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Existence of solutions for integral boundary value problems of singular Hadamard-type fractional differential equations on infinite interval

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Abstract

We consider the existence of solutions for the following Hadamard-type fractional differential equations:

$$\begin{cases} {}^{H}D^{\alpha}u(t)+q(t)f(t,u(t),{}^{H}D^{\beta_1}u(t),{}^{H}D^{\beta_2}u(t))=0, \quad 1<t<+\infty, \\ u(1)=0, \\ {}^{H}D^{\alpha-2}u(1)=\int_1^{+\infty}g_1(s)u(s)\frac{ds}{s}, \\ {}^{H}D^{\alpha-1}u(+\infty)=\int_1^{+\infty}g_2(s)u(s)\frac{ds}{s}, \end{cases}$$

where $2<\alpha\le 3$, $0<\beta_1\le\alpha-2<\beta_2\le\alpha-1$, $f:J\times\mathbb{R}^3\to\mathbb{R}$ satisfies the q-Carathéodory condition, $q,g_1,g_2:J\to\mathbb{R}^+$ are nonnegative, where $J=[1,+\infty)$. Nonlinear term f is dependent on the fractional derivative of lower order β_1, β_2, which creates additional complexity to verify the existence of solutions. The singularity occurring in our problem is associated with ${}^{H}D^{\beta_2}u\in C(1,+\infty)$ at the left endpoint $t=1$ (if $\beta_2<\alpha-1$).

MSC: 34B18; 34B10; 34B15

Keywords: Hadamard-type fractional differential equation; Carathéodory condition; Infinite interval; Fixed point theory

1 Introduction

Fractional differential equations arise from various applications including in a variety of fields of mathematical and natural science. Fractional boundary value problems can more accurately describe the nature of practical problems, they have solved a large number of applications in different kinds of fields such as viscoelasticity, biomedical engineering, mechanical, anomalous diffusion, etc. Therefore, they have become a research hot-spot. A variety of techniques, such as the method of mixed monotone operator, topological degree method, monotone iterative technique, etc., have been applied to obtain the existence of solutions for fractional boundary value problems (see [1–32]).

Apart from the common differential equation with Riemann–Liouville and Caputo fractional derivative, there are also several kinds of fractional derivatives: Hadamard, Erdelyi–Kober, Hilfer, and so on. Here we emphasize that the studies about Hadamard fractional differential equations are still at the early stage and need further investigation. More details and recent contributions to the topic can be found in [33–51] and the references therein.

In [36], the author used the fixed point index to study the existence of positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions:

$$\begin{cases} D^{\beta}u(t) + f_1(t,u(t),v(t)), & t \in [1,e], \\ D^{\beta}v(t) + f_2(t,u(t),v(t)), & t \in [1,e], \\ u(1) = v(1) = u'(1) = v'(1) = 0, \\ u(e) = \int_1^e h(s)v(s)\frac{ds}{s}, \\ v(e) = \int_1^e g(s)u(s)\frac{ds}{s}, \end{cases}$$

where $\beta \in (2,3], f_1, f_2$ are nonnegative continuous functions on $[1,e] \times \mathbb{R}^+ \times \mathbb{R}^+$.

Ardjouni in [45] employed the Schauder and Banach fixed point theorems and the method of upper and lower solutions to show the existence and uniqueness of a positive solution for nonlinear Hadamard fractional differential equations with integral boundary conditions:

$$\begin{cases} D_1^{\alpha}x(t) + f(t,x(t)) = D_1^{\beta}g(t,x(t)), & t \in [1,e], \\ x(1) = 0, \quad x(e) = \frac{1}{\Gamma(\alpha-\beta)}\int_1^e (\log\frac{e}{s})^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}, \end{cases}$$

where $1 < \alpha \le 2$, $0 < \beta \le \alpha - 1$, $g, f : [1,e] \times [0,\infty) \to [0,\infty)$ are given continuous functions, g is nondecreasing on x, and f does not require any monotone assumption.

In [46], Pei et al. investigated the following boundary value problem of Hadamard fractional integro-differential equations on infinite domain:

$$\begin{cases} {}^{H}D^{\alpha}u(t) + f(t,u(t),{}^{H}I^{\gamma}u(t),{}^{H}D^{\alpha-1}u(t)) = 0, & 1 < \alpha < 2, t \in (1,\infty), \\ u(1) = 0, \quad {}^{H}D^{\alpha-1}u(\infty) = \sum_{i=1}^{m}\lambda_i {}^{H}I^{\beta_i}u(\eta), \end{cases}$$

where $\gamma, \beta_i, \lambda_i \ge 0$ $(i = 1,2,\dots,m)$ are given constants and η, β_i, λ_i satisfy $\Gamma(\alpha) > \sum_{i=1}^{m}\frac{\lambda_i\Gamma(\alpha)}{\Gamma(\alpha+\beta_i)} \times (\log t)^{\alpha+\beta_i-1}$. The nonlinear term f is nondecreasing with respect to the second, third, and last variables. By use of the monotone iterative method, the authors obtained not only the existence of positive solutions for Hadamard fractional integro-differential equations on infinite intervals, but also the minimal and maximal positive solutions and two explicit monotone iterative sequences converging to the extremal solution.

El-Sayed and Gaafar [47] established the existence of positive solutions to the following singular nonlinear Hadamard-type fractional differential equations with infinite-point boundary conditions or integral boundary condition:

$$\begin{aligned} &{}_{H}D^{\gamma}v(t) + f\bigl(t,v(t),{}_{H}D^{\delta}v(t),v'(t)\bigr) = 0, \quad \text{a.e. } t \in (1,e), \\ &v(1) = 0, \quad v(e) = v_0 + \lambda\int_1^e v\bigl(\Phi(\xi)\bigr)\frac{\Phi'(\xi)}{\Phi(\xi)}\,d\xi, \end{aligned}$$

or

$$v(1) = 0, \quad v(e) = v_0 + \lambda \sum_{j=1}^{\infty} a_j v\big(\Phi(\eta_j)\big),$$

where $1 < \gamma < 2$, $0 < \delta < 1$, $1 \le \gamma - \delta < 2$, λ, v_0, a_j are nonnegative constants. $f : [1, e] \times \mathbb{R}^+ \times \mathbb{R}^2$ is an L_p Carathéodory positive function.

In [2], Hao et al. considered a boundary value problem of fractional differential equation inclusions of Riemann–Liouville type on the infinite interval:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t), D_{0+}^{\alpha-2} u(t), D_{0+}^{\alpha-1} u(t)) = 0, & t \in (0, \infty), \\ u(0) = u'(0) = 0, \quad D^{\alpha-1} u(\infty) = \xi I_{0+}^{\beta} u(\eta), \end{cases}$$

where $2 < \alpha \le 3$, $\beta > 0$, $\xi, \eta \ge 0$, $\Gamma(\alpha + \beta) > \xi \eta^{\alpha+\beta-1}$, $f \in C([0, +\infty] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Under suitable growth conditions of the nonlinear term f, by using the Schauder fixed point theorem and Banach contraction mapping principle, the authors showed the existence and uniqueness results of solutions.

Inspired by the works mentioned above, we will study the existence of solutions for the following boundary value problem for Hadamard fractional differential equations:

$$^{H}D^{\alpha} u(t) + q(t) f\big(t, u(t), {}^{H}D^{\beta_1} u(t), {}^{H}D^{\beta_2} u(t)\big) = 0, \quad 1 < t < +\infty, \tag{1.1}$$

supplemented with Hadamard integral boundary conditions

$$\begin{cases} u(1) = 0, \\ {}^{H}D^{\alpha-2} u(1) = \int_1^{+\infty} g_1(s) u(s) \frac{ds}{s}, \\ {}^{H}D^{\alpha-1} u(+\infty) = \int_1^{+\infty} g_2(s) u(s) \frac{ds}{s}, \end{cases} \tag{1.2}$$

$2 < \alpha \le 3$, $0 < \beta_1 \le \alpha - 2 < \beta_2 \le \alpha - 1$, $q \in C(J, \mathbb{R}^+)$, $J = [1, +\infty)$, and $0 < \int_1^{+\infty} q(s) \frac{ds}{s} < +\infty$, $f : J \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the q-Carathéodory condition, $g_1, g_2 : J \to \mathbb{R}^+$ are nonnegative functions. ${}^{H}D^{\alpha}$ represents a Hadamard fractional derivative of order α.

We say that f satisfies the q-Carathéodory condition on $J \times \mathbb{R}^3 \to \mathbb{R}$, if

(1) for each $(u, v, w) \in \mathbb{R}^3$, the mapping $t \to f(t, u, v, w)$ is measurable on J;

(2) for a.e. $t \in J$, the mapping $(u, v, w) \to f(t, (1 + (\log t)^{\alpha-1}) u, (1 + (\log t)^{\alpha-\beta_1-1}) v, (1 + (\log t)^{\alpha-\beta_2}) \frac{w}{(\log t)^{2+\beta_2-\alpha}})$ is continuous on $\mathbb{R}^3$;

(3) for each $r > 0$, there exists a nonnegative function f_r satisfying $\int_1^{\infty} q(s) f_r(s) \frac{ds}{s} < +\infty$ such that, for any $u, v, w \in \mathbb{R}$ with $\max\{u, v, w\} \le r$,

$$\left| f\left(t, \big(1 + (\log t)^{\alpha-1}\big) u, \big(1 + (\log t)^{\alpha-\beta_1-1}\big) v, \frac{1 + (\log t)^{\alpha-\beta_2}}{(\log t)^{2+\beta_2-\alpha}} w\right) \right| \le f_r(t), \quad \text{for a.e. } t \in J.$$

Compared with [2], in this paper, the nonlinear term f contains lower Hadamard fractional derivatives ${}^{H}D^{\beta_1}$, ${}^{H}D^{\beta_2}$, which are not only the particular case ${}^{H}D^{\alpha-1}$, ${}^{H}D^{\alpha-2}$. Furthermore, when $\beta_2 < \alpha - 1$, $\lim_{t\to 1^+} ({}^{H}D^{\beta_2} u)(t) = \infty$, the singularity creates additional complexity to verify the existence of solutions.

This paper is organized as follows. Section 2 contains some important lemmas, which play a key role in the study, and presents some properties of Green's functions that are used to define an operator. In Sect. 3, the existence of solution for (1.1), (1.2) is established by using the fixed point theory in cones. In Sect. 4, the main results are illustrated by an example.

2 Some preliminaries and lemmas

Let $\mathrm{AC}_\delta^n(J,\mathbb{R}) = \{y : J \to \mathbb{R}, \delta^{n-1}y \in \mathrm{AC}(J,\mathbb{R})\}$, where $\delta = t\frac{d}{dt}$ and $\mathrm{AC}(J,\mathbb{R})$ is the space of absolutely continuous functions from J into $\mathbb{R}$.

Definition 2.1 ([1]) The Hadamard fractional integral of order $\alpha > 0$ for a function $g \in L^1(J,\mathbb{R})$ is defined as

$$ {}^H I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}, $$

where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 ([1]) The Hadamard derivative of fractional order $\alpha > 0$ for a function $g \in \mathrm{AC}_\delta^n(J,\mathbb{R})$ is defined as

$$ {}^H D^\alpha g(t) = \delta^n\left({}^H I^{n-\alpha} g\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} g(s)\frac{ds}{s}, $$

where $n-1 < \alpha < n$, $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1 ([1]) *If $\beta - 1 > \gamma > 0$, then*
(1) ${}^H I^\gamma \log(t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}(\log t)^{\beta+\gamma-1}$,
(2) ${}^H D^\gamma \log(t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}(\log t)^{\beta-\gamma-1}$.

Lemma 2.2 ([1]) *For $\alpha > 0$, $n = [\alpha]+1$ and $x \in C(J)\cap L^1(J)$, the solution of Hadamard fractional differential equation ${}^H D^\alpha x(t) = 0$ is $x(t) = \sum_{i=1}^n c_i(\log t)^{\alpha-i}$, where $c_i \in \mathbb{R}$ $(i = 1,2,\ldots,n)$.*

Lemma 2.3 ([1]) *Let $\alpha > 0$. If $u \in L^1(J)$, then the equality ${}^H D^\alpha({}^H I^\alpha u)(t) = u(t)$ holds a.e. on J.*

Lemma 2.4 ([1]) *If $u \in C(J)$ and ${}^H D^\alpha u \in L^1(J)$, then*

$$ {}^H I^\alpha\left({}^H D^\alpha u\right)(t) = u(t) + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} + \cdots + c_n(\log t)^{\alpha-n}, $$

where $c_i \in \mathbb{R}$ $(i = 1,2,3\ldots,n)$, $n = [\alpha]+1$.

For further analysis, we introduce the following denotations:

$$ l_1 = \frac{1}{\Gamma(\alpha)}\int_1^{+\infty} g_1(t)(\log t)^{\alpha-1}\frac{dt}{t}, \qquad l_2 = \frac{1}{\Gamma(\alpha-1)}\int_1^{+\infty} g_2(t)(\log t)^{\alpha-2}\frac{dt}{t}, $$
$$ \delta_1 = \frac{1}{\Gamma(\alpha-1)}\int_1^{+\infty} g_1(t)(\log t)^{\alpha-2}\frac{dt}{t}, \qquad \delta_2 = \frac{1}{\Gamma(\alpha)}\int_1^{+\infty} g_2(t)(\log t)^{\alpha-1}\frac{dt}{t}, $$

$$\delta = \frac{1}{(1-\delta_1)(1-\delta_2)-l_1 l_2}, \qquad \varphi_\lambda(t) = \frac{(\log t)^{\lambda-1}}{\Gamma(\lambda)},$$

$$G_\lambda(t,s) = \frac{1}{\Gamma(\lambda)} \begin{cases} (\log t)^{\lambda-1} - (\log \frac{t}{s})^{\lambda-1}, & 1 \le s \le t < +\infty, \\ (\log t)^{\lambda-1}, & 1 \le t \le s < +\infty, \end{cases} \tag{2.1}$$

$$T_{1,\lambda}(t) = \varphi_\lambda(t)\delta l_2 + \varphi_{\lambda-1}(t)\delta(1-\delta_2), \qquad T_{2,\lambda}(t) = \varphi_\lambda(t)\delta(1-\delta_1) + \varphi_{\lambda-1}(t)\delta l_1, \tag{2.2}$$

$$K_\lambda(t,s) = G_\lambda(t,s) + T_{1,\lambda}(t)\int_1^\infty g_1(t)G_\alpha(t,s)\frac{dt}{t} + T_{2,\lambda}(t)\int_1^\infty g_2(t)G_\alpha(t,s)\frac{dt}{t}. \tag{2.3}$$

We will use the following conditions:

($\mathbf{H}_1$) $q \in C(J,\mathbb{R}^+)$, $f : J \times \mathbb{R}^3 \to \mathbb{R}$ satisfies q-Carathéodory condition;

($\mathbf{H}_2$) $g_1, g_2 \in L^1(J,\mathbb{R}^+)$, $l_1, \delta_2 < +\infty$, and $(1-\delta_1)(1-\delta_2) - l_1 l_2 > 0$.

Lemma 2.5 *Let $h \in C(J) \cap L^1(J)$ with $0 < \int_1^\infty h(s)\frac{ds}{s} < \infty$, then the solution of Hadamard type fractional differential equation*

$$^H D^\alpha u(t) + h(t) = 0, \tag{2.4}$$

subject to the same condition (1.2) can be expressed by

$$u(t) = \int_1^{+\infty} K_\alpha(t,s)h(s)\frac{ds}{s},$$

where $K_\alpha(t,s)$ is denotation (2.3) with $\lambda = \alpha$.

Proof Due to Lemma 2.4, the solution of Hadamard fractional differential equation $^H D^\alpha u(t) + h(t) = 0$ can be written as

$$u(t) = -{}^H I^\alpha h(t) + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} + c_3(\log t)^{\alpha-3}, \tag{2.5}$$

where $c_i \in \mathbb{R}$ $(i = 1,2,3)$ are arbitrary constants. From $u(1) = 0$, we have $c_3 = 0$. By Lemma 2.1, we have

$$^H D^{\alpha-2}u(t) = -{}^H I^2 h(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(2)}(\log t) + c_2\frac{\Gamma(\alpha-1)}{\Gamma(1)}.$$

Using the condition $^H D^{\alpha-2}u(1) = \int_1^{+\infty} g_1(s)u(s)\frac{ds}{s}$, we conclude that $c_2 = \frac{1}{\Gamma(\alpha-1)}\int_1^{+\infty} g_1(t) \times u(t)\frac{dt}{t}$.

Similarly, we get

$$^H D^{\alpha-1}u(t) = -{}^H I^1 h(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(1)}.$$

From the condition $^H D^{\alpha-1}u(+\infty) = \int_1^{+\infty} g_2(s)u(s)\frac{ds}{s}$, we conclude that

$$c_1 = \frac{1}{\Gamma(\alpha)}\int_1^{+\infty} h(t)\frac{dt}{t} + \frac{1}{\Gamma(\alpha)}\int_1^{+\infty} g_2(t)u(t)\frac{dt}{t}.$$

Consequently,

$$
\begin{aligned}
u(t) &= -{}^{H}I^{\alpha}h(t) + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} \\
&= -\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} h(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^{\infty} (\log t)^{\alpha-1} h(s)\frac{ds}{s} \\
&\quad + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)}\int_1^{\infty} g_2(s)u(s)\frac{ds}{s} + \frac{(\log t)^{\alpha-2}}{\Gamma(\alpha-1)}\int_1^{\infty} g_1(s)u(s)\frac{ds}{s} \\
&= \int_1^{\infty} G_\alpha(t,s)h(s)\frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)}\int_1^{\infty} g_2(s)u(s)\frac{ds}{s} \\
&\quad + \frac{(\log t)^{\alpha-2}}{\Gamma(\alpha-1)}\int_1^{\infty} g_1(s)u(s)\frac{ds}{s}.
\end{aligned}
\tag{2.6}
$$

Multiplying both sides of (2.6) by $\frac{g_1(t)}{t}$ and integrating from 1 to $+\infty$, we get

$$
\begin{aligned}
\int_1^{+\infty} u(t)g_1(t)\frac{dt}{t} &= \int_1^{+\infty} g_1(t)\left(\int_1^{\infty} G_\alpha(t,s)h(s)\frac{ds}{s}\right)\frac{dt}{t} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^{+\infty} g_1(t)(\log t)^{\alpha-1}\frac{dt}{t}\int_1^{\infty} g_2(s)u(s)\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\int_1^{+\infty} g_1(t)(\log t)^{\alpha-2}\frac{dt}{t}\int_1^{\infty} g_1(s)u(s)\frac{ds}{s}.
\end{aligned}
\tag{2.7}
$$

Similarly,

$$
\begin{aligned}
\int_1^{+\infty} u(t)g_2(t)\frac{dt}{t} &= \int_1^{+\infty} g_2(t)\left(\int_1^{\infty} G_\alpha(t,s)h(s)\frac{ds}{s}\right)\frac{dt}{t} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^{+\infty} g_2(t)(\log t)^{\alpha-1}\frac{dt}{t}\int_1^{\infty} g_2(s)u(s)\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\int_1^{+\infty} g_2(t)(\log t)^{\alpha-2}\frac{dt}{t}\int_1^{\infty} g_1(s)u(s)\frac{ds}{s}.
\end{aligned}
\tag{2.8}
$$

For convenience, we denote

$$
\begin{aligned}
X_1 &= \int_1^{+\infty} g_1(t)u(t)\frac{dt}{t}, \qquad X_2 = \int_1^{+\infty} g_2(t)u(t)\frac{dt}{t}, \\
A_1 &= \int_1^{+\infty}\left(\int_1^{\infty} g_1(t)G_\alpha(t,s)\frac{dt}{t}\right)h(s)\frac{ds}{s}, \qquad A_2 = \int_1^{+\infty}\left(\int_1^{\infty} g_2(t)G_\alpha(t,s)\frac{dt}{t}\right)h(s)\frac{ds}{s}.
\end{aligned}
$$

From (2.7), (2.8) we can deduce

$$
\begin{pmatrix} 1-\delta_1 & -l_1 \\ -l_2 & 1-\delta_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.
$$

Thus,

$$
\begin{aligned}
X_1 &= \frac{(1-\delta_2)A_1 + l_1A_2}{(1-\delta_1)(1-\delta_2) - l_1l_2} = \delta\big((1-\delta_2)A_1 + l_1A_2\big), \\
X_2 &= \frac{l_2A_1 + (1-\delta_1)A_2}{(1-\delta_1)(1-\delta_2) - l_1l_2} = \delta\big(l_2A_1 + (1-\delta_1)A_2\big).
\end{aligned}
$$

Substituting X_1, X_2 into (2.6), we can conclude

$$\begin{aligned} u(t) &= \int_1^\infty G_\alpha(t,s)h(s)\frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)}\delta\big(l_2A_1 + (1-\delta_1)A_2\big) \\ &\quad + \frac{(\log t)^{\alpha-2}}{\Gamma(\alpha-1)}\delta\big((1-\delta_2)A_1 + l_1A_2\big) \\ &= \int_1^\infty G_\alpha(t,s)h(s)\frac{ds}{s} + \left[\frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)}\delta l_2 + \frac{(\log t)^{\alpha-2}}{\Gamma(\alpha-1)}\delta(1-\delta_2)\right]A_1 \\ &\quad + \left[\frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)}\delta(1-\delta_1) + \frac{(\log t)^{\alpha-2}}{\Gamma(\alpha-1)}\delta l_1\right]A_2 \\ &= \int_1^\infty G_\alpha(t,s)h(s)\frac{ds}{s} + T_{1,\alpha}(t)A_1 + T_{2,\alpha}(t)A_2 \\ &= \int_1^\infty \left[G_\alpha(t,s) + T_{1,\alpha}(t)\int_1^\infty g_1(t)G_\alpha(t,s)\frac{dt}{t} + T_{2,\alpha}(t)\int_1^\infty g_2(t)G_\alpha(t,s)\frac{dt}{t}\right]h(s)\frac{ds}{s} \\ &= \int_1^\infty K_\alpha(t,s)h(s)\frac{ds}{s}. \end{aligned}$$

□

Lemma 2.6 *The functions* G_λ, φ_λ, $T_{1,\lambda}$, $T_{2,\lambda}$ *defined in* (2.1), (2.2) *satisfy*

(1) G_λ *is continuous in* $J \times J$ *and* $0 \le G_\lambda(t,s) \le \frac{1}{\Gamma(\lambda)}(\log t)^{\lambda-1}$, $\forall t,s \in J \times J$, $\lambda \ge 1$;

(2) $\frac{G_\lambda(t,s)}{1+(\log t)^{\lambda-1}} \le \frac{1}{\Gamma(\lambda)}$, $\forall t,s \in J \times J$, $\lambda \ge 1$;

(3) *If* $\lambda \ge 2$, *for any* $t \in J$, $\frac{\varphi_\lambda(t)}{1+(\log t)^{\lambda-1}} \le \frac{1}{\Gamma(\lambda)}$, $\frac{\varphi_{\lambda-1}(t)}{1+(\log t)^{\lambda-1}} \le \frac{1}{\Gamma(\lambda-1)}$;

(4) *If* $1 < \lambda < 2$, *for any* $t,s \in J$,

$$\frac{(\log t)^{2-\lambda}}{1+(\log t)^\lambda}G_\lambda(t,s) \le \frac{1}{\Gamma(\lambda)}, \qquad \frac{(\log t)^{2-\lambda}}{1+(\log t)^\lambda}\varphi_\lambda(t) \le \frac{1}{\Gamma(\lambda)},$$
$$\frac{(\log t)^{2-\lambda}}{1+(\log t)^\lambda}\varphi_{\lambda-1}(t) \le \frac{1}{\Gamma(\lambda-1)};$$

(5) *For any* $t \in J$, $\lambda \ge 2$,

$$\frac{T_{1,\lambda}(t)}{1+(\log t)^{\lambda-1}} \le \frac{\delta l_2}{\Gamma(\lambda)} + \frac{\delta(1-\delta_2)}{\Gamma(\lambda-1)}, \qquad \frac{T_{2,\lambda}(t)}{1+(\log t)^{\lambda-1}} \le \frac{\delta(1-\delta_1)}{\Gamma(\lambda)} + \frac{\delta l_1}{\Gamma(\lambda-1)}.$$

Proof By the definition, conclusions (1) and (2) can be easily obtained.

(3) If $\lambda \ge 2$, for $\forall t \in J$, $\frac{\varphi_\lambda(t)}{1+(\log t)^{\lambda-1}} = \frac{(\log t)^{\lambda-1}}{1+(\log t)^{\lambda-1}}\frac{1}{\Gamma(\lambda)} \le \frac{1}{\Gamma(\lambda)}$,

$$\begin{aligned} \frac{\varphi_{\lambda-1}(t)}{1+(\log t)^{\lambda-1}} &= \frac{(\log t)^{\lambda-2}}{1+(\log t)^{\lambda-1}}\frac{1}{\Gamma(\lambda-1)} \\ &\le \begin{cases} \frac{(\log t)^{\lambda-1}}{1+(\log t)^{\lambda-1}}\frac{1}{\Gamma(\lambda-1)}, & t \ge e, \\ \frac{1}{1+(\log t)^{\lambda-1}}\frac{1}{\Gamma(\lambda-1)}, & 1 \le t \le e \end{cases} \\ &\le \frac{1}{\Gamma(\lambda-1)}. \end{aligned}$$

(4) If $1 < \lambda < 2$, for any $t,s \in J$, $\frac{(\log t)^{2-\lambda}}{1+(\log t)^\lambda}G_\lambda(t,s) \le \frac{\log t}{1+(\log t)^\lambda}\frac{1}{\Gamma(\lambda)} \le \frac{1}{\Gamma(\lambda)}$,

$$\frac{(\log t)^{2-\lambda}}{1+(\log t)^\lambda}\varphi_\lambda(t) = \frac{\log t}{1+(\log t)^\lambda}\frac{1}{\Gamma(\lambda)} \le \frac{1}{\Gamma(\lambda)},$$

$$\frac{(\log t)^{2-\lambda}}{1+(\log t)^{\lambda}}\varphi_{\lambda-1}(t)=\frac{1}{1+(\log t)^{\lambda}}\frac{1}{\Gamma(\lambda-1)}\leq\frac{1}{\Gamma(\lambda-1)}.$$

On the basis of (3), conclusion (5) can be easily deduced. □

Remark 1 From Lemma 2.1 and $0<\beta_1\leq\alpha-2<\beta_2\leq\alpha-1$, we can calculate

$$
\begin{aligned}
&{}^{H}D^{\beta_1}T_{1,\alpha}(t)=T_{1,\alpha-\beta_1}(t),\qquad {}^{H}D^{\beta_1}T_{2,\alpha}(t)=T_{2,\alpha-\beta_1}(t),\\
&{}^{H}D^{\beta_2}T_{1,\alpha}(t)=\begin{cases}T_{1,\alpha-\beta_2}(t), & \beta_2<\alpha-1,\\ \delta l_2, & \beta_2=\alpha-1,\end{cases}\\
&{}^{H}D^{\beta_2}T_{2,\alpha}(t)=\begin{cases}T_{2,\alpha-\beta_2}(t), & \beta_2<\alpha-1,\\ \delta(1-\delta_1), & \beta_2=\alpha-1.\end{cases}
\end{aligned}
$$

Define the function spaces

$$X=\left\{u\in C(J):\sup_{t\in J}\frac{|u(t)|}{1+(\log t)^{\alpha-1}}<+\infty\right\}$$

with the form $\|u\|_X=\sup_{t\in J}\frac{|u(t)|}{1+(\log t)^{\alpha-1}}$ and

$$
\begin{aligned}
Y=\Bigg\{&u\in X:{}^{H}D^{\beta_1}u\in C(J),{}^{H}D^{\beta_2}u\in C(1,+\infty),\sup_{t\in J}\frac{|{}^{H}D^{\beta_1}u(t)|}{1+(\log t)^{\alpha-\beta_1-1}}<+\infty,\\
&\sup_{t\in J}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left|{}^{H}D^{\beta_2}u(t)\right|<+\infty\Bigg\}
\end{aligned}
$$

with the norm

$$\|u\|_Y=\max\left\{\sup_{t\in J}\frac{|u(t)|}{1+(\log t)^{\alpha-1}},\sup_{t\in J}\frac{|{}^{H}D^{\beta_1}u(t)|}{1+(\log t)^{\alpha-\beta_1-1}},\sup_{t\in J}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left|{}^{H}D^{\beta_2}u(t)\right|\right\}.$$

By a standard method, we can show that $(Y,\|\cdot\|_Y)$ is a Banach space.

According to the same method in [46, 49], we can get the following lemma.

Lemma 2.7 *Let $U\subset Y$ be a bounded set, then U is relatively compact in Y if the following conditions hold*:

(1) *For any $u\in U$, $\frac{u(t)}{1+(\log t)^{\alpha-1}}$, $\frac{{}^{H}D^{\beta_1}u(t)}{1+(\log t)^{\alpha-\beta_1-1}}$, and $\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}({}^{H}D^{\beta_2}u)(t)$ are equicontinuous on any compact interval of J*;

(2) *For any $\varepsilon>0$, there exists a constant $L=L(\varepsilon)>0$ such that*

$$\left|\frac{u(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{u(t_2)}{1+(\log t_2)^{\alpha-1}}\right|<\varepsilon,\qquad\left|\frac{{}^{H}D^{\beta_1}u(t_1)}{1+(\log t_1)^{\alpha-\beta_1-1}}-\frac{{}^{H}D^{\beta_1}u(t_2)}{1+(\log t_2)^{\alpha-\beta_1-1}}\right|<\varepsilon,$$

and

$$\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}\left({}^{H}D^{\beta_2}u\right)(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}\left({}^{H}D^{\beta_2}u\right)(t_2)\right|<\varepsilon,$$

for any $t_1,t_2\geq L$ and $u\in U$.

Lemma 2.8 (Schauder's fixed point theorem) *Let C be a compact, closed, bounded, and convex subset of a Banach space X. Suppose that $T : C \to C$ is a continuous and compact mapping. Then T has at least one fixed point in C.*

3 Main results

In this section, we shall establish the existence result of at least one solution of (1.1), (1.2). For convenience, we denote $F_u(s) = f(s, u(s), {}^H D^{\beta_1} u(s), {}^H D^{\beta_2} u(s))$.

We define an operator $A : Y \to C(J, \mathbb{R})$ as follows:

$$Au(t) = \int_1^{+\infty} K_\alpha(t,s) q(s) F_u(s) \frac{ds}{s}, \quad t \in J. \tag{3.1}$$

By Lemma 2.5, we can show that $u(t)$ is a solution of boundary value problem (1.1), (1.2) if and only if it is the fixed point of A.

From Lemma 2.1 and Remark 1, we have

$$\begin{aligned}
{}^H D^{\beta_1} Au(t) &= {}^H D^{\beta_1} \left(\int_1^{+\infty} K_\alpha(t,s) q(s) F_u(s) \frac{ds}{s} \right) \\
&= {}^H D^{\beta_1} \left(-\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} q(s) F_u(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)} \int_1^\infty q(s) F_u(s) \frac{ds}{s} \right. \\
&\quad + T_{1,\alpha}(t) \int_1^\infty \left(\int_1^\infty g_1(t) G_\alpha(t,s) \frac{dt}{t} \right) q(s) F_u(s) \frac{ds}{s} \\
&\quad \left. + T_{2,\alpha}(t) \int_1^\infty \left(\int_1^\infty g_2(t) G_\alpha(t,s) \frac{dt}{t} \right) q(s) F_u(s) \frac{ds}{s} \right) \\
&= -{}^H I^{\alpha-\beta_1} q(t) F_u(t) + \frac{(\log t)^{\alpha-\beta_1-1}}{\Gamma(\alpha-\beta_1)} \int_1^\infty q(s) F_u(s) \frac{ds}{s} \\
&\quad + T_{1,\alpha-\beta_1}(t) \int_1^\infty \left(\int_1^\infty g_1(t) G_\alpha(t,s) \frac{dt}{t} \right) q(s) F_u(s) \frac{ds}{s} \\
&\quad + T_{2,\alpha-\beta_1}(t) \int_1^\infty \left(\int_1^\infty g_2(t) G_\alpha(t,s) \frac{dt}{t} \right) q(s) F_u(s) \frac{ds}{s} \\
&= \int_1^{+\infty} K_{\alpha-\beta_1}(t,s) q(s) F_u(s) \frac{ds}{s}.
\end{aligned} \tag{3.2}$$

Similarly, if $\beta_2 < \alpha - 1$,

$${}^H D^{\beta_2} Au(t) = \int_1^{+\infty} K_{\alpha-\beta_2}(t,s) q(s) F_u(s) \frac{ds}{s}; \tag{3.3}$$

if $\beta_2 = \alpha - 1$,

$$\begin{aligned}
{}^H D^{\beta_2} Au(t) &= {}^H D^{\beta_2} \left(\int_1^{+\infty} K_\alpha(t,s) q(s) F_u(s) \frac{ds}{s} \right) \\
&= {}^H D^{\beta_2} \left(-\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} q(s) F_u(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)} \int_1^\infty q(s) F_u(s) \frac{ds}{s} \right. \\
&\quad + T_{1,\alpha}(t) \int_1^\infty \left(\int_1^\infty g_1(t) G_\alpha(t,s) \frac{dt}{t} \right) q(s) F_u(s) \frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+ T_{2,\alpha}(t) \int_1^{\infty} \left(\int_1^{\infty} g_2(t) G_\alpha(t,s) \frac{dt}{t} \right) q(s) F_u(s) \frac{ds}{s} \Bigg) \\
&= -{}^H I^1 q(t) F_u(t) + \int_1^{\infty} q(s) F_u(s) \frac{ds}{s} \\
&\quad + \delta l_2 \int_1^{\infty} \left(\int_1^{\infty} g_1(t) G_\alpha(t,s) \frac{dt}{t} \right) q(s) F_u(s) \frac{ds}{s} \\
&\quad + \delta(1-\delta_1) \int_1^{\infty} \left(\int_1^{\infty} g_2(t) G_\alpha(t,s) \frac{dt}{t} \right) q(s) F_u(s) \frac{ds}{s} \\
&= \int_1^{+\infty} K^{\beta_2}(t,s) q(s) F_u(s) \frac{ds}{s},
\end{aligned} \tag{3.4}
$$

where $K^{\beta_2}(t,s) = G(t,s) + \delta l_2 \int_1^\infty g_1(t) G_\alpha(t,s) \frac{dt}{t} + \delta(1-\delta_1) \int_1^\infty g_2(t) G_\alpha(t,s) \frac{dt}{t}$,

$$
G(t,s) = \begin{cases} 0, & 1 \le s \le t < +\infty, \\ 1, & 1 \le t \le s < +\infty. \end{cases}
$$

Remark 2 In (2.3) we choose $\lambda = \alpha$, $\lambda = \alpha - \beta_1$ respectively, from Lemma 2.6, we can easily get the following inequalities:

$$
\begin{aligned}
\frac{K_\alpha(t,s)}{1+(\log t)^{\alpha-1}} &\le \frac{1}{\Gamma(\alpha)} + l_1 \left(\frac{\delta l_2}{\Gamma(\alpha)} + \frac{\delta(1-\delta_2)}{\Gamma(\alpha-1)} \right) \\
&\quad + \delta_2 \left(\frac{\delta(1-\delta_1)}{\Gamma(\alpha)} + \frac{\delta l_1}{\Gamma(\alpha-1)} \right) \\
&\triangleq \Delta_1, \quad \forall t,s \in J,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\frac{K_{\alpha-\beta_1}(t,s)}{1+(\log t)^{\alpha-\beta_1-1}} &\le \frac{1}{\Gamma(\alpha-\beta_1)} + l_1 \left(\frac{\delta l_2}{\Gamma(\alpha-\beta_1)} + \frac{\delta(1-\delta_2)}{\Gamma(\alpha-\beta_1-1)} \right) \\
&\quad + \delta_2 \left(\frac{\delta(1-\delta_1)}{\Gamma(\alpha-\beta_1)} + \frac{\delta l_1}{\Gamma(\alpha-\beta_1-1)} \right) \\
&\triangleq \Delta_2, \quad \forall t,s \in J,
\end{aligned} \tag{3.6}
$$

when $1 < \alpha - \beta_2 < 2$, for all $t,s \in J$, according to Lemma 2.6(2), (4), let $\lambda = \alpha - \beta_2$,

$$
\begin{aligned}
\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}} K_{\alpha-\beta_2}(t,s) &= \frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}} G_{\alpha-\beta_2}(t,s) \\
&\quad + \frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}} T_{1,\alpha-\beta_2}(t) \int_1^{+\infty} g_1(s) G_\alpha(t,s) \frac{dt}{t} \\
&\quad + \frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}} T_{2,\alpha-\beta_2}(t) \int_1^{+\infty} g_2(s) G_\alpha(t,s) \frac{dt}{t} \\
&\le \frac{1}{\Gamma(\alpha-\beta_2)} + \left[\frac{\delta l_2}{\Gamma(\alpha-\beta_2)} + \frac{\delta(1-\delta_2)}{\Gamma(\alpha-\beta_2-1)} \right] l_1 \\
&\quad + \left[\frac{\delta(1-\delta_1)}{\Gamma(\alpha-\beta_2)} + \frac{\delta l_1}{\Gamma(\alpha-\beta_2-1)} \right] \delta_2 \\
&\triangleq \Delta_3
\end{aligned} \tag{3.7}
$$

when $\alpha - \beta_2 = 1$, we have the following inequality:

$$\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}K^{\beta_2}(t,s) = \frac{\log t}{1+\log t}K^{\beta_2}(t,s) \le 1 + \delta l_1 l_2 + \delta(1-\delta_1)\delta_2 \triangleq \Delta_4. \tag{3.8}$$

Lemma 3.1 *The operator $A : Y \to Y$ is well defined.*

Proof For any $u \in Y$, $u \neq \theta$, let $\|u\|_Y = r > 0$. In view of the definition of Y,

$$\sup_{t\in J}\frac{|u(t)|}{1+(\log t)^{\alpha-1}} \le r, \qquad \sup_{t\in J}\frac{|^H D^{\beta_1}u(t)|}{1+(\log t)^{\alpha-\beta_1-1}} \le r,$$
$$\sup_{t\in J}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left|^H D^{\beta_2}u(t)\right| \le r.$$

From the continuity of K_α, q, F_u, we know $Au(t)$, $^H D^{\beta_1}Au(t)$, and $^H D^{\beta_2}Au(t)$ are continuous.

$$\begin{aligned}\left|F_u(s)\right| &= \left|f\left(s,u(s),{}^H D^{\beta_1}u(s),{}^H D^{\beta_2}u(s)\right)\right| \\ &= \left|f\left(s,\left(1+(\log s)^{\alpha-1}\right)\frac{u(s)}{1+(\log s)^{\alpha-1}},\left(1+(\log s)^{\alpha-\beta_1-1}\right)\frac{^H D^{\beta_1}u(s)}{1+(\log s)^{\alpha-\beta_1-1}},\right.\right. \\ &\quad \left.\left.\frac{1+(\log s)^{\alpha-\beta_2}}{(\log s)^{2+\beta_2-\alpha}}\frac{(\log s)^{2+\beta_2-\alpha}}{1+(\log s)^{\alpha-\beta_2}}{}^H D^{\beta_2}u(s)\right)\right| \\ &\le f_r(s).\end{aligned}$$

Using (3.5)–(3.8) and the Caratheodory condition, we can deduce

$$\begin{aligned}\sup_{t\in J}\left|\frac{Au(t)}{1+(\log t)^{\alpha-1}}\right| &= \sup_{t\in J}\left|\int_1^{+\infty}\frac{K_\alpha(t,s)}{1+(\log t)^{\alpha-1}}q(s)F_u(s)\frac{ds}{s}\right| \le \Delta_1\int_1^{+\infty}q(s)f_r(s)\frac{ds}{s} < +\infty, \\ \sup_{t\in J}\left|\frac{^H D^{\beta_1}Au(t)}{1+(\log t)^{\alpha-\beta_1-1}}\right| &= \sup_{t\in J}\left|\int_1^{+\infty}\frac{K_{\alpha-\beta_1}(t,s)}{1+(\log t)^{\alpha-\beta_1-1}}q(s)F_u(s)\frac{ds}{s}\right| \\ &\le \Delta_2\int_1^{+\infty}q(s)f_r(s)\frac{ds}{s} < +\infty, \\ \sup_{t\in J}\left|\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left(^H D^{\beta_2}Au_n\right)(t)\right| &= \sup_{t\in J}\left|\int_1^{+\infty}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}K_{\alpha-\beta_2}(t,s)q(s)F_u(s)\frac{ds}{s}\right| \\ &\le \Delta_3\int_1^{+\infty}q(s)f_r(s)\frac{ds}{s} < +\infty, \quad (\beta_2 < \alpha - 1),\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in J}\left|\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left(^H D^{\beta_2}Au_n\right)(t)\right| &= \sup_{t\in J}\left|\int_1^{+\infty}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}K^{\beta_2}(t,s)q(s)F_u(s)\frac{ds}{s}\right| \\ &\le \Delta_4\int_1^{+\infty}q(s)f_r(s)\frac{ds}{s} < +\infty, \quad (\beta_2 = \alpha - 1).\end{aligned}$$

From the definition of space Y, we know $Au \in Y$, which means operator A is well defined. □

Theorem 3.1 *Assume that conditions* $(\mathbf{H}_1)$–$(\mathbf{H}_2)$ *are satisfied. Further assume that the following condition* $(\mathbf{H}_3)$ *holds*:

$(\mathbf{H}_3)$ *There exists a constant* $R > 0$ *such that* $f_R(t)$ *satisfies*

$$\Delta \int_1^{+\infty} q(s) f_R(s) \frac{ds}{s} \le R,$$

where $\Delta = \max\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$. *Then boundary value problem* (1.1), (1.2) *has at least one solution.*

Proof The proof is divided into the following steps:

Step 1. We will prove that $A : Y \to Y$ is completely continuous.

Firstly we will show that A is a continuous operator. Let $u_n, u \in Y$ $(n = 1, 2, \dots)$ with $\|u_n - u\|_Y \to 0$, $n \to \infty$. For any $t \in J$, we have

$$\begin{aligned}
&\frac{u_n(t)}{1 + (\log t)^{\alpha-1}} \to \frac{u(t)}{1 + (\log t)^{\alpha-1}}, \qquad \frac{{}^H D^{\beta_1} u_n(t)}{1 + (\log t)^{\alpha-\beta_1-1}} \to \frac{{}^H D^{\beta_1} u(t)}{1 + (\log t)^{\alpha-\beta_1-1}}, \\
&\frac{(\log t)^{2+\beta_2-\alpha}}{1 + (\log t)^{\alpha-\beta_2}} \bigl({}^H D^{\beta_2} u_n(t)\bigr) \to \frac{(\log t)^{2+\beta_2-\alpha}}{1 + (\log t)^{\alpha-\beta_2}} \bigl({}^H D^{\beta_2} u(t)\bigr), \quad n \to \infty.
\end{aligned}$$

Meanwhile, there exists a constant $r > 0$ such that $\|u_n\|_Y \le r$, $\|u\|_Y \le r$, which means

$$\begin{aligned}
&\sup_{t\in J} \left| \frac{u_n(t)}{1 + (\log t)^{\alpha-1}} \right| \le r, \qquad \sup_{t\in J} \left| \frac{u(t)}{1 + (\log t)^{\alpha-1}} \right| \le r, \\
&\sup_{t\in J} \left| \frac{{}^H D^{\beta_1} u_n(t)}{1 + (\log t)^{\alpha-\beta_1-1}} \right| \le r, \qquad \sup_{t\in J} \left| \frac{{}^H D^{\beta_1} u(t)}{1 + (\log t)^{\alpha-\beta_1-1}} \right| \le r, \\
&\sup_{t\in J} \left| \frac{(\log t)^{2+\beta_2-\alpha}}{1 + (\log t)^{\alpha-\beta_2}} \bigl({}^H D^{\beta_2} u_n\bigr)(t) \right| \le r, \qquad \sup_{t\in J} \left| \frac{(\log t)^{2+\beta_2-\alpha}}{1 + (\log t)^{\alpha-\beta_2}} \bigl({}^H D^{\beta_2} u\bigr)(t) \right| \le r.
\end{aligned}$$

Due to the Caratheodory condition, for a.e. $s \in J$, we have

$$\bigl|F_{u_n}(s) - F_u(s)\bigr| \to 0, \quad (n \to \infty) \qquad \bigl|F_{u_n}(s) - F_u(s)\bigr| \le 2 f_r(s).$$

By the Lebesgue dominated convergence theorem, we obtain

$$\int_1^{+\infty} \bigl|F_{u_n}(s) - F_u(s)\bigr| \frac{ds}{s} \to 0, \quad n \to \infty.$$

Following the above method, we can also deduce

$$\begin{aligned}
\left| \frac{Au_n(t) - Au(t)}{1 + (\log t)^{\alpha-1}} \right| &= \left| \int_1^{+\infty} \frac{K_\alpha(t,s)}{1 + (\log t)^{\alpha-1}} q(s) \bigl(F_{u_n}(s) - F_u(s)\bigr) \frac{ds}{s} \right| \\
&\le \Delta_1 \int_1^{+\infty} q(s) \bigl|F_{u_n}(s) - F_u(s)\bigr| \frac{ds}{s} \to 0, \quad n \to +\infty, \\
\left| \frac{{}^H D^{\beta_1} Au_n(t) - {}^H D^{\beta_1} Au(t)}{1 + (\log t)^{\alpha-\beta_1-1}} \right| &= \left| \int_1^{+\infty} \frac{K_{\alpha-\beta}(t,s)}{1 + (\log t)^{\alpha-\beta-1}} q(s) \bigl(F_{u_n}(s) - F_u(s)\bigr) \frac{ds}{s} \right| \\
&\le \Delta_2 \int_1^{+\infty} q(s) \bigl|F_{u_n}(s) - F_u(s)\bigr| \frac{ds}{s} \to 0, \quad n \to +\infty,
\end{aligned}$$

$$\begin{aligned}&\left|\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left({}^HD^{\beta_2}Au_n\right)(t)-\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left({}^HD^{\beta_2}Au\right)(t)\right|\\&\quad\le\Delta_3\int_1^{+\infty}q(s)\left|F_{u_n}(s)-F_u(s)\right|\frac{ds}{s}\to 0,\quad n\to+\infty\ (\beta<\alpha-1),\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left({}^HD^{\beta_2}Au_n\right)(t)-\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left({}^HD^{\beta_2}Au\right)(t)\right|\\&\quad\le\Delta_4\int_1^{+\infty}q(s)\left|F_{u_n}(s)-F_u(s)\right|\frac{ds}{s}\to 0,\quad n\to+\infty\ (\beta=\alpha-1).\end{aligned}$$

Thus, we conclude $\|Au_n-Au\|_Y\to 0$, $(n\to+\infty)$, which means that $A:Y\to Y$ is a continuous operator.

Next, we will show that A is a compact operator. Let B be a nonempty bounded subset of Y. There exists a positive number r_1 such that $\|u\|_Y\le r_1$, $\forall u\in B$, and there exists $f_{r_1}\in L^1(J)$ such that $|F_u(s)|\le f_{r_1}(s)$.

① For any $u\in B$, we can get

$$\begin{aligned}&\sup_{t\in J}\left|\frac{Au(t)}{1+(\log t)^{\alpha-1}}\right|\le\Delta_1\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}<+\infty,\\&\sup_{t\in J}\left|\frac{{}^HD^{\beta_1}Au(t)}{1+(\log t)^{\alpha-\beta_1-1}}\right|\le\Delta_2\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}<+\infty,\\&\sup_{t\in J}\left|\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left({}^HD^{\beta_2}Au_n\right)(t)\right|\le\Delta_3\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}<+\infty,\quad(\beta_2<\alpha-1),\end{aligned}$$

and

$$\sup_{t\in J}\left|\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\left({}^HD^{\beta_2}Au\right)(t)\right|\le\Delta_4\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}<+\infty,\quad(\beta_2=\alpha-1).$$

Therefore $\|Au\|_Y\le\Delta\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}<+\infty$ and $A(B)$ is bounded in Y.

② For any $b>1$, let $J_1\triangleq[1,b]$. Because $\frac{G_\alpha(t,s)}{1+(\log t)^{\alpha-1}}$ is continuous on $J_1\times J_1$ and $\frac{(\log t)^{\alpha-1}}{1+(\log t)^{\alpha-1}}$, $\frac{T_{1,\alpha}(t)}{1+(\log t)^{\alpha-1}}$, $\frac{T_{2,\alpha}(t)}{1+(\log t)^{\alpha-1}}$ are continuous on J_1, then they are uniformly continuous. So, for any $\varepsilon>0$, there is a constant $\delta_1>0$ such that, for all $t_1,t_2,s_1,s_2\in J_1$, with $|t_1-t_2|<\delta_1$, $|s_1-s_2|<\delta_1$,

$$\begin{aligned}&\left|\frac{G_\alpha(t_1,s)}{1+(\log t_1)^{\alpha-1}}-\frac{G_\alpha(t_2,s)}{1+(\log t_2)^{\alpha-1}}\right|<\varepsilon,\qquad\left|\frac{(\log t_1)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}-\frac{(\log t_2)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}\right|<\Gamma(\alpha)\varepsilon,\\&\left|\frac{T_{1,\alpha}(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{T_{1,\alpha}(t_2)}{1+(\log t_2)^{\alpha-1}}\right|<\varepsilon,\qquad\left|\frac{T_{2,\alpha}(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{T_{2,\alpha}(t_2)}{1+(\log t_2)^{\alpha-1}}\right|<\varepsilon.\end{aligned}$$

For $\forall u\in B$, $t_1,t_2\in J_1$, with $t_1<t_2$, $|t_1-t_2|<\delta_1$,

$$\begin{aligned}&\left|\frac{Au(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{Au(t_2)}{1+(\log t_2)^{\alpha-1}}\right|\\&\quad=\left|\int_1^{+\infty}\left(\frac{G_\alpha(t_1,s)}{1+(\log t_1)^{\alpha-1}}-\frac{G_\alpha(t_2,s)}{1+(\log t_2)^{\alpha-1}}\right)q(s)F_u(s)\frac{ds}{s}\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_1^{+\infty}\left(\frac{T_{1,\alpha}(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{T_{1,\alpha}(t_2)}{1+(\log t_2)^{\alpha-1}}\right)\left(\int_1^{+\infty}g_1(t)G_\alpha(t,s)\frac{dt}{t}\right)q(s)F_u(s)\frac{ds}{s}\\
&+\int_1^{+\infty}\left(\frac{T_{2,\alpha}(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{T_{2,\alpha}(t_2)}{1+(\log t_2)^{\alpha-1}}\right)\left(\int_1^{+\infty}g_2(t)G_\alpha(t,s)\frac{dt}{t}\right)q(s)F_u(s)\frac{ds}{s}\Bigg|\\
\le{}&\int_1^{t_2}\left|\frac{G_\alpha(t_1,s)}{1+(\log t_1)^{\alpha-1}}-\frac{G_\alpha(t_2,s)}{1+(\log t_2)^{\alpha-1}}\right|q(s)f_{r_1}(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{+\infty}\left|\frac{(\log t_1)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}-\frac{(\log t_2)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}\right|q(s)f_{r_1}(s)\frac{ds}{s}\\
&+\varepsilon l_1\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}+\varepsilon\delta_2\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}\\
\le{}&\varepsilon(1+l_1+\delta_2)\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}.
\end{aligned}
$$

Similarly, $\frac{G_{\alpha-\beta_1}(t,s)}{1+(\log t)^{\alpha-\beta_1-1}}$ is continuous on $J_1\times J_1$, and $\frac{(\log t)^{\alpha-\beta_1-1}}{1+(\log t)^{\alpha-\beta_1-1}}$, $\frac{T_{1,\alpha-\beta_1}(t)}{1+(\log t)^{\alpha-\beta_1-1}}$, $\frac{T_{2,\alpha-\beta_1}(t)}{1+(\log t)^{\alpha-\beta_1-1}}$ are continuous on J_1. For above $\varepsilon>0$, there is a constant $\delta_2>0$ such that, for all $t_1,t_2\in J_1$ with $t_1<t_2$, $|t_1-t_2|<\delta_2$,

$$
\left|\frac{{}^HD^{\beta_1}Au(t_1)}{1+(\log t_1)^{\alpha-\beta_1-1}}-\frac{{}^HD^{\beta_1}Au(t_2)}{1+(\log t_2)^{\alpha-\beta_1-1}}\right|\le\varepsilon(1+l_1+\delta_2)\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}.
$$

When $\beta_2<\alpha-1$, $\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}G_{\alpha-\beta_2}(t,s)$ is continuous on $J_1\times J_1$, and $\frac{\log t}{1+(\log t)^{\alpha-\beta_2}}$, $\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}T_{1,\alpha-\beta_2}(t)$, $\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}T_{2,\alpha-\beta_2}(t)$ are continuous on J_1. For above $\varepsilon>0$, there is a constant $\delta_3>0$ such that, for all $t_1,t_2\in J_1$ with $t_1<t_2$, $|t_1-t_2|<\delta_3$,

$$
\begin{aligned}
&\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}\big({}^HD^{\beta_2}Au\big)(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}\big({}^HD^{\beta_2}Au\big)(t_2)\right|\\
&\quad\le\int_1^{t_2}\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}G_{\alpha-\beta_2}(t_1,s)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}G_{\alpha-\beta_2}(t_2,s)\right|q(s)f_{r_1}(s)\frac{ds}{s}\\
&\qquad+\frac{1}{\Gamma(\alpha-\beta_2)}\int_{t_2}^{+\infty}\left|\frac{\log t_1}{1+(\log t_1)^{\alpha-\beta_2}}-\frac{\log t_2}{1+(\log t_2)^{\alpha-\beta_2}}\right|q(s)f_{r_1}(s)\frac{ds}{s}\\
&\qquad+\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}T_{1,\alpha-\beta_2}(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}T_{1,\alpha-\beta_2}(t_2)\right|l_1\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}\\
&\qquad+\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}T_{2,\alpha-\beta_2}(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}T_{2,\alpha-\beta_2}(t_2)\right|\delta_2\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}\\
&\quad\le\varepsilon(1+l_1+\delta_2)\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s},\quad(\beta_2<\alpha-1).
\end{aligned}
$$

When $\beta_2=\alpha-1$, $\frac{\log t}{1+\log t}G(t,s)$ is continuous on $J_1\times J_1$ and $\frac{\log t}{1+\log t}$ is continuous on J_1, there exists $\delta_4>0$ such that, for all $t_1,t_2\in J_1$ with $t_1<t_2$, $|t_1-t_2|<\delta_4$,

$$
\begin{aligned}
&\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}\big({}^HD^{\beta_2}Au\big)(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}\big({}^HD^{\beta_2}Au\big)(t_2)\right|\\
&\quad\le\int_1^{+\infty}\left|\frac{\log t_1}{1+\log t_1}G(t_1,s)-\frac{\log t_2}{1+\log t_2}G(t_2,s)\right|q(s)f_{r_1}(s)\frac{ds}{s}\\
&\qquad+\left|\frac{\log t_1}{1+\log t_1}-\frac{\log t_2}{1+\log t_2}\right|\delta l_1l_2\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{\log t_1}{1+\log t_1}-\frac{\log t_2}{1+\log t_2}\right|\delta(1-\delta_1)\delta_2\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}\\
&\leq\varepsilon\bigl(1+\delta l_1l_2+\delta(1-\delta_1)\delta_2\bigr)\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}.
\end{aligned}
$$

Hence, $\frac{Au(t)}{1+(\log t)^{\alpha-1}}$, $\frac{{}^HD^{\beta_1}Au(t)}{1+(\log t)^{\alpha-\beta_1-1}}$, $\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}({}^HD^{\beta_2}Au)(t)$ are equicontinuous on J_1.

③ Now we indicate that the second condition (2) of Lemma 2.7 holds. Since

$$
\lim_{t\to+\infty}\frac{T_{1,\alpha}(t)}{1+(\log t)^{\alpha-1}}=\frac{\delta l_2}{\Gamma(\alpha)},\qquad \lim_{t\to+\infty}\frac{T_{2,\alpha}(t)}{1+(\log t)^{\alpha-1}}=\frac{\delta(1-\delta_1)}{\Gamma(\alpha)},
$$

for any $\varepsilon>0$, there exists a constant $T_1>1$ such that $\forall t_1,t_2>T_1$,

$$
\left|\frac{T_{1,\alpha}(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{T_{1,\alpha}(t_2)}{1+(\log t_2)^{\alpha-1}}\right|<\varepsilon,\qquad \left|\frac{T_{2,\alpha}(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{T_{2,\alpha}(t_2)}{1+(\log t_2)^{\alpha-1}}\right|<\varepsilon.
$$

On the other hand, for $\forall u\in B$, we have

$$
\int_1^{+\infty}q(s)\bigl|F_u(s)\bigr|\frac{ds}{s}\leq\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}<+\infty.
$$

Hence, for given $\varepsilon>0$, there exists a constant $L>0$ such that $\int_L^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}<\varepsilon$. Similarly, due to

$$
\lim_{t\to+\infty}\frac{G_\alpha(t,s)}{1+(\log t)^{\alpha-1}}=0,\quad 1\leq s\leq L,
$$

there exists a constant $T_2>L$ such that, for any $t_1,t_2>T_2$, $1\leq s\leq L$, we have

$$
\left|\frac{G_\alpha(t_1,s)}{1+(\log t_1)^{\alpha-1}}-\frac{G_\alpha(t_2,s)}{1+(\log t_2)^{\alpha-1}}\right|<\varepsilon.
$$

Let $\forall t_1,t_2>\max\{T_1,T_2\}$, by Lemma 2.6,

$$
\begin{aligned}
&\left|\frac{Au(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{Au(t_2)}{1+(\log t_2)^{\alpha-1}}\right|\\
&\leq\int_1^L\left|\frac{G_\alpha(t_1,s)}{1+(\log t_1)^{\alpha-1}}-\frac{G_\alpha(t_2,s)}{1+(\log t_2)^{\alpha-1}}\right|q(s)\bigl|F_u(s)\bigr|\frac{ds}{s}\\
&\quad+\int_L^{+\infty}\left|\frac{G_\alpha(t_1,s)}{1+(\log t_1)^{\alpha-1}}-\frac{G_\alpha(t_2,s)}{1+(\log t_2)^{\alpha-1}}\right|q(s)\bigl|F_u(s)\bigr|\frac{ds}{s}\\
&\quad+\int_1^{+\infty}\left|\frac{T_{1,\alpha}(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{T_{1,\alpha}(t_2)}{1+(\log t_2)^{\alpha-1}}\right|\left(\int_1^{+\infty}g_1(t)G_\alpha(t,s)\frac{dt}{t}\right)q(s)\bigl|F_u(s)\bigr|\frac{ds}{s}\\
&\quad+\int_1^{+\infty}\left|\frac{T_{2,\alpha}(t_1)}{1+(\log t_1)^{\alpha-1}}-\frac{T_{2,\alpha}(t_2)}{1+(\log t_2)^{\alpha-1}}\right|\left(\int_1^{+\infty}g_2(t)G_\alpha(t,s)\frac{dt}{t}\right)q(s)\bigl|F_u(s)\bigr|\frac{ds}{s}\\
&\leq\varepsilon\int_1^L q(s)f_{r_1}(s)\frac{ds}{s}+\frac{2}{\Gamma(\alpha)}\varepsilon+\varepsilon l_1\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}+\varepsilon\delta_2\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}\\
&\leq\varepsilon(1+l_1+\delta_2)\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}+\frac{2}{\Gamma(\alpha)}\varepsilon.
\end{aligned}
$$

Similarly,

$$\lim_{t\to+\infty}\frac{T_{1,\alpha-\beta_1}(t)}{1+(\log t)^{\alpha-\beta_1-1}}=\frac{\delta l_2}{\Gamma(\alpha-\beta_1)},\qquad \lim_{t\to+\infty}\frac{T_{2,\alpha-\beta_1}(t)}{1+(\log t)^{\alpha-\beta_1-1}}=\frac{\delta(1-\delta_1)}{\Gamma(\alpha-\beta_1)},$$

there exists a constant $T_3>1$ such that $\forall t_1,t_2>T_3$,

$$\left|\frac{T_{1,\alpha-\beta_1}(t_1)}{1+(\log t_1)^{\alpha-\beta_1-1}}-\frac{T_{1,\alpha-\beta_1}(t_2)}{1+(\log t_2)^{\alpha-\beta_1-1}}\right|<\varepsilon,$$
$$\left|\frac{T_{2,\alpha-\beta_1}(t_1)}{1+(\log t_1)^{\alpha-\beta_1-1}}-\frac{T_{2,\alpha-\beta_1}(t_2)}{1+(\log t_2)^{\alpha-\beta_1-1}}\right|<\varepsilon.$$

From $\lim_{t\to+\infty}\frac{G_{\alpha-\beta_1}(t,s)}{1+(\log t)^{\alpha-\beta_1-1}}=0$, $1\leq s\leq L$, there exists a constant $T_4>L$ such that, for any $t_1,t_2>T_4$, $1\leq s\leq L$, we have $|\frac{G_{\alpha-\beta_1}(t_1,s)}{1+(\log t_1)^{\alpha-\beta_1-1}}-\frac{G_{\alpha-\beta_1}(t_2,s)}{1+(\log t_2)^{\alpha-\beta_1-1}}|<\varepsilon$. Let $\forall t_1,t_2>\max\{T_3,T_4\}$,

$$\left|\frac{{}^HD^{\beta_1}Au(t_1)}{1+(\log t_1)^{\alpha-\beta_1-1}}-\frac{{}^HD^{\beta_1}Au(t_2)}{1+(\log t_2)^{\alpha-\beta_1-1}}\right|\leq\varepsilon(1+l_1+\delta_2)\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}+\frac{2}{\Gamma(\alpha-\beta_1)}\varepsilon.$$

When $\beta_2<\alpha-1$,

$$\lim_{t\to+\infty}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}T_{1,\alpha-\beta_2}(t)=0,$$
$$\lim_{t\to+\infty}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}T_{2,\alpha-\beta_2}(t)=0,$$

there exists a constant $T_5>1$ such that $\forall t_1,t_2>T_5$,

$$\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}T_{1,\alpha-\beta_2}(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}T_{1,\alpha-\beta_2}(t_2)\right|<\varepsilon,$$
$$\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}T_{2,\alpha-\beta_2}(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}T_{2,\alpha-\beta_2}(t_2)\right|<\varepsilon.$$

From $\lim_{t\to+\infty}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}G_{\alpha-\beta_2}(t,s)=0$, $1\leq s\leq L$, there exists a constant $T_6>L$ such that, for any $t_1,t_2>T_6$, $1\leq s\leq L$, we have

$$\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}G_{\alpha-\beta_2}(t_1,s)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}G_{\alpha-\beta_2}(t_1,s)\right|<\varepsilon.$$

Let $\forall t_1,t_2>\max\{T_5,T_6\}$,

$$\begin{aligned}&\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}\big({}^HD^{\beta_2}Au\big)(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}\big({}^HD^{\beta_2}Au\big)(t_2)\right|\\&\quad\leq\varepsilon(1+l_1+\delta_2)\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}+\frac{2}{\Gamma(\alpha-\beta_2)}\varepsilon.\end{aligned}$$

When $\beta_2=\alpha-1$, $\lim_{t\to+\infty}\frac{\log t}{1+\log t}=1$, there exists a constant $T_7>1$ such that $\forall t_1,t_2>T_7$,

$$\left|\frac{\log t_1}{1+\log t_1}-\frac{\log t_2}{1+\log t_2}\right|<\varepsilon,$$

$\lim_{t\to+\infty}\frac{\log t}{1+\log t}G(t,s)=0$, $1\le s\le L$, there exists a constant $T_8>L$ such that, for any $t_1,t_2>T_8$, $1\le s\le L$, we have

$$\left|\frac{\log t_1}{1+\log t_1}G(t_1,s)-\frac{\log t_2}{1+\log t_2}G(t_2,s)\right|<\varepsilon.$$

Let $\forall t_1,t_2>\max\{T_7,T_8\}$,

$$\begin{aligned}&\left|\frac{(\log t_1)^{2+\beta_2-\alpha}}{1+(\log t_1)^{\alpha-\beta_2}}\big({}^HD^{\beta_2}Au\big)(t_1)-\frac{(\log t_2)^{2+\beta_2-\alpha}}{1+(\log t_2)^{\alpha-\beta_2}}\big({}^HD^{\beta_2}Au\big)(t_2)\right|\\&\quad\le\varepsilon\big(1+\delta l_1l_2+\delta(1-\delta_1)\delta_2\big)\int_1^{+\infty}q(s)f_{r_1}(s)\frac{ds}{s}+2\varepsilon.\end{aligned}$$

Combining ② and ③, from Lemma 2.7, we have $A(B)$ is relatively compact. Therefore we conclude that A is a compact operator. Hence, A is completely continuous in Y.

Step 2. $A(\Omega_R)\subset\Omega_R$, where $\Omega_R=\{u\in Y:\|u\|\le R\}$.

For any $u\in\Omega_R$, we know that

$$\frac{|u(t)|}{1+(\log t)^{\alpha-1}}\le R,\qquad\frac{|{}^HD^{\beta_1}u(t)|}{1+(\log t)^{\alpha-\beta_1-1}}\le R,\qquad\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\big|{}^HD^{\beta_2}u(t)\big|\le R.$$

From condition (H_3) and (3.5)–(3.8), we have

$$\begin{aligned}\left|\frac{Au(t)}{1+(\log t)^{\alpha-1}}\right|&=\left|\int_1^{+\infty}\frac{K_\alpha(t,s)}{1+(\log t)^{\alpha-1}}q(s)F_u(s)\frac{ds}{s}\right|\\&\le\Delta_1\int_1^{+\infty}q(s)f_R(s)\frac{ds}{s}\le R.\end{aligned}\tag{3.9}$$

$$\begin{aligned}\left|\frac{{}^HD^{\beta_1}Au(t)}{1+(\log t)^{\alpha-\beta_1-1}}\right|&=\left|\int_1^{+\infty}\frac{K_{\alpha-\beta_1}(t,s)}{1+(\log t)^{\alpha-\beta_1-1}}q(s)F_u(s)\frac{ds}{s}\right|\\&\le\Delta_2\int_1^{+\infty}q(s)f_R(s)\frac{ds}{s}\le R,\end{aligned}\tag{3.10}$$

$$\begin{aligned}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\big|{}^HD^{\beta_2}Au(t)\big|&=\left|\int_1^{+\infty}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}K_{\alpha-\beta_2}(t,s)q(s)F_u(s)\frac{ds}{s}\right|\\&\le\Delta_3\int_1^{+\infty}q(s)f_R(s)\frac{ds}{s}\le R,\quad(\beta_2<\alpha-1)\end{aligned}\tag{3.11}$$

$$\begin{aligned}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}\big|{}^HD^{\beta_2}Au(t)\big|&=\left|\int_1^{+\infty}\frac{(\log t)^{2+\beta_2-\alpha}}{1+(\log t)^{\alpha-\beta_2}}K^{\beta_2}(t,s)q(s)F_u(s)\frac{ds}{s}\right|\\&\le\Delta_4\int_1^{+\infty}q(s)f_R(s)\frac{ds}{s}\le R,\quad(\beta_2=\alpha-1),\end{aligned}\tag{3.12}$$

which means that $\|Au\|_Y\le R$, $A(\Omega_R)\subset\Omega_R$ holds.

Step 3. We will show that A has at least one solution in Y.

By step 1 and step 2, we have $A:\Omega_R\to\Omega_R$ is completely continuous. Ω_R is a nonempty, closed, bounded, and convex subset of Y. According to the Schauder fixed point theorem, we conclude that A has at least one fixed point in Ω_R, then boundary value problem (1.1), (1.2) has at least one solution. □

Let $J_0=(1,+\infty)$, $L(s)=\max\{1+(\log s)^{\alpha-1},1+(\log s)^{\alpha-\beta_1-1},\frac{1+(\log s)^{\alpha-\beta_2}}{(\log s)^{2+\beta_2-\alpha}}\}$, $s\in J_0$.

$$Y_1=\left\{y\in L\bigl(J_0,\mathbb{R}^+\bigr):\int_1^{+\infty}y(s)L(s)q(s)\frac{ds}{s}<+\infty\right\}.$$

Corollary 3.1 *Assume that conditions* $(\mathbf{H}_2)$–$(\mathbf{H}_3)$ *are satisfied. Further assume that the following condition* $(\mathbf{H}_4)$ *holds*:

$(\mathbf{H}_4)$ *$f(t,u,v,w)$ is continuous on $J\times\mathbb{R}^3$, and there exist nonnegative functions $a,b,c,d\in Y_1$ such that, for all $(u,v,w)\in\mathbb{R}^3$ and $t\in J$,*

$$\bigl|f(t,u,v,w)\bigr|\le a(t)+b(t)|u|+c(t)|v|+d(t)|w|.$$

Then boundary value problem (1.1) *and* (1.2) *has at least one solution*.

Proof Let $\Omega_R=\{u\in Y:\|u\|_Y\le R\}$. For any $u\in\Omega_R$, $s\in J$, we have

$$\begin{aligned}
\bigl|F_u(s)\bigr|&=\bigl|f\bigl(s,u(s),{}^HD^{\beta_1}u(s),{}^HD^{\beta_2}u(s)\bigr)\bigr|\\
&\le a(s)+b(s)u(s)+c(s){}^HD^{\beta_1}u(s)+d(s){}^HD^{\beta_2}u(s)\\
&=a(s)+b(s)\bigl(1+(\log s)^{\alpha-1}\bigr)\frac{u(s)}{1+(\log s)^{\alpha-1}}\\
&\quad+c(s)\bigl(1+(\log s)^{\alpha-\beta-1}\bigr)\frac{{}^HD^{\beta}u(s)}{1+(\log s)^{\alpha-\beta_1-1}}\\
&\quad+d(s)\frac{1+(\log s)^{\alpha-\beta_2}}{(\log s)^{2+\beta_2-\alpha}}\frac{(\log s)^{2+\beta_2-\alpha}}{1+(\log s)^{\alpha-\beta_2}}\bigl({}^HD^{\beta_2}u\bigr)(s)\\
&\le a(s)+R\biggl[b(s)\bigl(1+(\log s)^{\alpha-1}\bigr)+c(s)\bigl(1+(\log s)^{\alpha-\beta_1-1}\bigr)+d(s)\frac{1+(\log s)^{\alpha-\beta_2}}{(\log s)^{2+\beta_2-\alpha}}\biggr]\\
&\triangleq f_R(s).
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\int_1^{+\infty}a(s)q(s)\frac{ds}{s}\le\int_1^{+\infty}a(s)L(s)q(s)\frac{ds}{s}<+\infty,\\
&\int_1^{+\infty}b(s)\bigl(1+(\log s)^{\alpha-1}\bigr)q(s)\frac{ds}{s}\le\int_1^{+\infty}b(s)L(s)q(s)\frac{ds}{s}<+\infty,\\
&\int_1^{+\infty}c(s)\bigl(1+(\log s)^{\alpha-\beta_1-1}\bigr)q(s)\frac{ds}{s}\le\int_1^{+\infty}c(s)L(s)q(s)\frac{ds}{s}<+\infty,\\
&\int_1^{+\infty}d(s)\frac{1+(\log s)^{\alpha-\beta_2}}{(\log s)^{2+\beta_2-\alpha}}q(s)\frac{ds}{s}\le\int_1^{+\infty}d(s)L(s)q(s)\frac{ds}{s}<+\infty,
\end{aligned}$$

hence $\int_1^{+\infty}q(s)f_R(s)\frac{ds}{s}<+\infty$.

By Theorem 3.1, we get that problem (1.1), (1.2) has at least one solution. □

4 Example

Example 4.1 Consider the Hadamard-type fractional boundary value problem on unbounded domain:

$$\begin{cases} {}^{H}D^{\frac{5}{2}}u(t) + e^{-t}\frac{t^2(\log t)^{\frac{3}{4}}u(t)({}^{H}D^{\frac{1}{4}}u(t))({}^{H}D^{\frac{5}{4}}u(t))}{(1+(\log t)^{\frac{3}{2}})((1+(\log t)^{\frac{5}{4}})(1+(\log t)^{\frac{5}{4}})} = 0, \quad 1 < t < +\infty, \\ u(1) = 0, \\ {}^{H}D^{\frac{1}{2}}u(1) = \int_1^{+\infty}\frac{u(s)}{s^3(\log s)^{\frac{1}{2}}}\frac{ds}{s}, \\ {}^{H}D^{\frac{3}{2}}u(+\infty) = \int_1^{+\infty}\frac{u(s)}{s^3(\log s)^{\frac{1}{2}}}\frac{ds}{s}. \end{cases} \tag{4.1}$$

Let $\alpha = \frac{5}{2}$, $\beta_1 = \frac{1}{4}$, $\beta_2 = \frac{5}{4}$, $q(t) = e^{-t}$, $g_1(t) = g_2(t) = \frac{1}{t^3(\log t)^{\frac{1}{2}}}$, $f_r(t) = t^2r^3$,

$$f(t,u,v,w) = \frac{t^2(\log t)^{\frac{3}{4}}uvw}{(1+(\log t)^{\frac{3}{2}})(1+(\log t)^{\frac{5}{4}})(1+(\log t)^{\frac{5}{4}})}.$$

Hence

$$f\left(t, \left(1+(\log t)^{\frac{3}{2}}\right)u, \left(1+(\log t)^{\frac{5}{4}}\right)v, \frac{1+(\log t)^{\frac{5}{4}}}{(\log t)^{\frac{3}{4}}}w\right) = t^2uvw \le t^2r^3 = f_r(t),$$
$$\forall t \in J_1, 0 \le u,v,w \le r.$$

$\int_1^{+\infty} f_r(t)q(t)\frac{dt}{t} = r^3\int_1^{+\infty} t^2e^{-t}\,dt = 5e^{-1}r^3$.

$$l_1 = \delta_2 = \frac{1}{\Gamma(\frac{5}{2})}\int_1^{+\infty}\frac{1}{t^3(\log t)^{\frac{1}{2}}}(\log t)^{\frac{3}{2}}\frac{dt}{t} = \frac{1}{9\Gamma(\frac{5}{2})} \approx 0.0836 < 1,$$
$$l_2 = \delta_1 = \frac{1}{\Gamma(\frac{3}{2})}\int_1^{+\infty}\frac{1}{t^3(\log t)^{\frac{1}{2}}}(\log t)^{\frac{1}{2}}\frac{dt}{t} = \frac{1}{3\Gamma(\frac{3}{2})} \approx 0.3761 < 1,$$

and $\frac{1}{\delta} = (1-\delta_1)(1-\delta_2) - l_1l_2 \approx 0.5403 > 0$.

$$\begin{aligned} \Delta_1 &= \frac{1}{\Gamma(\alpha)} + l_1\left(\frac{\delta l_2}{\Gamma(\alpha)} + \frac{\delta(1-\delta_2)}{\Gamma(\alpha-1)}\right) + \delta_2\left(\frac{\delta(1-\delta_1)}{\Gamma(\alpha)} + \frac{\delta l_1}{\Gamma(\alpha-1)}\right) \\ &= \frac{1}{3\Gamma(\frac{3}{2})}(2+5l_1\delta) \approx 1.0432, \\ \Delta_2 &= \frac{1}{\Gamma(\alpha-\beta_1)} + l_1\left(\frac{\delta l_2}{\Gamma(\alpha-\beta_1)} + \frac{\delta(1-\delta_2)}{\Gamma(\alpha-\beta_1-1)}\right) \\ &\quad + \delta_2\left(\frac{\delta(1-\delta_1)}{\Gamma(\alpha-\beta_1)} + \frac{\delta l_1}{\Gamma(\alpha-\beta_1-1)}\right) \\ &= \frac{1}{5\Gamma(\frac{5}{4})}(4+9l_1\delta) \approx 1.1899, \end{aligned}$$

$$\begin{aligned}\Delta_3 &= \frac{1}{\Gamma(\alpha-\beta_2)} + l_1\left(\frac{\delta l_2}{\Gamma(\alpha-\beta_2)} + \frac{\delta(1-\delta_2)}{\Gamma(\alpha-\beta_2-1)}\right) \\ &\quad + \delta_2\left(\frac{\delta(1-\delta_1)}{\Gamma(\alpha-\beta_2)} + \frac{\delta l_1}{\Gamma(\alpha-\beta_2-1)}\right) \\ &= \frac{4}{5\Gamma(\frac{1}{4})}\left(1+\frac{5}{4}l_1\delta\right) \approx 1.3166, \\ \Delta_4 &= 1 + \delta l_1 l_2 + \delta(1-\delta_1)\delta_2 = 1 + \delta l_1 \approx 1.1547.\end{aligned}$$

Therefore, conditions $(H_1)(H_2)$ hold. Choose $\Delta = \max\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\} = \Delta_3$, $R \le \sqrt{\frac{e}{5\Delta}}$, then $\Delta \int_1^{+\infty} f_R(t)e^{-t}\frac{dt}{t} = 5\Delta e^{-1}R^3 \le R$, so condition (H_3) holds. By Theorem 3.1, BVP (4.1) has at least one solution.

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