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Identification of chaos in fractional chaotic map

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Abstract

This paper utilizes the 0–1 test algorithm to identify chaos in a fractional chaotic map. A fractional Burgers map is proposed by means of the Caputo-like delta difference operator. The bifurcation diagrams, phase trajectories and 0–1 test results of the fractional Burgers map are presented, respectively. This work extends the 0–1 test algorithm to the discrete fractional chaotic map.

Keywords: Fractional chaotic map; Chaos; Bifurcation; 0–1 test

1 Introduction

The fractional systems have recently received increasing attention, because fractional calculus can accurately explain many realistic problems. Many important results, such as chaos and bifurcation, on the continuous fractional systems can be found in [1–7]. However, the studies on discrete fractional systems are still in their infancy, especially in chaos dynamics. Some pioneering work for the discrete fractional systems can be found in [8–14]. In addition, researchers get more interesting results for discrete fractional chaotic systems by means of [15–20]. Although many results have been presented, the identification of chaos of the fractional chaotic map is still an open topic.

Very recently, Wu et al. used Lyapunov exponents to identify chaos for the fractional chaotic map in the [21]. However, this method needs to construct the Jacobian matrix of fractional chaotic map. In fact, the construction of the Jacobi matrix is relatively tedious work in the case of a fractional map. In order to avoid the construction of a Jacobian matrix, in this paper we extend the 0–1 test algorithm [22–28] to a fractional chaotic map, which provides another reference for the study of the fractional chaotic map.

This article is structured as follows. The basic definitions of discrete fractional calculus are introduced in Sect. 2. In Sect. 3, the fractional Burgers map is obtained by means of Caputo-like delta difference. In Sect. 4, the chaotic behaviors of the fractional Burgers map are investigated. In Sect. 5, the conclusions arising from this study are summarized.

2 Preliminaries

The basic definitions of discrete fractional calculus briefly described in this section. See references [10–12] for more details. The falling $\mathbb{N}_a = \{a, a + 1, a + 2, \dots\}$ denotes the time scale.

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Definition 1 Let $x(t) : \mathbb{N}_a \rightarrow \mathbb{R}$ and $\nu > 0$, the fractional sum of order ν is defined by

$$\Delta_a^{-\nu} x(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} x(s), \quad t \in \mathbb{N}_{a+\nu}, \tag{1}$$

where a is the starting point, $\sigma(s) = s + 1$, and $t^{(\nu)}$ is the falling function defined as $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$.

Definition 2 For $\nu > 0$, the ν th Caputo-like delta difference can be defined as

$${}^C \Delta_a^\nu x(t) := \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t - \sigma(s))^{(n-\nu-1)} \Delta^n x(s), \quad t \in \mathbb{N}_{a+n-\nu}, \tag{2}$$

where $n = [\nu] + 1$.

Theorem 1 ([13]) *For the delta fractional difference equation*

$$\begin{cases} {}^C \Delta_a^\nu x(t) = f(t + \nu - 1, x(t + \nu - 1)), \\ \Delta^k x(a) = x_k, \quad m = [\nu] + 1, \quad k = 0, 1, 2, \dots, m - 1, \end{cases} \tag{3}$$

the equivalent discrete integral equation can be obtained as

$$x(t) = x_0(t) + \frac{1}{\Gamma(\nu)} \sum_{s=a+m-\nu}^{t-\nu} (t - \sigma(s))^{(\nu-1)} \times f(s + \nu - 1, x(s + \nu - 1)), \quad t \in \mathbb{N}_{a+m}, \tag{4}$$

where the initial iteration reads

$$x_0(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k x(a). \tag{5}$$

Particularly, if for the initial point $a = 0$, and $0 < \nu < 1$, Eq. (4) is rewritten by

$$x(t) = x(0) + \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} (t - \sigma(s))^{(\nu-1)} \times f(s + \nu - 1, x(s + \nu - 1)), \quad t \in \mathbb{N}_m. \tag{6}$$

Using $(t - \sigma(s))^{(\nu-1)} = \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)}$, and $s + \nu = j$, Eq. (6) is further simplified to

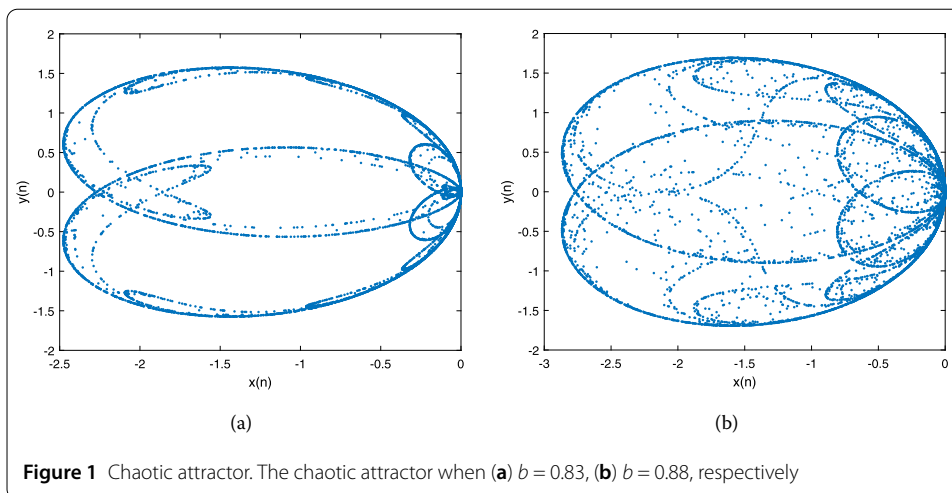
$$x(t) = x(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^t \frac{\Gamma(t-j+\nu)}{\Gamma(t-j+1)} \times f(j-1, x(j-1)), \quad t \in \mathbb{N}_m. \tag{7}$$

3 The fractional Burgers map and its solution

The fractional version of Burgers map is obtained by using the Caputo-like delta operator in this section.

Consider the falling Burgers map

$$\begin{cases} x(t+1) = (1-a)x(t) - y(t)^2, \\ y(t+1) = (1+b)y(t) + y(t)x(t). \end{cases} \tag{8}$$



By taking $a = 1$ and $b = 0.83$, system (8) exhibits a chaotic behavior, which is shown in Fig. 1(a). When b is further increased to 0.88, a more complex chaotic attractor is plotted in Fig. 1(b).

Based on the first-order difference operator Δ , system (8) can be rewritten as

$$\begin{cases} \Delta x(t + 1) = (1 - a)x(t) - y(t)^2 - x(t), \\ \Delta y(t + 1) = (1 + b)y(t) + y(t)x(t) - y(t). \end{cases} \tag{9}$$

The Caputo-like delta operator ${}^C\Delta_a^\nu$ is applied to system (9), a novel fractional Burgers map is obtained as

$$\begin{cases} {}^C\Delta_a^\nu x(t) = (1 - a)x(t + \nu - 1) - y(t + \nu - 1)^2 - x(t + \nu - 1), \\ {}^C\Delta_a^\nu y(t) = (1 + b)y(t + \nu - 1) + y(t + \nu - 1)x(t + \nu - 1) - y(t + \nu - 1). \end{cases} \tag{10}$$

According to Theorem 1 in Sect. 2, taking the initial point as 0, the numerical solution of system (10) is given as

$$\begin{cases} x(t) = x(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^t \frac{\Gamma(t-j+\nu)}{\Gamma(t-j+1)} \times ((1 - a)x(j - 1) - y(j - 1)^2 - x(j - 1)), \\ y(t) = y(0) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^t \frac{\Gamma(t-j+\nu)}{\Gamma(t-j+1)} \times ((1 + b)y(j - 1) + y(j - 1)x(j - 1) - y(j - 1)). \end{cases} \tag{11}$$

4 The chaotic dynamics of fractional Burgers map

The chaos of fractional Burgers map is identified by using the recently proposed 0–1 test.

4.1 The notion of 0–1 test

Let us briefly describe the 0–1 test as follows (see [22] for more details). We consider a set of data $\{\phi_k, k = 1, 2, 3, \dots\}$, which is obtained from a dynamical system. According to [22], we define a new coordinates (p_k, s_k) . Here p_k and s_k are the falling functions defined as

$$\begin{cases} p_k = \sum_{\lambda=1}^k \phi_k \cos(\theta_\lambda), \\ s_k = \sum_{\lambda=1}^k \phi_k \sin(\theta_\lambda), \end{cases} \tag{12}$$

where

$$\theta_\lambda = \lambda c + \sum_{i=1}^\lambda \phi_i, \quad \lambda = 1, 2, \dots, k, \tag{13}$$

and $c \in [\frac{\pi}{5}, \frac{4\pi}{5}]$ is a random number. Further, the mean square displacement is defined by

$$M_c(k) = \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{\lambda=1}^N [(p_{\lambda+k} - p_\lambda) - (s_{\lambda+k} - s_\lambda)]^2, \tag{14}$$

where $k \ll N$, we calculate $M_c(k)$ for $k < k_{\text{cut}} = \text{round}(N/10)$, as recommended in the [24]. For a given c , the modified $D_c(k)$ can be defined as

$$D_c(k) = M_c(k) - \langle \phi \rangle^2 \frac{1 - \cos(kc)}{1 - \cos(c)}, \tag{15}$$

where $\langle \phi \rangle$ is the mean value of $\{\phi_k, k = 1, 2, 3, \dots\}$, which can be calculated by

$$\langle \phi \rangle = \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \phi_k. \tag{16}$$

According to Eq. (15), we can define the asymptotic growth rate K_c by means of the correlation and regression methods, respectively. If the correlation method is used, the K_c is defined as

$$K_c = \frac{\text{cov}(\varepsilon, X)}{\sqrt{(\text{var}(\varepsilon)) \text{var}(X)}}, \tag{17}$$

where $\varepsilon = 1, 2, 3, \dots$ and $X = (D_c(1), D_c(2), D_c(3), \dots), D_c(k_{\text{cut}})$. If the regression method is employed, K_c is calculated by

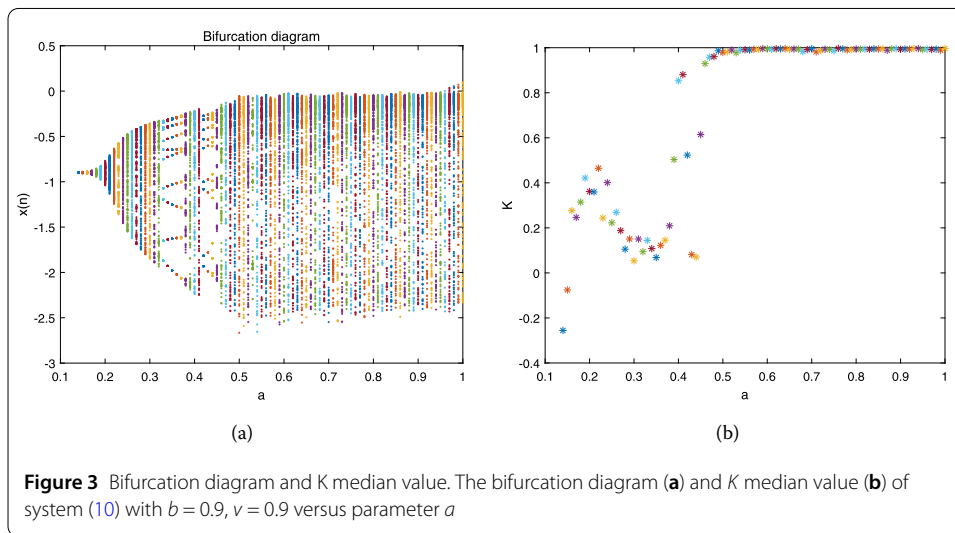
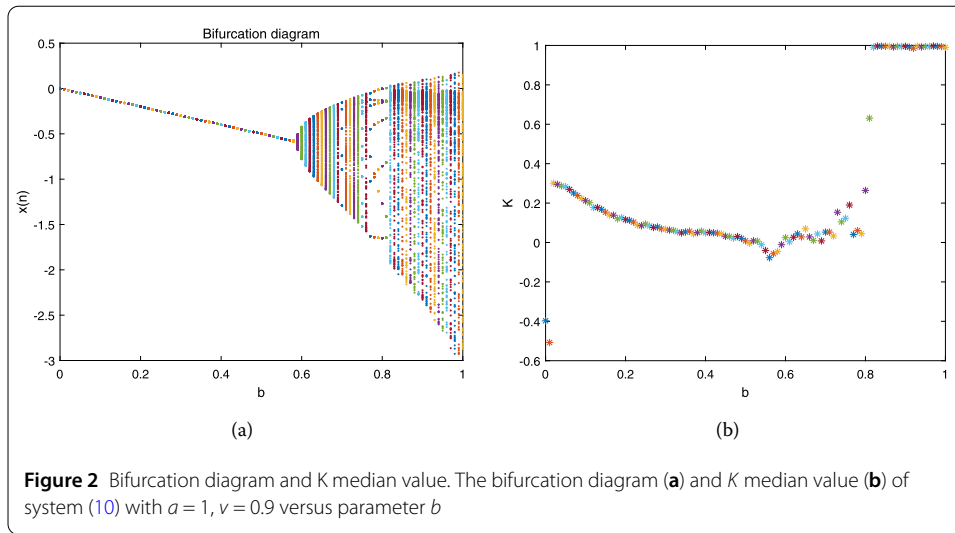
$$K_c = \lim_{k \rightarrow \infty} \frac{\log D_c(k)}{\log k}. \tag{18}$$

Finally, according to the K_c , we can get the median value $K = \text{median}(K_c)$. The idea of the 0–1 test can be described as (a) the dynamical behavior is regular when K is equal to 0; (b) the dynamical behavior is chaotic when K is equal to 1. In addition, we can also observe (p, s) trajectories, the regular dynamics correspond to bounded motions whereas Brownian-like motions correspond to chaotic dynamics.

4.2 Testing for chaos

In this section, identification of chaos of the fractional Burgers map is researched by using the bifurcation diagrams, phase trajectories and 0–1 test.

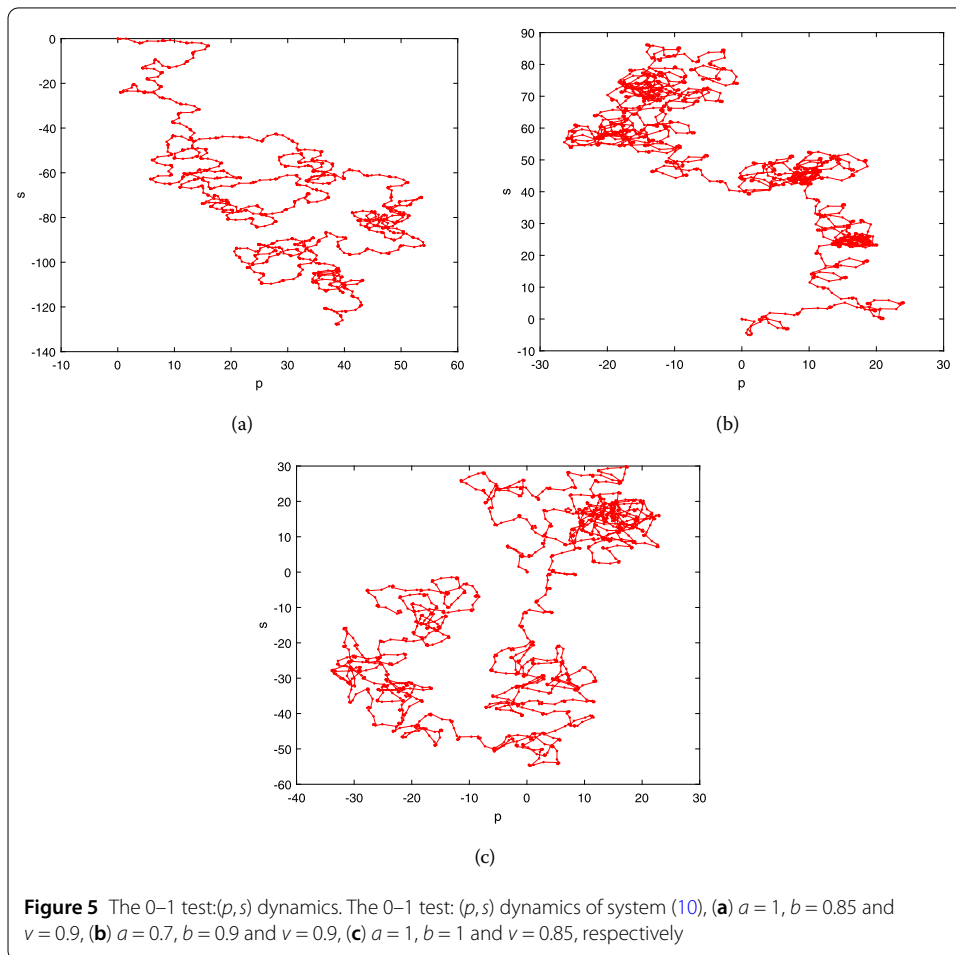
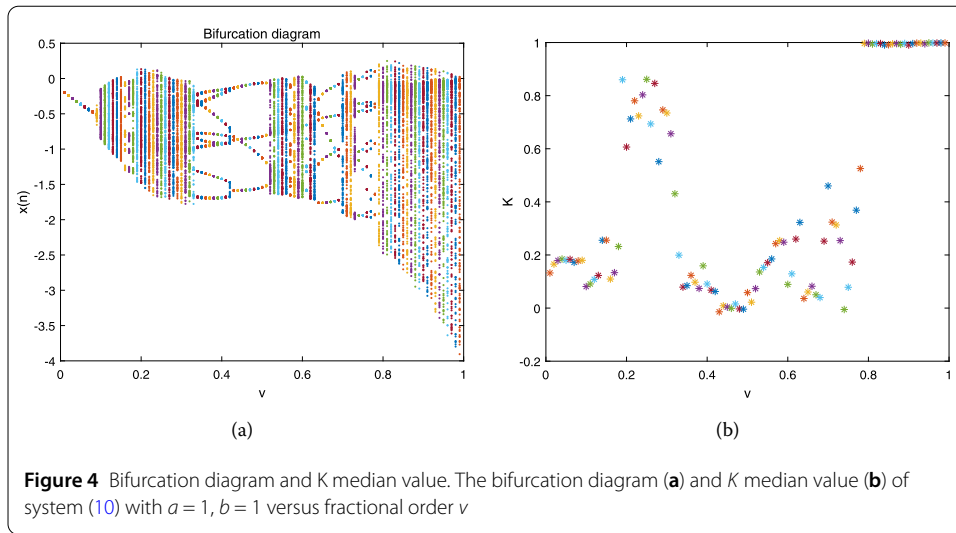
Let $a = 1$, $\nu = 0.9$ and b is fixed. The step size of the b is set at 0.01, in this paper, we can obtain the bifurcation diagram of system (10) versus $b \in [0, 1]$ as shown in Fig. 2(a). The K median value of the time series $x(n)$ of system (10) versus $b \in [0, 1]$ is drawn in Fig. 2(b). Let $b = 0.9$, $\nu = 0.9$, and the a is fixed. The step size of the a is set at 0.01, we can obtain the bifurcation diagram of system (10) versus $a \in [0, 1]$ as shown in Fig. 3(a). The K median



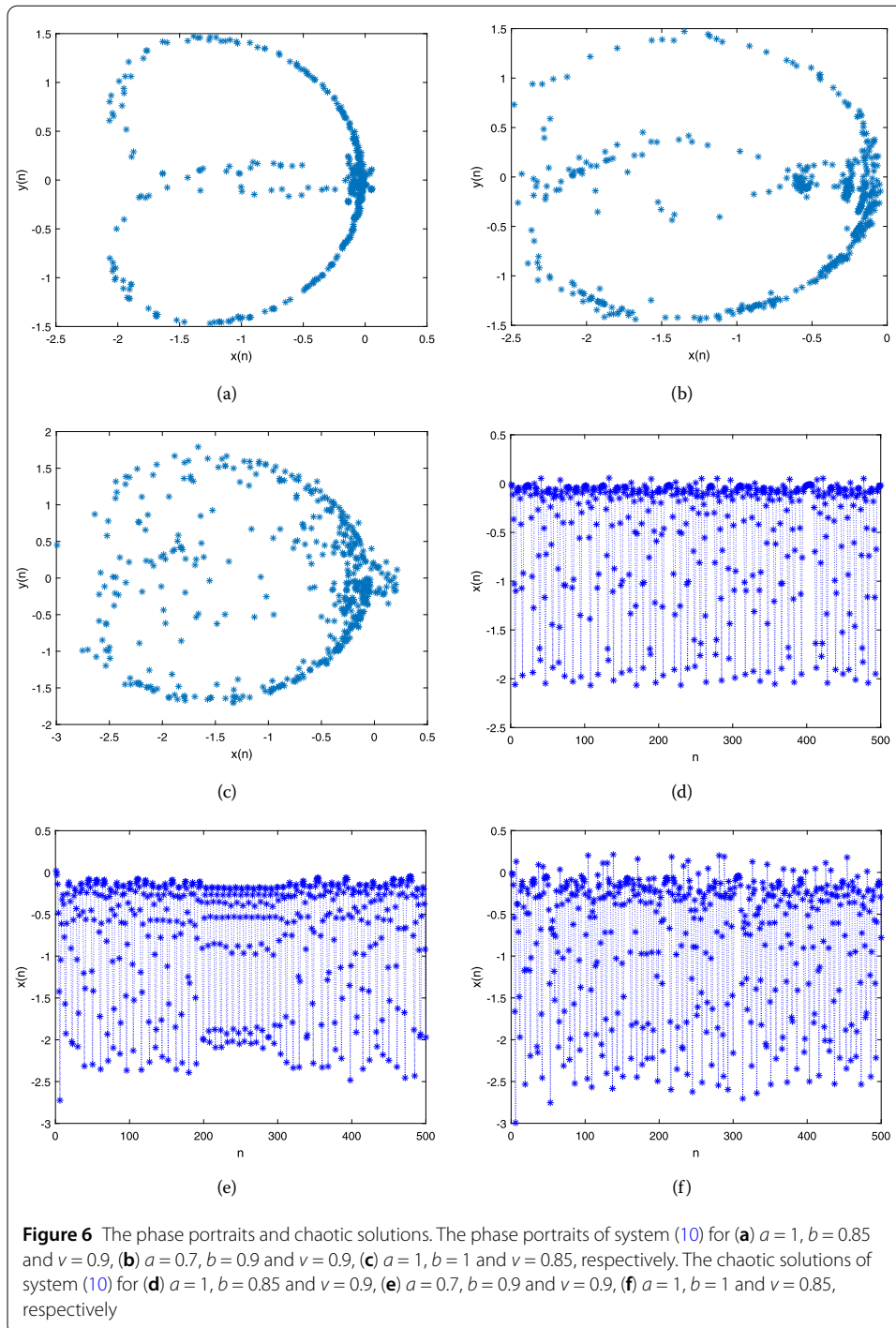
value of the time series $x(n)$ of system (10) versus $a \in [0, 1]$ is drawn in Fig. 3(b). Then, let $a = 1, b = 1$, and the fractional over ν is fixed. The step size of the ν is set at 0.01, we can obtain the bifurcation diagram of system (10) versus $\nu \in (0, 1)$ as shown in Fig. 4(a). The K median value of the time series $x(n)$ of system (10) versus $\nu \in (0, 1)$ is drawn in Fig. 4(b). From Figs. 2, 3 and 4, we can find that system (10) shows a different chaotic dynamics when we change the parameters b and a , and the fractional order ν . From Fig. 2, system (10) implies chaotic behaviors ($(K \cong 1)$) when $b \in [0.82, 1]$. From Fig. 3, system (10) implies chaotic behaviors ($(K \cong 1)$) when $a \in [0.46, 1]$. From Fig. 4, system (10) implies chaotic behaviors ($(K \cong 1)$) when $\nu \in [0.79, 1]$.

In addition, the (p, s) dynamics is used to further verify the chaotic dynamics of system (10). The test results are shown in Fig. 5 which shows Brownian-like trajectories. The phase trajectories and chaotic solutions corresponding to Fig. 5 are shown in Fig. 6.

Based on the analyses above, we can find that the dynamics demonstrated in the 0–1 test results are consistent with the bifurcation diagrams and phase trajectories.

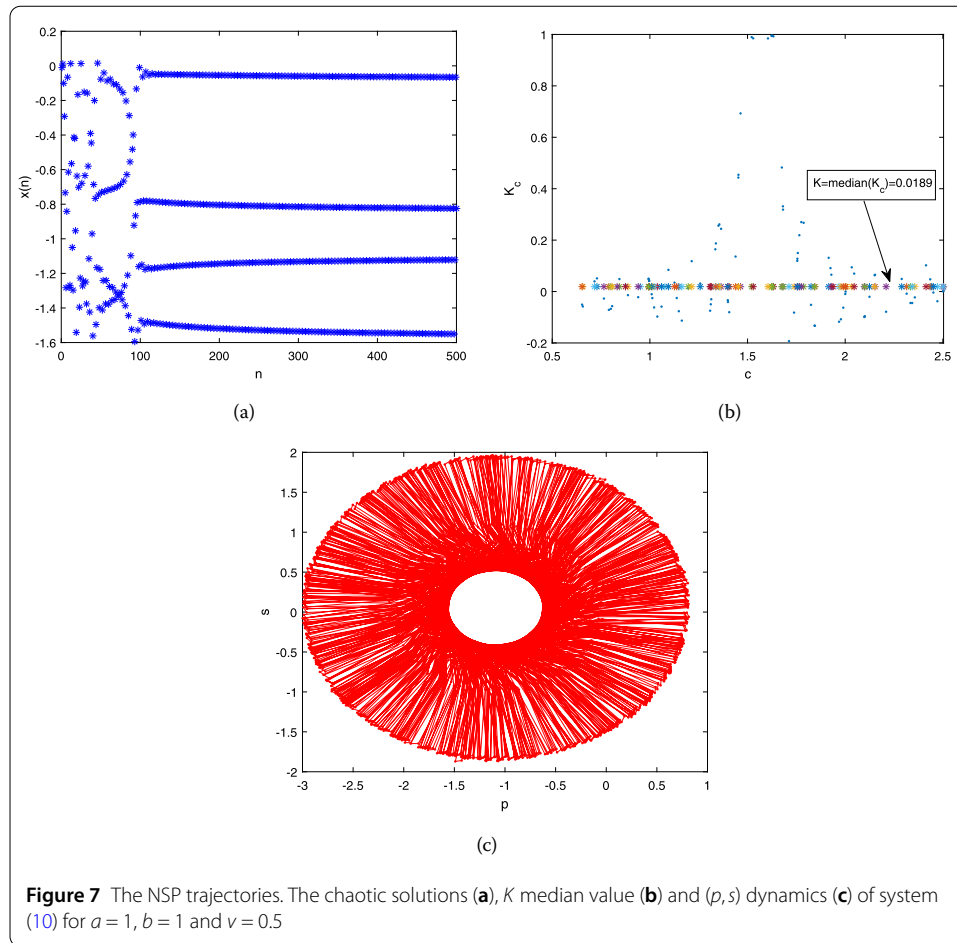


Besides, system (10) exhibits many NSP (numerically stable periodic) trajectories. As an example, we take $a = 1, b = 1$ and $\nu = 0.5$, a NSP orbit of period-4 is obtained as shown in Fig. 7(a). Figure 7(b) is the K median value of the time series $x(n)$ of system (10). The (p, s) dynamics is shown in Fig. 7(c).



5 Conclusions

Using the bifurcation diagrams, phase trajectories and 0–1 test, the identification of chaos of the fractional Burgers map is investigated in this paper. Two remarkable results are obtained as follows. Firstly, the extremely rich dynamical behaviors of the fractional Burgers map are revealed. The fractional Burgers map presents regular motions, NSP (numerically stable periodic) orbits and chaotic behaviors when we choose the different fractional order v . Compared with the Burgers map, the fractional Burgers map enlarges the parameter



space and extends the range of chaos. Secondly, this study identified chaos of the fractional Burgers map by novelly using the 0–1 test, which is different from the Lyapunov exponent method in [21]. These results show that the 0–1 test is a convenient tool to diagnose chaos in fractional chaotic map.

In fact, the system is inevitably affected by uncertain factors, such as external noise and a random parameter, therefore in further research we will focus on the identification of chaos in the stochastic fractional chaotic map.

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Competing interests

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Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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