b\frac{\xi}{\xi - \varpi}\left\{\int_{l_0}^{\rho(l)} j(l, l_1)\Delta l_1\right\}^{\Delta} e_{G_2}(l_1, l_0). \tag{50}$$

We can notice from Theorem 1.90 of [23] and $N^{\Delta}(l) \geq 0$ that

$$\left[\frac{\xi}{\xi - \varpi}N^{\frac{\xi - \varpi}{\xi}}(l)\right]^{\Delta} = N^{\Delta}(l)\int_0^1 \left[N(l) + h\psi(l)N^{\Delta}(l)\right]^{-\frac{\varpi}{\xi}} dh$$

$$= \frac{N^{\Delta}(l)}{N^{\frac{\varpi}{\xi}}(l)}\int_0^1 \left[1 + h\psi(l)\frac{N^{\Delta}(l)}{N(l)}\right]^{-\frac{\varpi}{\xi}} dh \leq \frac{N^{\Delta}(l)}{N^{\frac{\varpi}{\xi}}(l)},$$

which, together with (50), implies that

$$\left[N^{\frac{\xi - \varpi}{\xi}}(l)\right]^{\Delta} \leq b\left\{\int_{l_0}^{\rho(l)} j(l, l_1)\Delta l_1\right\}^{\Delta} e_{G_2}(l_1, l_0).$$

Integrate this inequality and using $N(l_0) = b$ and (46), we get the required inequality (43). □

Remark 3.10 Theorem 3.9 becomes Theorem 2.1 of [21] by letting $\xi = 1$, $\varpi = 0$, $\rho(l) \leq t$, $j(l, l_1) = k(x, t)$, $b = g(x)$, $x(l) = u(t)$ with $\mathbb{T} = \mathbb{R}$.

Remark 3.11 As a particular case of delta derivative on time scales, if $\xi = 1$, $\varpi = 0$, $\rho(l) \leq b$, $h = 0$, $j(l, l_1) = n(t)$, $b = m(t)$, and $x(l) = u(t)$ in Theorem 3.9, then it reduces to Lemma 3.1 due to Boukerrioua et al. [4] with $l(t) = 1$.

Corollary 3.12 *Let* (P2), (P4), *and* $j(l, l_1), j^{\Delta}(l, l_1), h(l, l_1), h^{\Delta}(l, l_1) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+)$. *Suppose that*

$$x^{\xi}(l) \leq b + \int_{l_0}^{\rho(l)} j(l, l_1)x^{\xi}(l_1)\Delta l_1 \int_{l_0}^{\rho(l)} h(l, l_1)x(l_1)\Delta l_1, \quad l \in \mathbb{T}_0.$$

Then

$$x(l) \leq \frac{b^{\frac{1}{\xi}}}{[1 - \frac{1}{\xi}b^{\frac{1}{\xi}}\int_{l_0}^{\rho(l)}(j(l, l_1)h(l, l_1))\Delta l_1]},$$

where $\xi \neq 0$ *and* $b \geq 0$ *are constants.*

Proof The proof of Corollary 3.12 is the same that of Theorem 3.9 with appropriate alterations. □

4 Application

This segment indicates a prompt use of Theorem 3.9 for analyzing the boundedness and uniqueness of the delay integral equations on time scales. Consider the following class of nonlinear delay dynamic integral equations:

$$
\begin{cases}
(x^{\xi}(l))^{\Delta} = M(l, l_1, x(l), \int_{l_0}^{l} E(l, l_1, x(l_1)\Delta l_1), \\
x(l_0) = b^{\frac{1}{\xi - \varpi}}.
\end{cases}
\tag{51}
$$

The global existence on the solutions of (51) can be explored by the following corollary.

Corollary 4.1 *Assume that*

$$
\left| M(l, l_1, x, y) \right| \leq j(l, l_1)|x|^{\xi - 1}\left[|x| + h(l, l_1)|x|^{\xi - 1}|y| \right]
\tag{52}
$$

and

$$
\left| E(l, l_1, x) \right| \leq m(l, l_1)|x|
\tag{53}
$$

for $l \in \mathbb{T}_0$, $x, y \in \mathbb{R}$. If $x(l)$ is a solution of (4), then

$$
\left| x(l) \right| \leq b\left(1 + \int_{l_0}^{l} j(l, l_1)e_{G_2}(l_1, l_0)\Delta l_1 \right),
\tag{54}
$$

where $M \in C_{rd}([\beta, l_0] \cap \mathbb{T} \times \mathbb{R}^2, \mathbb{R})$, $E \in C_{rd}([\beta, l_0] \cap \mathbb{T} \times \mathbb{R}, \mathbb{R})$, j, h, m, x, ρ are defined as in (P1), (P2), (P4), b and $\xi > 1$ are constants, and G_2 is as in (44).

Proof Clearly, equation (51) by employing (52) and (53) transforms into

$$
\left| x^{\xi}(l) \right| \leq |b| + \int_{l_0}^{l} |M(l, l_1, x(l_1), \int_{l_0}^{l_1} E(l_1, q_1, x(q_1)\Delta q_1)|\Delta l_1
$$

$$
\leq |b| + \int_{l_0}^{l} j(l, l_1)\left| x(l_1) \right|^{\xi - 1}
$$

$$
\times \left[\left| x(l_1) \right| + \int_{l_0}^{l_1} h(l_1, q_1)\left| x(q_1) \right|^{\xi - 1}\Delta q_1 \int_{l_0}^{l_1} m(l_1, q_1)\left| x(q_1) \right|\Delta q_1 \right]\Delta l_1.
\tag{55}
$$

We argue as in the case of Theorem 3.9 with $\varpi = \xi - 1$ in order to get (54) from (55). The proof is done.   □

Next, we look at the delay dynamic equation (4) with $x(l_0) = x_0$ and $\xi = 3$.

Example 4.2 Let

$$
\left| M(l, l_1, x_1, y_1) - M(l, l_1, x_2, y_2) \right|
$$

$$
\leq j(l, l_1)\left| x_1^3 - x_2^3 \right|
$$

$$
\times \left[\left| x_1^2 - x_1^2 \right| + h(l, l_1)|x_1 - x_2||y_1 - y_2| \right],
\tag{56}
$$

$$\left| E(l, l_1, x_1) - E(l, l_1, x_2) \right| \le m(l, l_1) \left| x_1^2 - x_2^2 \right|. \tag{57}$$

Then (51) has at most one solution.

Proof Two solutions $x_1(l)$, $x_2(l)$ of (51) are equivalent to

$$x_1^3(l) - x_2^3(l) = \int_{l_0}^{l} M(l, l_1, x_1(l_1), \int_{l_0}^{l_1} E(l_1, q_1, x_1(q_1)\Delta q_1) \Delta l_1$$

$$- \int_{l_0}^{l} M(l, l_1, x_2(l_1), \int_{l_0}^{l_1} E(l_1, q_1, x_2(q_1)\Delta q_1) \Delta l_1,$$

which by hypotheses (56) and (57) leads to

$$\left| x_1^3(l) - x_2^3(l) \right|$$

$$\le \int_{l_0}^{l} \left| M\left(l, l_1, x_1(l_1), \int_{l_0}^{l_1} E(l_1, q_1, x_1(q_1))\Delta q_1 \right) \Delta l_1 \right.$$

$$\left. - M(l, l_1, x_2(l_1), \int_{l_0}^{l_1} E(l_1, q_1, x_2(q_1)\Delta q_1) \right| \Delta l_1,$$

$$\le \int_{l_0}^{l} j(l, l_1) \left| x_1^3(l_1) - x_2^3(l_1) \right| \left[\left| x_1(l_1) - x_2(l_1) \right| \right.$$

$$\left. + \int_{l_0}^{l_1} h(l_1, q_1) \left| x_1(q_1) - x_2(q_1) \right| \Delta q_1 \int_{l_0}^{l_1} m(l_1, q_1) \left| x_1^2(q_1) - x_2^2(q_1) \right| \Delta q_1 \right] \Delta l_1$$

$$\le \int_{l_0}^{l} j(l, l_1) \sqrt{\left| x_1^3(l_1) - x_2^3(l_1) \right|} \left[\sqrt{\left| x_1^3(l_1) - x_2^3(l_1) \right|} + \int_{l_0}^{l_1} h(l_1, q_1) \right.$$

$$\times \left[\sqrt{\left| x_1^3(l_1) - x_2^3(l_1) \right|} \Delta q_1 \int_{l_0}^{l_1} m(l_1, q_1) \left[\sqrt{\left| x_1^3(l_1) - x_2^3(l_1) \right|} \Delta q_1 \right] \Delta l_1.$$

The earlier inequality by some changes in the method of Theorem 3.9 with $\xi = 1$ and $\varpi = \frac{1}{2}$ applied to the function $|x_1^3(l) - x_2^3(l)|$ produces

$$\left| x_1^3(l) - x_2^3(l) \right| \le 0, \quad l \in \mathbb{T}_0.$$

Therefore $x_1(l) = x_2(l)$. Along these lines the delay dynamic equation (51) has one positive solution.   □

Example 4.3 If

$$\left| M(l, l_1, x, y) \right| \le j(l, l_1)|x| \left[|x|^2 + h(l, l_1)|x||y| \right] \tag{58}$$

and

$$\left| E(l, l_1, x) \right| \le m(l, l_1)|x|^2, \tag{59}$$

then the solution $x(l)$ indicates

$$\left| x(l) \right| \le |x_0| \sqrt{1 + \int_{l_0}^{l} j(l, l_1) e_{G_2}(l_1, l_0) \Delta l_1}, \quad l \in \mathbb{T}_0. \tag{60}$$

Proof Equation (51) with (58), (59), and $\xi = 3$ can be reconstructed as

$$
\begin{aligned}
\left|x(l)\right|^3 &\le |x_0|^3 + \int_{l_0}^{l} |M\left(l, l_1, x(l_1), \int_{l_0}^{l_1} E\left(l_1, q_1, x(q_1)\Delta q_1\right)|\Delta l_1 \right.\\
&\le |x_0|^3 + \int_{l_0}^{l} j(l, l_1)|x(l_1)|\right)|\left[\left|x(l_1)\right|^2 + \int_{l_0}^{l_1} h(l_1, q_1)\left|x(q_1)\right|\Delta q_1 \right.\\
&\quad \times \left. \int_{l_0}^{l_1} m(l_1, q_1)\left|x(q_1)\right|^2 \Delta q_1 \right]\Delta l_1\\
&\le \left|x_0^2\right|^{\frac{3}{2}} + \int_{l_0}^{l} j(l, l_1)\left|x(l_1)\right|\left[\left|x(l_1)\right|^2 + \int_{l_0}^{l_1} h(l_1, q_1)\left|x(q_1)\right|\Delta q_1 \right.\\
&\quad \times \left. \int_{l_0}^{l_1} m(l_1, q_1)\left|x(q_1)\right|^2 \Delta q_1 \right]\Delta l_1.
\end{aligned}
\tag{61}
$$

The required estimate (60) can be retrieved by intently looking at the arguments of Theorem 3.9 with $\xi = 3$ and $\varpi = 1$ and making few modifications to (61). □

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The author declares that there are no financial and nonfinancial competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
1. Hilger, S.: Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math. **18**, 18–56 (1990)
2. Bohner, M., Georgiev, S.G.: Multivariable Dynamic Calculus on Time Scales. Springer, Cham (2016)
3. Bohner, M.: Partial differentiation on time scales. In: Multivariable Dynamic Calculus in Time Scales, pp. 303–447 (2016)
4. Boukerrioua, K., Meziri, I., Chiheb, T.: Some refinements of certain Gamidov integral inequalities on time scales and applications. Kragujev. J. Math. **42**(1), 131–152 (2018)
5. Agarwal, R., Bohner, M., Peterson, A.: Inequalities on time scales, a survey. Math. Inequal. Appl. **4**, 535–557 (2001)
6. Gu, J., Meng, F.W.: Some new nonlinear Volterra–Fredholm type dynamic integral inequalities on time scales. Appl. Math. Comput. **245**(1), 235–242 (2014)
7. Pachpatte, D.P.: Explicit estimates on integral inequalities with time scales. J. Inequal. Pure Appl. Math. **17**(4), Article ID 143 (2006)
8. Li, W.N., Sheng, W.: Some nonlinear integral inequalities on time scales. J. Inequal. Appl. **2007**, Article ID 70465 (2007)
9. Saker, S.H.: Some nonlinear dynamic inequalities on time scales and applications. J. Math. Inequal. **4**(4), 561–579 (2010)
10. Saker, S.H.: Some nonlinear dynamic inequalities on time scales. Math. Inequal. Appl. **14**(3), 633–645 (2011)
11. Sarikaya, M.Z., Ozkan, U.M., Yildirim, H.: Time scale integral inequalities similar to Qi's inequality. J. Inequal. Pure Appl. Math. **7**(4), Article ID 128 (2006)

12. Sun, Y.G., Hassan, T.: Some nonlinear dynamic integral inequalities on time scales. Appl. Math. Comput. **220**(4), 221–225 (2013)
13. Diblík, J., Ruzickova, M., Vaclavíkov, B.: Bounded solutions of dynamic equations on time scales. Int. J. Difference Equ. **3**, 61–69 (2008)
14. Khan, Z.: On some explicit bounds of integral inequalities related to time scales. Adv. Differ. Equ. **2019**, 243 (2019)
15. Ma, Q.H., Wang, J.W., Ke, X.H., Pecaric, J.: On the boundedness of a class of nonlinear dynamic equations of second order. Appl. Math. Lett. **26**(11), 1099–1105 (2013)
16. Nasser, B.B., Boukerrioua, M.A., Defoort, M., Djemia, M., Hammami, M.A., Laleg-Kirati, T.: Sufficient conditions for uniform exponential stability and *h*-stability of some classes of dynamic equations on arbitrary time scales. Nonlinear Anal. Hybrid Syst. **32**(1), 54–64 (2019)
17. Li, W.N.: Bounds for certain new integral inequalities on time scales. Adv. Differ. Equ. **2009**, Article ID 484185 (2009)
18. Du, L., Xu, R.: Some new Pachpatte type inequalities on time scales and their applications. J. Math. Inequal. **6**, 229–240 (2012)
19. Li, W.N.: Some Pachpatte type inequalities on time scales. Comput. Math. Appl. **57**, 275–282 (2009)
20. Li, W.N.: Some new dynamic inequalities on time scales. J. Math. Anal. Appl. **319**(2), 802–814 (2006)
21. Oguntuase, J.A.: On integral inequalities of Gronwall–Bellman–Bihari type in several variables. J. Inequal. Pure Appl. Math. **1**(2), Article ID 20 (2000)
22. Pachpatte, D.B.: On a nonlinear dynamic integrodifferential equation on time scales. J. Appl. Anal. **16**, 279–294 (2010)
23. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
24. Bohner, E.A., Bohner, M., Akin, F.: Pachpatte inequalities on time scales. J. Inequal. Pure Appl. Math. **6**, Article ID 6 (2005)
25. Li, W.N.: Some delay integral inequalities on time scales. Comput. Math. Appl. **59**, 1929–1936 (2010)
26. Pachpatte, D.B.: Estimates of certain integral inequalities on time scales. J. Math. **2013**, Article ID 902087 (2013)
27. Meng, F., Shao, J.: Some new Volterra–Fredholm type dynamic integral inequalities on time scales. Appl. Math. Comput. **223**, 444–451 (2013)
28. Haidong, L.: A class of retarded Volterra–Fredholm type integral inequalities on time scales and their applications. J. Inequal. Appl. **2017**, 293 (2017)
29. Nasser, B.B., Boukerrioua, M.A., Defoort, M., Djemia, M., Hammami, M.A.: Some generalizations of Pachpatte inequalities on time scales. IFAC-PapersOnLine **50**(1), 14884–14889 (2017)
30. Mi, Y.: A generalized Gronwall Bellman type delay integral inequality with two independent variables on time scales. J. Math. Inequal. **11**(4), 1151–1160 (2017)
31. Khan, Z.: Solvability for a class of integral inequalities with maxima on the theory of time scales and their applications. Bound. Value Probl. **2019**, 146 (2019)
32. Cooke, K.L., Gyori, I.: Numerical approximation of the solutions of delay differential equations on an infinite interval using piecewise constant arguments. Comput. Math. Appl. **28**, 81–92 (1994)