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On the existence of solutions for fractional boundary value problems on the ethane graph

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Abstract

A few researchers have studied fractional differential equations on star graphs. They use star graphs because their method needs a common point which has edges with other nodes while other nodes have no edges between themselves. It is natural that we feel that this method is incomplete. Our aim is extending the method on more generalized graphs. In this work, we investigate the existence of solutions for some fractional boundary value problems on the ethane graph. In this way, we consider a graph with labeled vertices by 0 or 1, inspired by a graph representation of the chemical compound of ethane, and define fractional differential equations on each edge of this graph. Also, we provide an example to illustrate our last main result.

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1 Introduction

During the recent decades, initial and boundary value problems have been used in investigating natural processes in the world around us. The wide variety of such boundary value problems has attracted the attention of many researchers to study various phenomena using mathematical tools and computer simulation software. In other words, some recent publications show the importance of fractional differential equations in modeling of a variety of applied sciences (see, for example, [1–9]) and numerical computations (see, for example, [10–15]). One of our aims in this work is extending theoretical field in order to increase our abilities in finding more effective applications on chemical reactions. If we can do so, computer software engineers will be able to produce some software in the future using which everybody could do chemical experiments without the use of minerals, and this will help the environment.

The fractional calculus plays an important role in this regard. By using some techniques, we can solve the mathematical modelings described by the fractional differential equations and obtain the corresponding solutions and then analyze the qualitative behaviors of a

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solution function under given boundary conditions. Existing works for such problems in general setting can be seen in the numerous published papers (see, for example, [16–51]).

The graph theory is a relatively new area of mathematics in which we are concerned with networks of points connected by lines. This structure can be found in many real constructions in the world around us. In other words, by development and expansion of some dynamical and industrial systems such as gas transmission lines, water pipelines, the expansive growth of computer networks, structure of molecules in medicine and biology, etc., new descriptive models have emerged for studying the related processes designed by specialists of these fields. Due to the graph structure of these networks, the study of mathematical models described by ordinary or fractional differential equations on graphs was considered. In fact, boundary value problems on a graph are defined as a problem consisting of a system of differential equations on the given graph with certain boundary conditions on nodes.

The starting point for the theory of differential equations on graphs is related to a work of Lumer in 1980 [52]. He investigated general evolution equations on ramification spaces by using local operators defined on such spaces. With a similar structure, Nicaise studied the propagation of nerve impulses [53]. In 1989, Zavgorodnij considered boundary value problems for linear differential equations on a geometric graph where solutions of the problem were coordinated at the interior vertices [54]. He constructed an adjoint boundary value problem and obtained a self-adjointness criterion [54]. In 2008, Gordeziani et al. discussed the existence and uniqueness of solutions for ordinary differential equations on graphs [55]. They used the double-sweep method for solving the boundary value problem and presented a numerical approach.

In most of the mentioned works, differential equations are considered on a graph, and solutions of them are obtained by computational and numerical approaches. But there are limited works in fractional boundary value problems on graphs in which the existence of solutions is proved by some techniques in fixed point theory [56, 57]. The first work in this regard is related to a paper of Graef et al. [56]. The authors introduced a star graph $G = V(G) \cup E(G)$ consisting of three vertices and two edges with $V(G) = \{v_0, v_1, v_2\}$ and $E(G) = \{e_1 = \overrightarrow{v_1v_0}, e_2 = \overrightarrow{v_2v_0}\}$, where v_0 is the junction node and $e_i = \overrightarrow{v_iv_0}$ is the edge connecting nodes v_i to v_0 with length $l_i = |\overrightarrow{v_iv_0}|$ for $i = 1, 2$. On each edge $e_i = \overrightarrow{v_iv_0}$, a local coordinate system with origin at vertices v_1 and v_2 and the coordinate $t \in (0, l_i)$ is considered. Graef et al. defined a system of nonlinear fractional differential equations on each $e_i = \overrightarrow{v_iv_0}$ by

$$-D_0^\alpha u_i(t) = g_i(t)h_i(t, u_i(t)) \quad (i = 1, 2, t \in (0, l_i))$$

with boundary value conditions

$$u_1(0) = u_2(0) = 0, \quad u_1(l_1) = u_2(l_2), \quad D_0^\beta u_1(l_1) + D_0^\beta u_2(l_2) = 0,$$

where $\alpha \in (1, 2]$, $\beta \in (0, \alpha)$, $g_i : [0, l_i] \rightarrow \mathbb{R}$ are continuous functions with $g_i(t) \neq 0$ on $[0, l_i]$ and also $h_i : [0, l_i] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Two fractional operators D_0^α and D_0^β denote Riemann–Liouville fractional derivatives. The Banach contraction principle and the Schauder fixed point theorem are applied to establish the existence results by authors.

In 2019, Mehandiratta et al. generalized the work of Graef to the case of a star graph consisting of $n + 1$ vertices and n edges [57]. Indeed, the authors considered a general star

graph G with $V(G) = \{v_0, v_1, \dots, v_n\}$ and $E(G) = \{e_1 = \overrightarrow{v_1v_0}, e_2 = \overrightarrow{v_2v_0}, \dots, e_n = \overrightarrow{v_nv_0}\}$, where $l_i = |\overrightarrow{v_iv_0}|$ is the length of each edge e_i connecting vertices v_i to v_0 ($i = 1, 2, \dots, n$). The authors investigated the following boundary value problems on each edge of the star graph G given by

$$\begin{cases} {}^c\mathcal{D}_0^\alpha u_i(t) = f_i(t, u_i(t), {}^c\mathcal{D}_0^\beta u_i(t)) & (i = 1, 2, \dots, n, t \in (0, l_i)), \\ u_i(0) = 0, \quad u_i(l_i) = u_j(l_j) & (i \neq j), \\ \sum_{i=1}^n u'_i(l_i) = 0, \end{cases} \tag{1}$$

where $\alpha \in (1, 2]$, $\beta \in (0, \alpha - 1]$, $f_i : [0, l_i] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and ${}^c\mathcal{D}_0^\gamma$ denotes the Caputo fractional derivative of order $\gamma \in \{\alpha, \beta\}$. They used transformations $x = \frac{t}{l_i} \in [0, 1]$ and $z(x) = u(t) = u(l_ix)$ for $t \in [0, l_i]$ and proved the relation ${}^c\mathcal{D}_0^\alpha u(t) = l_i^{-\alpha} ({}^c\mathcal{D}_0^\alpha z(x))$. Then, by applying these transformations, the system of fractional boundary value problems on graph (1) converts into the following system of fractional boundary value problems on the unit interval $[0, 1]$ given by

$${}^c\mathcal{D}_0^\alpha z_i(x) = l_i^\alpha h_i(x, z_i(x), l_i^{-\beta} {}^c\mathcal{D}_0^\beta z_i(x)) \quad (x \in [0, 1])$$

with boundary conditions $z_i(0) = 0, z_i(1) = z_j(1)$ for $i \neq j$ and $\sum_{i=1}^n l_i^{-1} z'_i(1) = 0$, where $z_i(x) = u_i(l_ix)$ and $h_i(x, u, w) = f_i(l_ix, u, w)$ for $i = 1, 2, \dots, n$.

Motivated by the aforementioned works, our aim is to generalize the above boundary value problems to a new problem on the ethane graph which is a general graph with respect to star graphs. More precisely, by considering the ethane graph with labeled vertices by 0 or 1 (see Fig. 6), we investigate the existence of solutions for the nonlinear fractional boundary value problem

$$\begin{cases} {}^c\mathcal{D}_0^\vartheta u_i(t) = h_i(t, u_i(t), u'_i(t)) & (t \in [0, 1], i = 1, 2, \dots, 7), \\ \lambda_1 u_i(0) + \lambda_2 u_i(1) = \lambda_3 \int_0^1 u_i(s) ds, \quad \lambda_1 u'_i(0) + \lambda_2 u'_i(1) = \lambda_3 \int_0^1 u_i(s) ds, \end{cases} \tag{2}$$

where $\vartheta \in (1, 2]$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, ${}^c\mathcal{D}_0^\vartheta$ denotes the Caputo fractional derivative and functions $h_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous for $i = 1, 2, \dots, 7$, where $n = 7$ is the number of edges of the graph representation of ethane with $|e_i| = 1$.

The boundary conditions in this problem show that the linear combinations of values of the unknown functions and their derivatives at two ends of each edge are proportional to a multiple of the integral of the unknown functions. Also, it is notable that the obtained solutions for the fractional boundary value problem (2) can be interpreted in different practical meanings of the organic chemistry. In other words, every solution function $u_i(t)$ on an arbitrary edge e_i may indicate an amount of the bond energy, the bond strength, the bond polarity, etc. This could lead to valuable applications in chemical reactions theory. Hence, we claim that this abstract idea could be useful for young researchers in their future work.

This paper is arranged as follows. In Sect. 2, some primitive notions for labeling the ethane graph by 0 or 1 are explained. Moreover, some necessary relations on the fractional calculus are recalled. In Sect. 3, the main existence results for the nonlinear fractional boundary value problem (2) are proved by known fixed point theorems. In the end of the paper, an example is provided to illustrate the validity of results.

Figure 1 A sketch of the star graph G with two edges

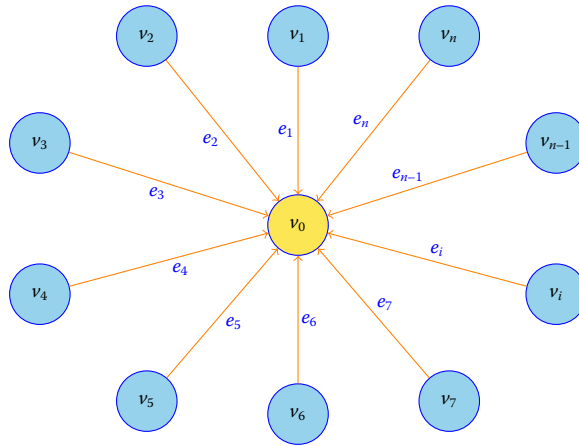
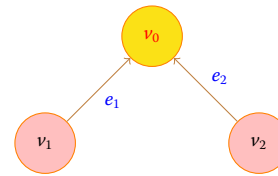


Figure 2 A sketch of the star graph G with n edges

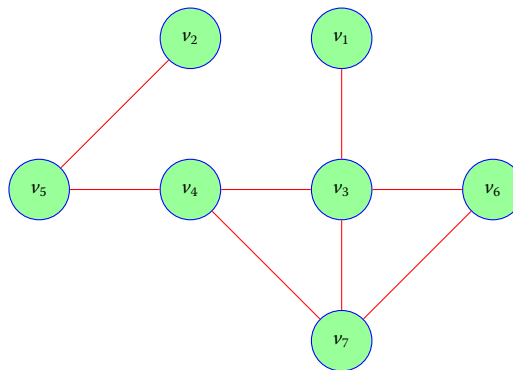


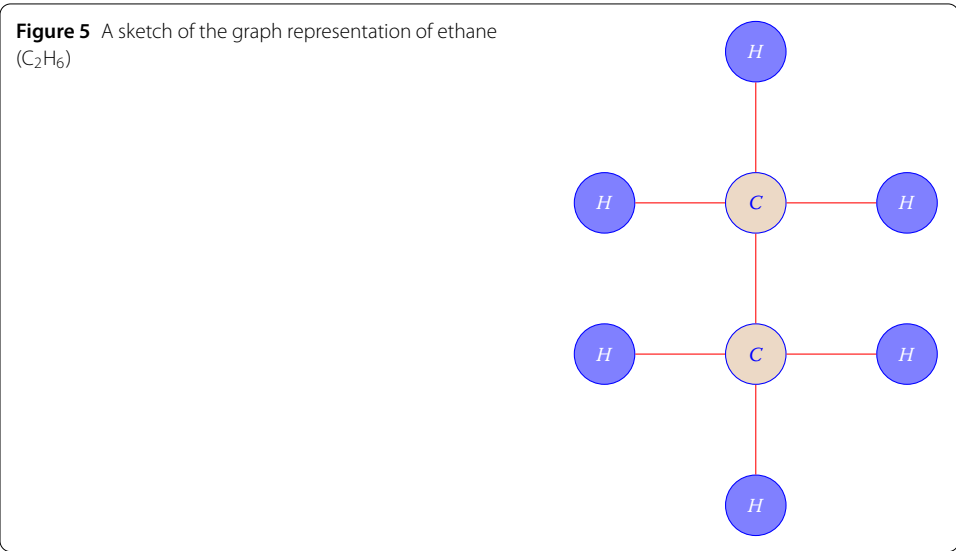
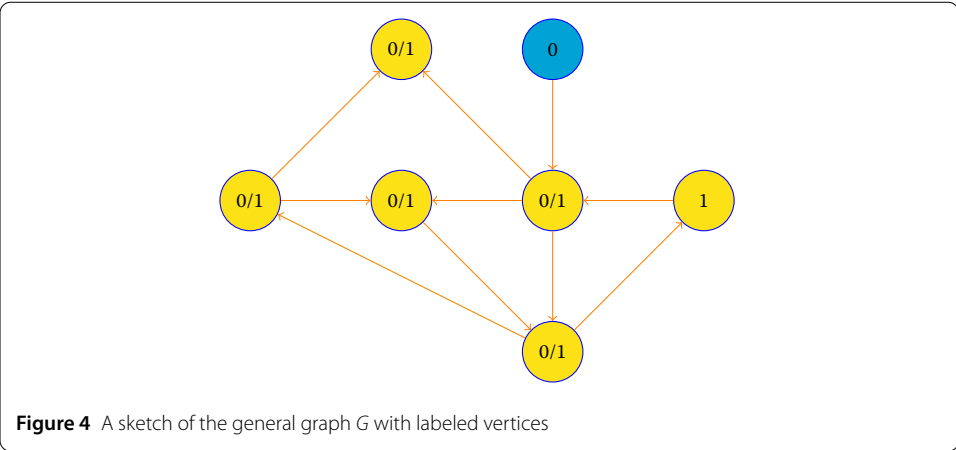
Figure 3 A sketch of the non-star graph G with more than one junction node

2 Preliminaries

In this section, we provide some primitive concepts about the new class of fractional boundary value problems (2) on the ethane graph. For this, we first state two important points about the used methods in [56] and [57].

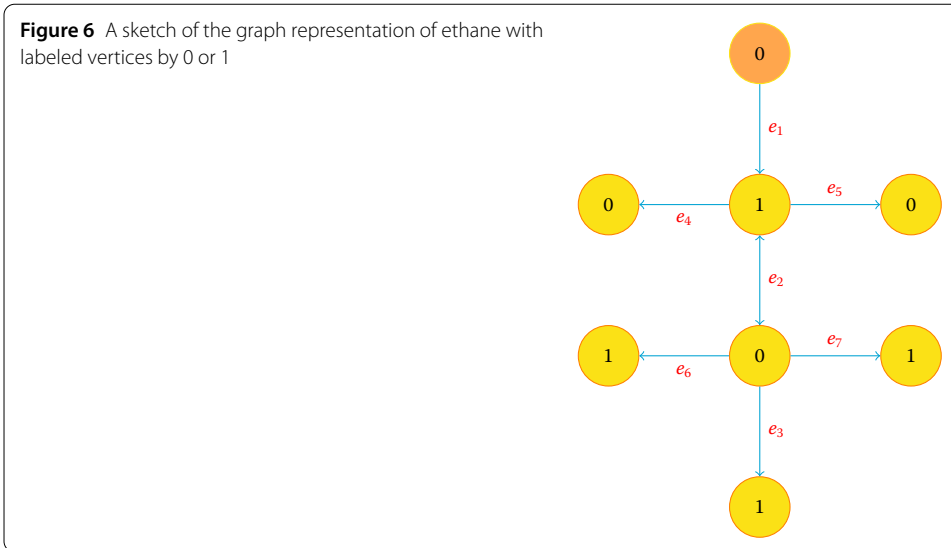
(1) In both articles, the authors assume the graph G as a star graph consisting of one junction node v_0 (Figs. 1 and 2), while in general cases, the graph G may not be in the form of a star graph and may have a general structure with more than one junction node. For example, see Fig. 3 where there exist five junction nodes.

(2) In both articles, the authors consider the length of each edge to be the variable values $|e_i| = l_i$ for $i = 1, 2, \dots, n$, where n denotes the number of edges for the graph G . Next, they convert $[0, l_i]$ into a unit interval $[0, 1]$ by using a change of variable for the normalization



of the length of all edges, while from the beginning, one can consider the length of all edges to be fixed value $|e_i| = 1$ without specifying boundary vertices of each edge as the origin. For this purpose, we propose a new method for labeling vertices. In this case, we can assign two labels 0 or 1 to each vertex of a graph. In other words, the label of each vertex depends on the direction of the corresponding edge. When we move along an arbitrary edge, the label of the starting vertex and the ending vertex are considered values 0 and 1 and vice versa. Hence, some vertices may have two labels 0 and 1 simultaneously, and the origin of each edge is not fixed, and it changes whenever the direction of the movement along the edge is changed. By this rule, we do not need to normalize the length of each edge by using the specific transformation, and also we are free to determine one of two vertices of the corresponding edge as the origin. For example, see Fig. 4 illustrating one of the possible cases for labeling. In this graph, we begin to move along edges from the blue vertex in the first step.

In this paper, we are going to study some existence results for a system of fractional differential equations on the ethane graph. We represent the ethane molecule as a graph with labeled vertices by 0 or 1. As you know, ethane is a chemical compound of hydrogen (H) and carbon (C) with chemical formula C_2H_6 (see Fig. 5). Ethane is the simplest hy-



drocarbon that contains a single carbon-carbon bond. We consider atoms of hydrogens and carbons as vertices of the graph and also the existing chemical bonds between atoms are considered as edges of the graph. This molecular graph is not a star graph, and so the method proposed in articles [56] and [57] for assigning the origin at boundary nodes except the junction node v_0 will not be useful because this graph has more than one junction node. Thus we have to use a different method. We can label vertices of the above graph in the form of labeled vertices by two values 0 or 1 and consider the length of each edge to be unit value $l_i = 1$ (see Fig. 6). For more complicate graphs, one can use doubled indexing for nodes of graphs.

Hence, in view of the ethane graph with labeled vertices by 0 or 1 as above, we can perform our purposes on the existence of solutions for the nonlinear fractional boundary value problem (2).

Now, we recall some basic notions and properties about the fractional calculus. Let $\vartheta > 0$. The Riemann–Liouville fractional integral of a function $u : [a, b] \rightarrow \mathbb{R}$ is defined by

$$\mathcal{I}_0^\vartheta u(t) = \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} u(s) ds$$

provided that the right-hand side integral exists [58, 59]. Let $n-1 \leq \vartheta < n$. Then $n = [\vartheta] + 1$. The Caputo fractional derivative of a function $u \in C^{(n)}([a, b], \mathbb{R})$ is defined by ${}^c\mathcal{D}_0^\vartheta u(t) = \int_0^t \frac{(t-s)^{n-\vartheta-1}}{\Gamma(n-\vartheta)} u^{(n)}(s) ds$ provided that the integral exists [58, 59]. It has been proved that the general solution for the homogeneous fractional differential equation ${}^c\mathcal{D}_0^\vartheta u(t) = 0$ is in the form $u(t) = b_0^* + b_1^*t + b_2^*t^2 + \dots + b_{n-1}^*t^{n-1}$, and we have

$$\mathcal{I}_0^\vartheta ({}^c\mathcal{D}_0^\vartheta u(t)) = u(t) + \sum_{j=0}^{n-1} b_j^* t^j = u(t) + b_0^* + b_1^*t + b_2^*t^2 + \dots + b_{n-1}^*t^{n-1},$$

where b_0^*, \dots, b_{n-1}^* are some real constants [60]. We need the next results which are known as Schaefer and Krasnoselskii fixed point theorems, respectively.

Lemma 1 ([61]) *Let \mathcal{X} be a Banach space and $\Upsilon : \mathcal{X} \rightarrow \mathcal{X}$ be a completely continuous operator. Then either the set $\{u \in \mathcal{X} : u = \mu \Upsilon u, \mu \in (0, 1)\}$ is unbounded or the operator Υ has at least one fixed point in \mathcal{X} .*

Lemma 2 ([61]) *Let \mathcal{E} be a closed, bounded, convex, and nonempty subset of a Banach space \mathcal{X} . Let Υ_1 and Υ_2 be two operators such that $\Upsilon_1 u + \Upsilon_2 v \in \mathcal{E}$ whenever $u, v \in \mathcal{E}$, Υ_1 is compact and continuous and Υ_2 is a contraction map. Then there exists $w \in \mathcal{E}$ such that $w = \Upsilon_1 w + \Upsilon_2 w$.*

3 Main results

In this section, we prove our main results on the ethane graph (Fig. 6). In this way, we consider the Banach spaces $\mathcal{X}_i = \{u_i : u_i, u'_i \in C[0, 1]\}$ with the norm $\|u_i\|_{\mathcal{X}_i} = \sup_{t \in [0, 1]} |u_i(t)| + \sup_{t \in [0, 1]} |u'_i(t)|$ for $i = 1, 2, \dots, 7$. Note that the product space $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_7)$ equipped with the norm $\|u = (u_1, u_2, \dots, u_7)\|_{\mathcal{X}} = \sum_{i=1}^7 \|u_i\|_{\mathcal{X}_i}$ is a Banach space.

Lemma 3 *Let $\varphi_1, \dots, \varphi_7 \in C[0, 1]$. Then u_i^* is a solution for the boundary value problem*

$$\begin{cases} {}^c D_0^\vartheta u_i(t) = \varphi_i(t) & (t \in [0, 1]), \\ \lambda_1 u_i(0) + \lambda_2 u_i(1) = \lambda_3 \int_0^1 u_i(s) ds, & \lambda_1 u'_i(0) + \lambda_2 u'_i(1) = \lambda_3 \int_0^1 u_i(s) ds, \end{cases} \tag{3}$$

if and only if u_i^ is a solution for the following fractional integral equation satisfying the boundary conditions:*

$$\begin{aligned} u_i(t) = & \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(s) ds \\ & + \frac{\Lambda_1 - \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 - \lambda_3)t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(s) ds \\ & + \frac{\Lambda_2 - \lambda_2 (\lambda_1 + \lambda_2 - \lambda_3)^2 t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_i(s) ds \\ & + \frac{\Lambda_3 + (\lambda_1 + \lambda_2)\lambda_3 (\lambda_1 + \lambda_2 - \lambda_3)t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(\tau) d\tau ds, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Lambda &= \lambda_1^2 + \lambda_2^2 + 2\lambda_1 \lambda_2 - \frac{3}{2}\lambda_1 \lambda_3 - \frac{1}{2}\lambda_2 \lambda_3 \neq 0, \\ \Lambda_1 &= \frac{3}{2}\lambda_2^2 \lambda_3 - \frac{1}{2}\lambda_2 \lambda_3^2 - \lambda_1^2 \lambda_2 - \lambda_2^3 - 2\lambda_1 \lambda_2^2 + \frac{3}{2}\lambda_1 \lambda_2 \lambda_3, \\ \Lambda_2 &= \lambda_1 \lambda_2^2 + \lambda_2^3 - \frac{3}{2}\lambda_2^2 \lambda_3 - \frac{1}{2}\lambda_1 \lambda_2 \lambda_3 + \frac{1}{2}\lambda_2 \lambda_3^2, \\ \Lambda_3 &= \lambda_1^2 \lambda_3 + \lambda_1 \lambda_2 \lambda_3 - \lambda_1 \lambda_3^2. \end{aligned}$$

Proof Let u_i^* be a solution for the fractional problem (3) ($i = 1, 2, \dots, 7$). Then there exist constants $b_0^{(i)}, b_1^{(i)} \in \mathbb{R}$ such that $u_i^*(t) = \mathcal{I}_0^\vartheta \varphi_i(t) + b_0^{(i)} + b_1^{(i)} t$. In other words,

$$u_i^*(t) = \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(s) ds + b_0^{(i)} + b_1^{(i)} t. \tag{5}$$

Thus, $u_i^{*'}(t) = \int_0^t \frac{(t-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_i(s) ds + b_1^{(i)}$ and so

$$\int_0^1 u_i^{*'}(s) ds = \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(\tau) d\tau ds + b_0^{(i)} + \frac{1}{2} b_1^{(i)}.$$

By using the boundary conditions, we obtain

$$\begin{aligned} b_0^{(i)} &= \frac{\Lambda_1}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(s) ds \\ &+ \frac{\Lambda_2}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_i(s) ds \\ &+ \frac{\Lambda_3}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(\tau) d\tau ds \end{aligned}$$

and

$$\begin{aligned} b_1^{(i)} &= -\frac{\lambda_2\lambda_3}{\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(s) ds - \frac{\lambda_2(\lambda_1 + \lambda_2 - \lambda_3)}{\Lambda} \\ &\times \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} \varphi_i(s) ds + \frac{(\lambda_1 + \lambda_2)\lambda_3}{\Lambda} \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \varphi_i(\tau) d\tau ds. \end{aligned}$$

Now, by substituting the values for $b_0^{(i)}$ and $b_1^{(i)}$ in equation (5), one can find that u_i^* is a solution for integral equation (4). For the converse part, by using some direct calculations, one can see that u_i^* is a solution for the fractional problem (3) whenever u_i^* is a solution for integral equation (5). This completes the proof. \square

Now, by considering Lemma 3, define the operator $\Upsilon : \mathcal{X} \rightarrow \mathcal{X}$ by

$$\Upsilon(u_1, u_2, \dots, u_7)(t) := (\Upsilon_1(u_1, u_2, \dots, u_7)(t), \dots, \Upsilon_7(u_1, u_2, \dots, u_7)(t)), \tag{6}$$

where

$$\begin{aligned} \Upsilon_i(u_1, u_2, \dots, u_7)(t) &= \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(s, u_i(s), u_i'(s)) ds \\ &+ \frac{\Lambda_1 - \lambda_2\lambda_3(\lambda_1 + \lambda_2 - \lambda_3)t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(s, u_i(s), u_i'(s)) ds \\ &+ \frac{\Lambda_2 - \lambda_2(\lambda_1 + \lambda_2 - \lambda_3)^2 t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} h_i(s, u_i(s), u_i'(s)) ds \\ &+ \frac{\Lambda_3 + (\lambda_1 + \lambda_2)\lambda_3(\lambda_1 + \lambda_2 - \lambda_3)t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \\ &\times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(\tau, u_i(\tau), u_i'(\tau)) d\tau ds \end{aligned}$$

for all $t \in [0, 1]$ and $u_i \in \mathcal{X}_i$. Put

$$\begin{aligned}
 \mathcal{K}_0^* &:= \frac{1}{\Gamma(\vartheta + 1)} + \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 1)} \\
 &\quad + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} \\
 &\quad + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 2)}, \\
 \mathcal{K}_1^* &:= \frac{1}{\Gamma(\vartheta)} + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 1)} + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} \\
 &\quad + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 2)}, \\
 \Delta_1^* &:= \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 1)} + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} \\
 &\quad + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 2)}, \\
 \Delta_2^* &:= \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 1)} + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} \\
 &\quad + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 2)}.
 \end{aligned} \tag{7}$$

Theorem 4 *Let $h_1, \dots, h_7 : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Assume that there exist constants $L_i > 0$ such that $|h_i(t, u_1, u_2)| \leq L_i$ for all $u_1, u_2 \in \mathbb{R}$ and $t \in [0, 1]$ ($i = 1, 2, \dots, 7$). Then the fractional boundary value problem (2) has a solution.*

Proof By considering definition of the operator \mathcal{Y} , it is clear that the fractional boundary value problem (2) has a solution if and only if \mathcal{Y} has a fixed point on the product space $\mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_7$. First, we show that the operator \mathcal{Y} is complete continuous. Since the functions h_1, \dots, h_7 are continuous, the operator $\mathcal{Y} : \mathcal{X} \rightarrow \mathcal{X}$ is continuous. Let M be a bounded subset of \mathcal{X} and $u = (u_1, u_2, \dots, u_7) \in \mathcal{X}$. Then we have

$$\begin{aligned}
 |\mathcal{Y}_i u(t)| &\leq \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| ds \\
 &\quad + \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| ds \\
 &\quad + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2 t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, u_i(s), u'_i(s))| ds \\
 &\quad + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 &\quad \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, u_i(\tau), u'_i(\tau))| d\tau ds \\
 &\leq \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} L_i ds + \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} L_i ds \\
 &\quad + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2 t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} L_i ds
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \int_0^s \frac{(s - \tau)^{\vartheta-1}}{\Gamma(\vartheta)} L_i \, d\tau \, ds \\
 \leq & L_i \left[\frac{1}{\Gamma(\vartheta + 1)} + \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 1)} \right. \\
 & \left. + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 2)} \right] = L_i \mathcal{K}_0^*
 \end{aligned}$$

for each $t \in [0, 1]$, where \mathcal{K}_0^* is given in (7). Also, we have

$$\begin{aligned}
 |\mathcal{Y}'_i u(t)| \leq & \int_0^t \frac{(t - s)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 & + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1 - s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 & + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1 - s)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 & + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \int_0^s \frac{(s - \tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, u_i(\tau), u'_i(\tau))| \, d\tau \, ds \\
 \leq & L_i \left[\frac{1}{\Gamma(\vartheta)} + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 1)} \right. \\
 & \left. + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta + 2)} \right] = L_i \mathcal{K}_1^*
 \end{aligned}$$

for all $t \in [0, 1]$, where \mathcal{K}_1^* is given in (7). This implies that $\|\mathcal{Y}_i u(t)\|_{\mathcal{X}_i} \leq L_i(\mathcal{K}_0^* + \mathcal{K}_1^*)$. Hence, $\|\mathcal{Y} u(t)\|_{\mathcal{X}} = \sum_{i=1}^7 \|\mathcal{Y}_i u(t)\|_{\mathcal{X}_i} \leq \sum_{i=1}^7 L_i(\mathcal{K}_0^* + \mathcal{K}_1^*) < \infty$. This shows that the operator \mathcal{Y} is uniformly bounded. Now, we prove that the operator \mathcal{Y} is equicontinuous. Now, let $u = (u_1, u_2, \dots, u_7) \in M$ and $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
 |\mathcal{Y}_i u(t_2) - \mathcal{Y}_i u(t_1)| \leq & \int_0^{t_1} \frac{[(t_2 - s)^{\vartheta-1} - (t_1 - s)^{\vartheta-1}]}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 & + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 & + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|(t_2 - t_1)}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1 - s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 & + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2(t_2 - t_1)}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1 - s)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 & + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|(t_2 - t_1)}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \int_0^s \frac{(s - \tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, u_i(\tau), u'_i(\tau))| \, d\tau \, ds.
 \end{aligned}$$

The right-hand side of the inequality converges to zero independently of $u \in M$ as $t_1 \rightarrow t_2$. Similarly, we have

$$\begin{aligned}
 |\mathcal{Y}'_i u(t_2) - \mathcal{Y}'_i u(t_1)| \leq & \int_0^{t_1} \frac{[(t_2 - s)^{\vartheta-2} - (t_1 - s)^{\vartheta-2}]}{\Gamma(\vartheta - 1)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 & + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_i(s, u_i(s), u'_i(s))| \, ds.
 \end{aligned}$$

Again, the right-hand side of the inequality converges to zero independently of $u \in M$ as $t_1 \rightarrow t_2$. Hence, $\|\mathcal{Y}u(t_2) - \mathcal{Y}u(t_1)\|_{\mathcal{X}} \rightarrow 0$ as $t_1 \rightarrow t_2$. This shows that \mathcal{Y} is an equicontinuous operator on the product space \mathcal{X} . Now by using the Arzela–Ascoli theorem, one can conclude that \mathcal{Y} is a completely continuous operator. Here, consider the subset

$$\Omega := \{(u_1, u_2, \dots, u_7) \in \mathcal{X} : (u_1, u_2, \dots, u_7) = \mu \mathcal{Y}(u_1, u_2, \dots, u_7), \mu \in (0, 1)\}$$

of \mathcal{X} . We prove that Ω is a bounded set. Let $(u_1, u_2, \dots, u_7) \in \Omega$. Then

$$(u_1, u_2, \dots, u_7) = \mu \mathcal{Y}(u_1, u_2, \dots, u_7),$$

and so $u_i(t) = \mu \mathcal{Y}_i(u_1, u_2, \dots, u_7)$ for all $t \in [0, 1]$ and $i = 1, 2, \dots, 7$. Thus,

$$\begin{aligned} |u_i(t)| &\leq \mu \left[\int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| \, ds \right. \\ &\quad + \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| \, ds \\ &\quad + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2 t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, u_i(s), u'_i(s))| \, ds \\ &\quad + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\ &\quad \left. \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, u_i(\tau), u'_i(\tau))| \, d\tau \, ds \right] \\ &\leq \mu L_i \left[\frac{1}{\Gamma(\vartheta+1)} + \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta+1)} \right. \\ &\quad \left. + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta+2)} \right] = \mu L_i \mathcal{K}_0^* \end{aligned}$$

and

$$\begin{aligned} |u'_i(t)| &\leq \mu \left[\int_0^t \frac{(t-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, u_i(s), u'_i(s))| \, ds \right. \\ &\quad + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s))| \, ds \\ &\quad + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, u_i(s), u'_i(s))| \, ds \\ &\quad + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, u_i(\tau), u'_i(\tau))| \, d\tau \, ds \left. \right] \\ &\leq \mu L_i \left[\frac{1}{\Gamma(\vartheta)} + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta+1)} \right. \\ &\quad \left. + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta+2)} \right] = \mu L_i \mathcal{K}_1^*. \end{aligned}$$

This implies that $\|u\|_{\mathcal{X}} = \sum_{i=1}^7 \|u_i\|_{\mathcal{X}_i} \leq \mu \sum_{i=1}^7 L_i(\mathcal{K}_0^* + \mathcal{K}_1^*) < \infty$ and so Ω is bounded. Now, by using Lemmas 1 and 3, the operator \mathcal{Y} has a fixed point in \mathcal{X} which is a solution for the fractional boundary value problem (2). □

Now, we review the fractional boundary value problem (2) under different conditions.

Theorem 5 Assume that $h_1, \dots, h_7 : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, there exist continuous functions $\sigma_1, \dots, \sigma_7 : [0, 1] \rightarrow \mathbb{R}$, $\delta_1, \dots, \delta_7 : [0, 1] \rightarrow \mathbb{R}^+$ and nondecreasing continuous functions $\phi_1, \dots, \phi_7 : [0, 1] \rightarrow \mathbb{R}^+$ such that $|h_i(t, u_1, u_2)| \leq \delta_i(t)\phi_i(|u_1| + |u_2|)$ and

$$|h_i(t, u_1, u_2) - h_i(t, v_1, v_2)| \leq \sigma_i(t)(|u_1 - v_1| + |u_2 - v_2|)$$

for all $t \in [0, 1]$, $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $i = 1, \dots, 7$. If $k := (\Delta_1^* + \Delta_2^*) \sum_{i=1}^7 \|\sigma_i\| < 1$, then the fractional boundary value problem (2) has a solution, where $\|\sigma_i\| = \sup_{t \in [0,1]} |\sigma_i(t)|$ and the constants Δ_1^* and Δ_2^* are given in (7).

Proof Put $\|\delta_i\| = \sup_{t \in [0,1]} |\delta_i(t)|$ and choose suitable constants ρ_i such that

$$\rho_i \geq \sum_{i=1}^7 \phi_i(\|u_i\|_{\mathcal{X}_i}) \|\delta_i\| \{ \mathcal{K}_0^* + \mathcal{K}_1^* \}, \tag{8}$$

where \mathcal{K}_i^* s are given in (7). Consider the sets $\mathcal{B}_{\rho_i} := \{u = (u_1, u_2, \dots, u_7) \in \mathcal{X} : \|u\|_{\mathcal{X}} \leq \rho_i\}$, where ρ_i is given in (8). It is clear that \mathcal{B}_{ρ_i} is a closed, bounded, convex, and nonempty subset of the product Banach space \mathcal{X} . Now, define the operators \mathcal{Y}_1 and \mathcal{Y}_2 on \mathcal{B}_{ρ_i} by

$$\begin{aligned} \mathcal{Y}_1(u_1, u_2, \dots, u_7)(t) &:= (\mathcal{Y}_1^{(1)}(u_1, u_2, \dots, u_7)(t), \dots, \mathcal{Y}_1^{(7)}(u_1, u_2, \dots, u_7)(t)), \\ \mathcal{Y}_2(u_1, u_2, \dots, u_7)(t) &:= (\mathcal{Y}_2^{(1)}(u_1, u_2, \dots, u_7)(t), \dots, \mathcal{Y}_2^{(7)}(u_1, u_2, \dots, u_7)(t)), \end{aligned}$$

where $(\mathcal{Y}_1^{(i)}u)(t) = \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(s, u_i(s), u_i'(s)) ds$ and

$$\begin{aligned} (\mathcal{Y}_2^{(i)}u)(t) &= \frac{\Lambda_1 - \lambda_2\lambda_3(\lambda_1 + \lambda_2 - \lambda_3)t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(s, u_i(s), u_i'(s)) ds \\ &+ \frac{\Lambda_2 - \lambda_2(\lambda_1 + \lambda_2 - \lambda_3)^2t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} h_i(s, u_i(s), u_i'(s)) ds \\ &+ \frac{\Lambda_3 + (\lambda_1 + \lambda_2)\lambda_3(\lambda_1 + \lambda_2 - \lambda_3)t}{(\lambda_1 + \lambda_2 - \lambda_3)\Lambda} \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(\tau, u_i(\tau), u_i'(\tau)) d\tau ds \end{aligned}$$

for all $t \in [0, 1]$ and $u = (u_1, u_2, \dots, u_7) \in \mathcal{B}_{\rho_i}$. Let $\phi_i^* = \sup_{u_i \in \mathcal{X}_i} \phi_i(\|u_i\|_{\mathcal{X}_i})$. Now, for every $u = (u_1, \dots, u_7), v = (v_1, \dots, v_7) \in \mathcal{B}_{\rho_i}$, we have

$$\begin{aligned} &|(\mathcal{Y}_1^{(i)}u + \mathcal{Y}_2^{(i)}v)(t)| \\ &\leq \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u_i'(s))| ds \\ &+ \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, v_i(s), v_i'(s))| ds \\ &+ \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, v_i(s), v_i'(s))| ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, v_i(\tau), v_i'(\tau))| \, d\tau \, ds \\
 \leq & \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} \delta_i(s) \phi_i(|u_i(s)| + |u_i'(s)|) \, ds \\
 & + \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \delta_i(s) \phi_i(|v_i(s)| + |v_i'(s)|) \, ds \\
 & + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2 t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} \delta_i(s) \phi_i(|v_i(s)| + |v_i'(s)|) \, ds \\
 & + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|t}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \delta_i(\tau) \phi_i(|v_i(\tau)| + |v_i'(\tau)|) \, d\tau \, ds \\
 \leq & \|\delta_i\| \phi_i^* \left[\frac{1}{\Gamma(\vartheta+1)} + \frac{|\Lambda_1| + |\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta+1)} \right. \\
 & \left. + \frac{|\Lambda_2| + |\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta+2)} \right] = \|\delta_i\| \phi_i^* \mathcal{K}_0^*,
 \end{aligned}$$

and

$$\begin{aligned}
 & |((\Upsilon_1^{(i)})' u + (\Upsilon_2^{(i)})' v)(t)| \\
 \leq & \int_0^t \frac{(t-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, u_i(s), u_i'(s))| \, ds \\
 & + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, v_i(s), v_i'(s))| \, ds \\
 & + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, v_i(s), v_i'(s))| \, ds \\
 & + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, v_i(\tau), v_i'(\tau))| \, d\tau \, ds \\
 \leq & \int_0^t \frac{(t-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} \delta_i(s) \phi_i(|u_i(s)| + |u_i'(s)|) \, ds \\
 & + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \delta_i(s) \phi_i(|v_i(s)| + |v_i'(s)|) \, ds \\
 & + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} \delta_i(s) \phi_i(|v_i(s)| + |v_i'(s)|) \, ds \\
 & + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \delta_i(\tau) \phi_i(|v_i(\tau)| + |v_i'(\tau)|) \, d\tau \, ds \\
 \leq & \|\delta_i\| \phi_i^* \left[\frac{1}{\Gamma(\vartheta)} + \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta+1)} + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta)} \right. \\
 & \left. + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|\Gamma(\vartheta+2)} \right] = \|\delta_i\| \phi_i^* \mathcal{K}_1^*.
 \end{aligned}$$

This yields $\|\Upsilon_1 u + \Upsilon_2 v\|_{\mathcal{X}} = \sum_{i=1}^7 \|\Upsilon_1^{(i)} u + \Upsilon_2^{(i)} v\|_{\mathcal{X}_i} \leq \sum_{i=1}^7 \|\delta_i\| \phi_i^*(\mathcal{K}_0^* + \mathcal{K}_1^*) \leq \rho_i$, and so $\|\Upsilon_1 u + \Upsilon_2 v\|_{\mathcal{X}} \leq \rho_i$ and $\Upsilon_1 u + \Upsilon_2 v \in \mathcal{B}_{\rho_i}$. On the other hand, it is clear that the continuity of Υ_1 follows from the continuity of functions h_i . Now, we show that the operator Υ_1 is uniformly bounded. For this, note that

$$\begin{aligned} |(\Upsilon_1^{(i)} u)(t)| &\leq \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u_i'(s))| \, ds \\ &\leq \frac{1}{\Gamma(\vartheta+1)} \|\delta_i\| \phi_i(|u_i(t)| + |u_i'(t)|), \end{aligned}$$

and $|((\Upsilon_1^{(i)})' u)(t)| \leq \int_0^t \frac{(t-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, u_i(s), u_i'(s))| \, ds \leq \frac{1}{\Gamma(\vartheta)} \|\delta_i\| \phi_i(|u_i(t)| + |u_i'(t)|)$ for all u in \mathcal{B}_{ρ_i} . Thus, $\|\Upsilon_1 u\|_{\mathcal{X}} = \sum_{i=1}^7 \|\Upsilon_1^{(i)} u\|_{\mathcal{X}_i} \leq \left\{ \frac{1}{\Gamma(\vartheta+1)} + \frac{1}{\Gamma(\vartheta)} \right\} \sum_{i=1}^7 \|\delta_i\| \phi_i(\|u_i\|_{\mathcal{X}_i})$. This shows that the operator Υ_1 is uniformly bounded on \mathcal{B}_{ρ_i} . Now, we show that the operator Υ_1 is compact on \mathcal{B}_{ρ_i} . Let $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned} &|(\Upsilon_1^{(i)} u)(t_2) - (\Upsilon_1^{(i)} u)(t_1)| \\ &= \left| \int_0^{t_2} \frac{(t_2-s)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(s, u_i(s), u_i'(s)) \, ds - \int_0^{t_1} \frac{(t_1-s)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(s, u_i(s), u_i'(s)) \, ds \right| \\ &\leq \left| \int_0^{t_1} \frac{[(t_2-s)^{\vartheta-1} - (t_1-s)^{\vartheta-1}]}{\Gamma(\vartheta)} h_i(s, u_i(s), u_i'(s)) \, ds \right| \\ &\quad + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\vartheta-1}}{\Gamma(\vartheta)} h_i(s, u_i(s), u_i'(s)) \, ds \right| \\ &\leq \int_0^{t_1} \frac{[(t_2-s)^{\vartheta-1} - (t_1-s)^{\vartheta-1}]}{\Gamma(\vartheta)} |h_i(s, u_i(s), u_i'(s))| \, ds \\ &\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u_i'(s))| \, ds \\ &\leq \left\{ \frac{t_2^\vartheta - t_1^\vartheta - (t_2 - t_1)^\vartheta}{\Gamma(\vartheta+1)} + \frac{(t_2 - t_1)^\vartheta}{\Gamma(\vartheta+1)} \right\} \|\delta_i\| \phi_i(\|u_i\|_{\mathcal{X}_i}). \end{aligned}$$

Hence, $|(\Upsilon_1^{(i)} u)(t_2) - (\Upsilon_1^{(i)} u)(t_1)| \rightarrow 0$ as $t_1 \rightarrow t_2$. Also, we have

$$\begin{aligned} &|((\Upsilon_1^{(i)})' u)(t_2) - ((\Upsilon_1^{(i)})' u)(t_1)| \\ &= \left| \int_0^{t_2} \frac{(t_2-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} h_i(s, u_i(s), u_i'(s)) \, ds - \int_0^{t_1} \frac{(t_1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} h_i(s, u_i(s), u_i'(s)) \, ds \right| \\ &\leq \left| \int_0^{t_1} \frac{[(t_2-s)^{\vartheta-2} - (t_1-s)^{\vartheta-2}]}{\Gamma(\vartheta-1)} h_i(s, u_i(s), u_i'(s)) \, ds \right| \\ &\quad + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} h_i(s, u_i(s), u_i'(s)) \, ds \right| \\ &\leq \int_0^{t_1} \frac{[(t_2-s)^{\vartheta-2} - (t_1-s)^{\vartheta-2}]}{\Gamma(\vartheta-1)} |h_i(s, u_i(s), u_i'(s))| \, ds \end{aligned}$$

$$\begin{aligned}
 & + \int_{t_1}^{t_2} \frac{(t_2 - s)^{(\vartheta-2)}}{\Gamma(\vartheta - 1)} |h_i(s, u_i(s), u'_i(s))| \, ds \\
 \leq & \left\{ \frac{t_2^{\vartheta-1} - t_1^{\vartheta-1} - (t_2 - t_1)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{(t_2 - t_1)^{\vartheta-1}}{\Gamma(\vartheta)} \right\} \|\delta_i\| \phi_i(\|u_i\|_{\mathcal{X}_i}),
 \end{aligned}$$

and so $\|((\mathcal{Y}_1^{(i)})'u)(t_2) - ((\mathcal{Y}_1^{(i)})'u)(t_1)\| \rightarrow 0$ as $t_1 \rightarrow t_2$. Hence, $\|(\mathcal{Y}_1 u)(t_2) - (\mathcal{Y}_1 u)(t_1)\|_{\mathcal{X}}$ tends to zero as $t_1 \rightarrow t_2$. Thus, \mathcal{Y}_1 is equicontinuous, and so \mathcal{Y}_1 is a relatively compact operator on \mathcal{B}_{ρ_i} . Now, by using the Arzela–Ascoli theorem, we conclude that the operator \mathcal{Y}_1 is compact on \mathcal{B}_{ρ_i} . Here, we prove that the operator \mathcal{Y}_2 is a contraction. Let $u, v \in \mathcal{B}_{\rho_i}$. Then we have

$$\begin{aligned}
 & |(\mathcal{Y}_2^{(i)} u)(t) - (\mathcal{Y}_2^{(i)} v)(t)| \\
 \leq & \frac{|\Lambda_1| + |\lambda_2| |\lambda_3| |\lambda_1 + \lambda_2 - \lambda_3| t}{|\lambda_1 + \lambda_2 - \lambda_3| |\Lambda|} \\
 & \times \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s)) - h_i(s, v_i(s), v'_i(s))| \, ds \\
 & + \frac{|\Lambda_2| + |\lambda_2| (\lambda_1 + \lambda_2 - \lambda_3)^2 t}{|\lambda_1 + \lambda_2 - \lambda_3| |\Lambda|} \\
 & \times \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta - 1)} |h_i(s, u_i(s), u'_i(s)) - h_i(s, v_i(s), v'_i(s))| \, ds \\
 & + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2| |\lambda_3| |\lambda_1 + \lambda_2 - \lambda_3| t}{|\lambda_1 + \lambda_2 - \lambda_3| |\Lambda|} \\
 & \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, u_i(\tau), u'_i(\tau)) - h_i(\tau, v_i(\tau), v'_i(\tau))| \, d\tau \, ds \\
 \leq & \frac{|\Lambda_1| + |\lambda_2| |\lambda_3| |\lambda_1 + \lambda_2 - \lambda_3| t}{|\lambda_1 + \lambda_2 - \lambda_3| |\Lambda|} \\
 & \times \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \sigma_i(s) (|u_i(s) - v_i(s)| + |u'_i(s) - v'_i(s)|) \, ds \\
 & + \frac{|\Lambda_2| + |\lambda_2| (\lambda_1 + \lambda_2 - \lambda_3)^2 t}{|\lambda_1 + \lambda_2 - \lambda_3| |\Lambda|} \\
 & \times \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta - 1)} \sigma_i(s) (|u_i(s) - v_i(s)| + |u'_i(s) - v'_i(s)|) \, ds \\
 & + \frac{|\Lambda_3| + |\lambda_1 + \lambda_2| |\lambda_3| |\lambda_1 + \lambda_2 - \lambda_3| t}{|\lambda_1 + \lambda_2 - \lambda_3| |\Lambda|} \\
 & \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \sigma_i(\tau) (|u_i(\tau) - v_i(\tau)| + |u'_i(\tau) - v'_i(\tau)|) \, d\tau \, ds
 \end{aligned}$$

and

$$\begin{aligned}
 & |((\mathcal{Y}_2^{(i)})' u)(t) - ((\mathcal{Y}_2^{(i)})' v)(t)| \\
 \leq & \frac{|\lambda_2| |\lambda_3| |\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3| |\Lambda|} \\
 & \times \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(s, u_i(s), u'_i(s)) - h_i(s, v_i(s), v'_i(s))| \, ds
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} |h_i(s, u_i(s), u'_i(s)) - h_i(s, v_i(s), v'_i(s))| \, ds \\
 & + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} |h_i(\tau, u_i(\tau), u'_i(\tau)) - h_i(\tau, v_i(\tau), v'_i(\tau))| \, d\tau \, ds \\
 \leq & \frac{|\lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \frac{(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \sigma_i(s) (|u_i(s) - v_i(s)| + |u'_i(s) - v'_i(s)|) \, ds \\
 & + \frac{|\lambda_2|(\lambda_1 + \lambda_2 - \lambda_3)^2}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \frac{(1-s)^{\vartheta-2}}{\Gamma(\vartheta-1)} \sigma_i(s) (|u_i(s) - v_i(s)| + |u'_i(s) - v'_i(s)|) \, ds \\
 & + \frac{|\lambda_1 + \lambda_2||\lambda_3||\lambda_1 + \lambda_2 - \lambda_3|}{|\lambda_1 + \lambda_2 - \lambda_3||\Lambda|} \\
 & \times \int_0^1 \int_0^s \frac{(s-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \sigma_i(\tau) (|u_i(\tau) - v_i(\tau)| + |u'_i(\tau) - v'_i(\tau)|) \, d\tau \, ds.
 \end{aligned}$$

Hence, we obtain

$$\begin{aligned}
 \sup_{t \in [0,1]} |(\mathcal{Y}_2^{(i)} u)(t) - (\mathcal{Y}_2^{(i)} v)(t)| & \leq \|\sigma_i\| \Delta_1^* \|u_i - v_i\|_{\mathcal{X}_i}, \\
 \sup_{t \in [0,1]} |((\mathcal{Y}_2^{(i)})' u)(t) - ((\mathcal{Y}_2^{(i)})' v)(t)| & \leq \|\sigma_i\| \Delta_2^* \|u_i - v_i\|_{\mathcal{X}_i}.
 \end{aligned}$$

Thus, we get

$$\|\mathcal{Y}_2 u - \mathcal{Y}_2 v\|_{\mathcal{X}} = \sum_{i=1}^7 \|\mathcal{Y}_2^{(i)} u - \mathcal{Y}_2^{(i)} v\|_{\mathcal{X}_i} \leq (\Delta_1^* + \Delta_2^*) \sum_{i=1}^7 \|\sigma_i\| \|u_i - v_i\|_{\mathcal{X}_i},$$

and so $\|\mathcal{Y}_2 u - \mathcal{Y}_2 v\|_{\mathcal{X}} \leq k \|u - v\|_{\mathcal{X}}$. Since $k < 1$, \mathcal{Y}_2 is a contraction on \mathcal{B}_{ρ_i} . Now, by using Lemma 2, we conclude that the operator \mathcal{Y} has a fixed point which is a solution for the fractional boundary value problem (2). □

Finally, one can easily prove the uniqueness of solutions for the fractional problem (2).

Theorem 6 *Let $h_1, \dots, h_7 : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Assume that there exist constants $L_i > 0$ such that $|h_i(t, u_1, u_2) - h_i(t, u'_1, u'_2)| \leq L_i(|u_1 - u'_1| + |u_2 - u'_2|)$ for all $u_1, u_2, u'_1, u'_2 \in \mathbb{R}$ and $t \in [0, 1]$ ($i = 1, 2, \dots, 7$). Then the fractional boundary value problem (2) has a unique solution if $\sum_{i=1}^7 L_i(\mathcal{K}_0^* + \mathcal{K}_1^*) < 1$.*

Now, we present an example to illustrate our last main result.

Example 1 Consider the system of fractional differential equations

$$\begin{cases} {}^c\mathcal{D}_0^{1.08} u_1(t) = \frac{t|u_1(t)|}{10(1+|u_1(t)|)} + 0.1t|\arcsin u_1'(t)|, \\ {}^c\mathcal{D}_0^{1.08} u_2(t) = \frac{t}{2000}|\arctan u_2(t)| + \frac{t|\arcsin u_2'(t)|}{2000+2000|\arcsin u_2'(t)|}, \\ {}^c\mathcal{D}_0^{1.08} u_3(t) = 0.01t|\sin u_3(t)| + 0.01t\frac{|u_3'(t)|^3}{2+|u_3'(t)|^3} \end{cases} \tag{9}$$

with integral boundary value conditions

$$\begin{cases} 0.7u_1(0) + 0.12u_1(1) = 0.1 \int_0^1 u_1(s) ds, \\ 0.7u_1'(0) + 0.12u_1'(1) = 0.1 \int_0^1 u_1'(s) ds, \\ 0.7u_2(0) + 0.12u_2(1) = 0.1 \int_0^1 u_2(s) ds, \\ 0.7u_2'(0) + 0.12u_2'(1) = 0.1 \int_0^1 u_2'(s) ds, \\ 0.7u_3(0) + 0.12u_3(1) = 0.1 \int_0^1 u_3(s) ds, \\ 0.7u_3'(0) + 0.12u_3'(1) = 0.1 \int_0^1 u_3'(s) ds, \end{cases} \tag{10}$$

where $\vartheta = 1.08, \lambda_1 = 0.7, \lambda_2 = 0.12, \lambda_3 = 0.1, n = 3$ and ${}^c\mathcal{D}_0^{1.08}$ denotes the Caputo fractional derivative of order $\vartheta = 1.08$. Define continuous functions $h_1, h_2, h_3 : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{cases} h_1(t, u(t), v(t)) = \frac{t|u(t)|}{10(1+|u(t)|)} + 0.1t|\arcsin v(t)|, \\ h_2(t, u(t), v(t)) = \frac{t}{2000}|\arctan u(t)| + \frac{t|\arcsin v(t)|}{2000+2000|\arcsin v(t)|}, \\ h_3(t, u(t), v(t)) = 0.01t|\sin u(t)| + 0.01t\frac{|v(t)|^3}{2+|v(t)|^3}. \end{cases}$$

Let $u_1, u_2, v_1, v_2 \in \mathbb{R}$. Then we have

$$\begin{aligned} &|h_1(t, u_1(t), v_1(t)) - h_1(t, u_2(t), v_2(t))| \\ &\leq \frac{t}{10} (|u_1(t) - u_2(t)| + |\arcsin v_1(t) - \arcsin v_2(t)|) \\ &\leq \frac{t}{10} (|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|), \\ &|h_2(t, u_1(t), v_1(t)) - h_2(t, u_2(t), v_2(t))| \\ &\leq \frac{t}{2000} (|\arctan u_1(t) - \arctan u_2(t)| + |\arcsin v_1(t) - \arcsin v_2(t)|) \\ &\leq \frac{t}{2000} (|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|), \end{aligned}$$

and

$$\begin{aligned} &|h_3(t, u_1(t), v_1(t)) - h_3(t, u_2(t), v_2(t))| \\ &\leq \frac{t}{100} (|\sin u_1(t) - \sin u_2(t)| + |v_1(t) - v_2(t)|) \\ &\leq \frac{t}{100} (|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|). \end{aligned}$$

Hence, $\sigma_1(t) = \frac{t}{10}$, $\sigma_2(t) = \frac{t}{2000}$, and $\sigma_3(t) = \frac{t}{100}$ where $\|\sigma_1\| = \frac{1}{10}$, $\|\sigma_2\| = \frac{1}{2000}$, and $\|\sigma_3\| = \frac{1}{100}$. Define the continuous and nondecreasing functions $\phi_i : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $\phi_1(u) = \phi_2(u) = \phi_3(u) = u$ for all $u \in \mathbb{R}^+$. Then we get

$$\begin{aligned} |h_1(t, u(t), u'(t))| &\leq \frac{t}{10} (|u| + |\arcsin u'|) \\ &\leq \frac{t}{10} (|u| + |u'|) = \frac{t}{10} \phi_1(|u| + |u'|), \\ |h_2(t, u(t), u'(t))| &\leq \frac{t}{2000} (|\arctan u| + |\arcsin u'|) \\ &\leq \frac{t}{2000} (|u| + |u'|) = \frac{t}{2000} \phi_2(|u| + |u'|), \\ |h_3(t, u(t), u'(t))| &\leq \frac{t}{100} (|\sin u| + |u'|) \\ &\leq \frac{t}{100} (|u| + |u'|) = \frac{t}{100} \phi_3(|u| + |u'|), \end{aligned}$$

where continuous functions $\delta_i : [0, 1] \rightarrow \mathbb{R}$ are defined by $\delta_1(t) = \frac{t}{10}$, $\delta_2(t) = \frac{t}{2000}$, and $\delta_3(t) = \frac{t}{100}$. According to the obtained values, we get $\Delta_1^* \simeq 0.4808$, $\Delta_2^* \simeq 0.2486$, and so $\Delta_1^* + \Delta_2^* \simeq 0.7294$. Hence, $k := (\Delta_1^* + \Delta_2^*) \sum_{i=1}^3 \|\sigma_i\| = (\Delta_1^* + \Delta_2^*) (\|\sigma_1\| + \|\sigma_2\| + \|\sigma_3\|) \simeq 0.08059 < 1$. Now, by using Theorem 5, we conclude that the fractional boundary value problems (9)–(10) have a solution.

4 Conclusion

With the development and expansion of some dynamical and industrial systems such as gas transmission lines, water pipelines, the expansive growth of computer networks, structure of molecules in medicine and biology, etc., new descriptive models have emerged for studying the related processes designed by specialists of these fields. Due to the graph representation of these networks, the study of mathematical models described by ordinary or fractional differential equations on special graphs was considered. In this paper, our aim was to extend the idea on more generalized graphs. In this way, we investigated the existence of solutions for some fractional boundary value problems on the ethane graph. We presented two distinct results by considering different conditions on the problems. Also, we provided an example to illustrate our last main result.

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