Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Some symmetries, similarity solutions and various conservation laws of a type of dispersive water waves

Yufeng Zhang[1*], Na Bai[2] and Hongyang Guan[3]

*Correspondence:
zyfxz@cumt.edu.cn
[1]College of Mathematics, China University of Mining and Technology, Xuzhou, P.R. China
Full list of author information is available at the end of the article

Abstract

We investigate the point symmetries, Lie–Bäcklund symmetries for a type of dispersive water waves. We obtain some Lie transformation groups, various group-invariant solutions, and some similarity solutions. Besides, we produce different formats of conservation laws of the dispersive water waves by using different schemes. Finally, we consider some special solutions of the stationary dispersive water-wave equations.

PACS Codes: 05.45.Yv; 02.30.Jr; 02.30.Ik

Keywords: Symmetry; Similarity solution; Conservation law

1 Introduction

The classical dispersiveless long wave equations

$$u_t + uu_x + h_x, \qquad h_t + (uh)_x = 0 \tag{1}$$

have a number of dispersive generalizations [1]. Kupershmidt [2] investigated the commuting hierarchy and the Hamiltonian structures of the following generation of (1):

$$\begin{cases} u_t = (\frac{1}{2}u^2 + h + \beta u_x)_x, \\ h_t = (uh + \alpha u_{xx} - \beta h_x)_x, \end{cases} \tag{2}$$

and further turned (2) into the following system:

$$\begin{cases} \bar{u}_t = (\frac{1}{2}\bar{u}^2 + \bar{h} + \mu\bar{u})_x, \\ \bar{h}_t = (\bar{u}\bar{h} - \mu\bar{h}_x)_x, \qquad \mu = \gamma + \beta = \pm\sqrt{(}\alpha^2 + \beta^2), \end{cases} \tag{3}$$

by using the invertible change of variables: $u = \bar{u}$, $h = \bar{h} + \gamma\bar{u}_x$. For $\alpha = \frac{1}{3}$, $\beta = 0$, system (2) was given by Broer [1]. For $\beta = 0$, in terms of the potential $\varphi : u = \varphi_x$, system (2) was derived by Kaup [3], who found its multisoliton solutions. Matveev and Yavor [4] algebro-geometrically found a large class of almost periodic solutions of system (3). In the paper, we want to apply the Lie group analysis method [5] to study the point symmetries and Lie–Bäcklund transformation symmetries of system (2). In fact, many symmetries, similarity reductions, and conservation laws were obtained by Lie group analysis [6–10]. Lou et al. [11, 12] applied the symmetry group method to study some coherent solutions of nonlocal KdV systems and primary branch solutions of a first-order autonomous system. In addition, Ma [13] obtained some new conservation laws of some discrete evolution equations by symmetries and adjoint symmetries. Qu and Ji [14] studied inhomogeneous nonlinear diffusion equations by invariant subspace and conditional Lie–Backlund symmetry methods. It was shown that the equations admit a class of invariant subspaces governed by the nonlinear ordinary differential equations, which is equivalent to a kind of higher-order conditional Lie–Backlund symmetries of the equations. Ji and Qu [15] used the conditional Lie–Backlund symmetry method to study the invariant subspaces of nonlinear diffusion equations with convection and source terms and obtained a complete list of canonical forms for such equations, which admit higher-order conditional Lie–Backlund symmetries and multidimensional invariant subspaces. Ma [16, 17] discussed the conservation laws of differential and discrete equations, respectively. Recent studies by Ma et al. [18–22] also show a remarkable richness of rational function solutions, called lumps, as well as interaction solutions and solutions of other kinds. In addition, the invariant solutions can be formulated from the invariant submanifold method in [23].

Ibragimov and Avdonina [24] applied the Lie group method to propose a new approach for looking for conservation laws and exact solutions to nonlinear self-adjoint differential equations. For a system of m differential equations

$$F_\alpha(x,u,u_{(1)},\dots,u_{(s)}) = 0, \quad \alpha = 1,2,\dots,m, \tag{4}$$

where $u_{(1)} = \{u_i^\alpha\},\dots,u_{(s)} = \{u^\alpha_{i_1\cdots i_s}\}$, the steps are as follows.

Step 1: Introducing the adjoint equations of (4) by using the variational derivative:

$$F^*_\alpha(x,u,v,u_{(1)},v_{(1)},\dots,u_{(s)},v_{(s)}) = \frac{\delta\varphi}{\delta u^\alpha} = 0, \tag{5}$$

where $\varphi = \sum_{\beta=1}^m v^\beta F_\beta(x,u,u_{(1)},\dots,u_{(s)})$.

Step 2: Let

$$v^\alpha = \psi^\alpha(x,u), \quad \alpha = 1,2,\dots,m, \tag{6}$$

and require the following relations to be satisfied:

$$F^*_\alpha\bigl(x,u,\psi(x,u),\dots,u_{(s)},\psi_{(s)}\bigr) = \lambda^\beta_\alpha F_\beta(x,u,\dots,u_{(s)}). \tag{7}$$

We say that system (4) is nonlinearly self-adjoint if (7) holds for the solutions of (4), where λ^β_α are functions dependent on $x,u,u_{(1)},\dots$.

Step 3: Assume that the infinitesimal symmetry of the nonlinear self-adjoint system (4) is given by

$$X = \xi^i(x,u,u_{(1)},\dots)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u,u_{(1)},\dots)\frac{\partial}{\partial u^\alpha}. \tag{8}$$

Then a conservation law for the system is presented by

$$D_i(C^i) = 0, \tag{9}$$

where the components of the conserved vector are the following:

$$\begin{aligned} C^i &= W^\alpha\left[\frac{\partial\varphi}{\partial u_i^\alpha} - D_j\left(\frac{\partial\varphi}{\partial u_{ij}^\alpha}\right) + D_jD_k\left(\frac{\partial\varphi}{\partial u_{ijk}^\alpha}\right) - \cdots\right] \\ &\quad + D_j(W^\alpha)\left[\frac{\partial\varphi}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial\varphi}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_jD_k(W^\alpha)\left[\frac{\partial\varphi}{\partial u_{ijk}^\alpha} - \cdots\right], \end{aligned} \tag{10}$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

By applying (9) and (10) some conservation laws of some nonlinear self-adjoint differential equations were obtained in [5–9]. In addition, Göktas and Hereman [25] proposed a new method for looking for conservation laws of nonlinear differential equations without using the symmetries of differential equations. The explicit steps are as follows:

(1) Consider the form of conservation laws

$$D_t(\rho) + D_x(J) = 0 \tag{11}$$

for system (4). Assuming the uniformity in rank in the ith equation, form the linear system

$$A_i = \{r_{i,1}, r_{i,2}, \ldots, r_{i,\alpha}\}$$

and then gather the A_i to form the global linear system $\mathcal{A} = \bigcup_{i=1}^{\alpha} A_i$.

(2) Solving for the unknown weights $w(u_i)$, $w(\partial_t)$.

(3) Set $\mathcal{V} = \{v_1, \ldots, v_Q\}$ to be the sorted list of all the variables with positive weights, excluding ∂_t. Form all monomials of rank R or less by taking combinations of the variables in $\mathcal{V}$ and form sets consisting of ordered pairs.

Set $\mathcal{B}_0 = \{(1;0)\}$. For $q = 1, 2, \ldots, Q$, $m = 0, 1, \ldots, M-1$, where M is the number of pairs in $\mathcal{B}_{q-1}$, form $\mathcal{B}_{q,m} = \bigcup_{i=1}^{p_{q,m}}\{(T_{q,s}; W_{q,s})\}$, $T_{q,s} = T_{q-1,m}v_q^s$, $W_{q,s} = W_{q-1,m} + sw(v_q)$, $p_{q,m} = [\![\frac{R - W_{q-1,m}}{w(v_q)}]\!]$.

Denote $\mathcal{B}_q = \bigcup_{m=0}^{M-1}\mathcal{B}_{q,m}$.

(4) Let $\mathcal{G} = \mathcal{B}_Q$, which consists of all possible combinations of powers of the variables that produce rank R or less. For each pair $(T_{Q,s}; W_{q,s})$ in $\mathcal{G}$, apply $\frac{\partial^l}{\partial x^l}$ to the term $T_{Q,s}$, where $l = R - W_{Q,s}$. Set $\mathcal{H}$ to contain the terms that result from computing the various $\frac{\partial^l}{\partial x^l}(T_{Q,s})$.

(5) Removing those terms in $\mathcal{H}$ that can be written as a total derivative with respect to x,or as a derivative up to terms kept previously in the set, we denote such a set by I.

(6) For all terms from I with desired rank R, let

$$\rho = \sum_{i=1}^{\sigma} c_i I(i), \tag{12}$$

where $I(i)$ is the ith element in I, σ is the number of the terms in I, and c_i are constants to be determined later.

(7) Computing $D_t(\rho)$ and replacing the terms with derivatives with respect to ∂_t by using the original system (4), we denote the result by E. Then act on E by the variational derivatives $\frac{\delta E}{\delta u^\alpha}$ ($\alpha = 1,\dots,m$). Next, solving the system $\frac{\delta E}{\delta u^\alpha} = 0$ ($\alpha = 1,\dots,m$), we obtain the relations among the coefficients c_i. Thus we get the density ρ in (12). Substituting c_i into the expression E, we can obtain the resulting fluxes of differential equations.

In the paper, we apply the Lie group analysis method and the above approach to investigate the conservation laws of system (2).

2 Point symmetries, Lie–Bäcklund symmetries, and conservation laws

By using the Lie-group analysis method we easily get the point symmetry of the system (2):

$$\begin{cases} V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = t\partial_x - \partial_u, \\ V_4 = t\partial_t + \frac{x}{2}\partial_x - \frac{u}{2}\partial_u - h\partial_h. \end{cases} \tag{13}$$

Set

$$\mathcal{L} = p(u_t - uu_x - h_x - \beta u_{xx}) + q(h_t - u_x h - uh_x - \alpha u_{xxx} + \beta h_{xx}).$$

Then the adjoint equations of system (2) are given by

$$\begin{cases} \frac{\delta\mathcal{L}}{\delta u} = -p_t + p_x u - \beta p_{xx} + q_x h + \alpha q_{xxx} = 0, \\ \frac{\delta\mathcal{L}}{\delta h} = -q_t + p_x + uq_x + \beta q_{xx} = 0, \end{cases} \tag{14}$$

where u, h are solutions to system (2). System (14) has the solutions

$$p = h, \qquad q = u.$$

Hence system (2) is strictly self-adjoint. Besides, system (14) has some special solutions for given u and h. For example, when $u = h$, (14) has the solution

$$p = -c_1 x + c_2, \qquad q = c_1(x - t) + c_3,$$

where c_1, c_2, c_3 are constants. In particular, for $\beta = 0$, system (2) reduces to

$$\left(u'\right)^2 = u^2 - \frac{1}{3}u^3. \tag{15}$$

Obviously, (15) is solvable. Since system (2) is nonlinearly self-adjoint, we can look for the conservation laws by using the Lie group method. For system (2), the conservation laws are of the following form:

$$C^1 = W^u \frac{\partial\mathcal{L}}{\partial u_t} + W^h \frac{\partial\mathcal{L}}{\partial h_t}, \tag{16}$$

$$\begin{aligned} C^2 = {} & W^u \left[\frac{\partial\mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right) + D_x^2\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right)\right] + D_x\left(W^u\right)\left[\frac{\partial\mathcal{L}}{\partial u_{xx}} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right)\right] \\ & + D_x^2\left(W^u\right)\frac{\partial\mathcal{L}}{\partial u_{xxx}} + W^h\left[\frac{\partial\mathcal{L}}{\partial h_x} - D_x\left(\frac{\partial\mathcal{L}}{\partial h_{xx}}\right)\right] + D_x\left(W^h\right)\frac{\partial\mathcal{L}}{\partial h_{xx}}. \end{aligned} \tag{17}$$

For the vector V_3, the conservation law is

$$D_t(C^1) + D_x(C^2) = 0. \tag{18}$$

Note that $W^u = -1 - tu_t$ and $W^h = -th_t$. Then we have

$$\begin{aligned}
C^1 &= (1 + tu_t)p + th_t q = h + D_t(thu) - uh, \\
C^2 &= (1 + tu_t)(2hu - \beta h_x + \beta u_x + \alpha u_{xxx}) - tu_{xt}(-\beta h + \beta u + \alpha u) + \alpha tu_{xxt}u \\
&\quad + th_t(h + u^2 + \beta u_x) - \beta th_{xt}u.
\end{aligned}$$

To cancel the trivial operation in computing conservation laws, we may assume that

$$C^1|_{(2)} = \tilde{C}^1 + D_x(H^2) + D_z(H^3) + \cdots. \tag{19}$$

Then the conserved vector $C = (X^1, C^2, \ldots, c^m) = 0$ can be written as

$$\tilde{C} = (\tilde{C}^1, \ldots, \tilde{C}^m) = 0 \tag{20}$$

with the components

$$\tilde{C}^1, \qquad \tilde{C}^2 = C^2 + D_1(H^2), \qquad \ldots, \qquad \tilde{C}^m = C^m + D_1(H^m). \tag{21}$$

Based on versions (18)–(21), we get the reduced forms of the components of the conserved density:

$$\begin{aligned}
\tilde{C}^1 &= h - uh, \\
\tilde{C}^2 &= C^2 + D_t(thu) \\
&= (1 + tu_t)(2hu - \beta h_x + \beta u_x + \alpha u_{xxx}) - tu_{xt}(-\beta h + \beta u + \alpha u) + \alpha tu_{xxt}u \\
&\quad + th_t(h + u^2 + \beta u_x) - \beta th_{xt}u.
\end{aligned}$$

In particular, when $u = h$, $p = -c_1x + c_2$, and $q = c_1(x - t) + c_3$, we can obtain the special components of the conserved density:

$$\begin{aligned}
\bar{C}^1 &= -c_1x + c_2 + tu_t(-c_1t + c_2 + c_3), \\
\bar{C}^2 &= (1 + tu_t)\big[(-c_1t + c_2 + c_3)u + 2c_1\beta\big] - tu_{xt}\big[(2c_1x - c_1t + c_3 - c_2)\beta \\
&\quad + \alpha c_1(x - t) + \alpha c_3\big] + \alpha tu_{xxt}\big[c_1(x - t) + c_3\big] \\
&\quad + tu_t\big[-c_1x + c_2 + \big(c_1(x - t) + c_3\big)u + \beta c_1\big] - \beta tu_{xt}\big[c_1(x - t) + c_3\big].
\end{aligned}$$

For the symmetry vector V_4, we have that

$$W^u = -\frac{1}{2}u - tu_t - \frac{1}{2}xu_x, \qquad W^h = -h - th_t - \frac{1}{2}xh_x.$$

Similarly as before, we can obtain the components of the density of system (2):

$$\begin{aligned}
C^1 &= \left(\frac{1}{2}u + tu_t + \frac{1}{2}xu_x\right)p + \left(h + th_t + \frac{1}{2}xh_x\right)q = \left(\frac{1}{2}u + tu_t + \frac{1}{2}xu_x\right)h \\
&\quad + \left(h + th_t + \frac{1}{2}xh_x\right)u, \\
C^2 &= \left(\frac{1}{2}u + tu_t + \frac{1}{2}xu_x\right)(pu + qh - \beta p_x + \beta q_x + \alpha u_{xxx}) \\
&\quad + \left(tu_{xt} + \frac{1}{2}xu_{xx}\right)(-\beta p + \beta q + \alpha q_{xxx}) - \alpha\left(tu_{xxt} + \frac{1}{2}u_{xx} + \frac{1}{2}xu_{xxx}\right)q_{xxx} \\
&\quad + \left(h + th_t + \frac{1}{2}xh_x\right)(p + qu + \beta q_{xx}) - \beta\left(th_{xt} + \frac{3}{2}h_x + \frac{1}{2}xh_{xx}\right)q \\
&= \left(\frac{1}{2}u + tu_t + \frac{1}{2}xu_x\right)(2hu - \beta h_x + \beta u_x + \alpha u_{xxx}) \\
&\quad + \left(tu_{xt} + \frac{1}{2}xu_{xx}\right)(-\beta h + \beta u + \alpha u_{xxx}) - \alpha\left(tu_{xxt} + \frac{1}{2}u_{xx} + \frac{1}{2}xu_{xxx}\right)u_{xxx} \\
&\quad + \left(h + th_t + \frac{1}{2}xh_x\right)\left(h + u^2 + \beta u_{xx}\right) - \beta\left(th_{xt} + \frac{3}{2}h_x + \frac{1}{2}xh_{xx}\right)u.
\end{aligned}$$

In what follows, we investigate the Lie–Bäcklund symmetries of system (2) and the resulting conservation laws.

Set

$$\begin{aligned}
X &= \eta^u(x,t,u,h,u_x,h_x,u_{xx},h_{xx},u_{xxx},h_{xxx})\partial_u \\
&\quad + \eta^h(x,t,u,h,u_x,h_x,u_{xx},h_{xx},u_{xxx},h_{xxx})\partial_h.
\end{aligned} \tag{22}$$

Substituting (22) into system (2), we infer the following Lie-Bäcklund symmetries by using the software Maple:

$$\begin{aligned}
X_1 &= h_x\partial_h + u_x\partial_u, \qquad X_2 = h_x\partial_h + (1 + tu_x)\partial_u, \\
X_3 &= (6u_x + uh_x + \alpha u_{xxx} - \beta h_{xx}\partial_h + (uu_x + \beta u_{xx} + h_x)\partial_u, \\
X_4 &= (2thu_x + 2tuh_x + 2\alpha tu_{xxx} - 2\beta th_{xx} + h_{xx} + 2h)\partial_h \\
&\quad + (2tuu_x + 2\beta tu_{xx} + 2th_x + u_{xx} + u)\partial_u, \\
X_5 &= \big(6huu_x + 3u^2h_x - 6\beta u_xh_x + 6\alpha uu_{xxx} - 6\beta uh_{xx} + 12\alpha u_xh_{xx} + 4\beta^2h_{xxx} \\
&\quad + 6hh_x + 4\alpha h_{xxx}\big)\partial_h + \big(3u^2u_x + 6\beta uu_{xx} + 6\beta u_x^2 + 4\beta^2u_{xxx} \\
&\quad + 6hu_x + 6uh_x + 4\alpha u_{xxx}\big)\partial_u, \\
X_6 &= \big(6tuhu_x + 3tu^2h_x - 6\beta th_xu_x + 6\alpha tuu_{xxx} - 6\beta tuh_{xx} + 12\alpha tu_xu_{xx} + 4\beta^2th_{xxx} \\
&\quad + 6thh_x + 2xhu_x + 2xuh_x + 4\alpha th_{xxx} + 2\alpha xu_{xxx} - 2\beta xh_{xx} + 4hu \\
&\quad - 10\beta h_x + 6\alpha u_{xx}\big)\partial_h + \big(3tu^2u_x + 6\beta tuu_{xx} + 6\beta th_x^2 + 4\beta^2tu_{xxx} + 6thu_x + 6tuh_x \\
&\quad + 4\alpha tu_{xxx} + 2xuu_x + 2\beta xu_{xx} + 2xh_x + u^2 + 4h\big)\partial_u.
\end{aligned}$$

Applying (10), we can deduce the components of the conserved density for system (2). For X_1, we have

$$
\begin{aligned}
C^1 &= u_x p + h_x q = u_x h + h_x u = D_x(uh), \quad \text{hence } \tilde{C}_1 = 0,\\
C^2 &= -2huu_x + \beta uh_x - \beta u_x^2 - \beta hu_{xx} + \beta uu_{xx} - \alpha uu_{xxx} - hh_x - u^2 h_x\\
&\quad - \beta u_x h_x + \beta uh_x h_{xx} + \beta uh_{xx}.
\end{aligned}
$$

For X_2, $W^u = \eta^u = 1 + tu_x$ and $W^h = \eta^h = h_x$; substituting into (16) and (17), we have that

$$
\begin{aligned}
C^1 &= W^u p + W^h q = (1 + tu_x)p + h_x q = (1 + tu_x)h + uh_x,\\
C^2 &= (\beta p_x - pu - qh - \alpha q_{xx})W^u + (\alpha q_x - \beta p)D_x\bigl(W^u\bigr) - \alpha q D_x^2\bigl(W^u\bigr)\\
&\quad - (p + qu + \beta q_x)W^h + \beta q D_x\bigl(W^h\bigr)\\
&= (\beta p_x - pu - qh - \alpha q_{xx})(1 + tu_x) + (\alpha q_x - \beta p)tu_{xx} - \alpha q tu_{xxx}\\
&\quad - (p + qu + \beta q_x)h_x + \beta q h_{xx}.
\end{aligned}
$$

Similarly, for X_3, we get that

$$
\begin{aligned}
C^1 &= W^u p + W^h q = p(uu_x + \beta u_{xx} + h_x) + q(6u_x + uh_x + \alpha u_{xxx} - \beta h_{xx}),\\
C^2 &= (\beta p_x - pu - qh - \alpha q_{xx})(uu_x + \beta u_{xx} + h_x) + (\alpha q_x - \beta p)D_x(uu_x + \beta u_{xx} + h_x)\\
&\quad - \alpha q D_x^2(uu_x + \beta u_{xx} + h_x) - (p + qu + \beta q_x)(6u_x + uh_x + \alpha u_{xxx} - \beta h_{xx})\\
&\quad + \beta q D_x(6u_x + uh_x + \alpha u_{xxx} - \beta h_{xx}).
\end{aligned}
$$

For X_4, we have

$$
\begin{aligned}
C^1 &= p(2tuu_x + 2\beta tu_{xx} + 2th_x + u_{xx} + u)\partial_u\\
&\quad + q(2thu_x + 2tuh_x + 2\alpha tu_{xxx} - 2\beta th_{xx} + h_{xx} + 2h),\\
C^2 &= (\beta p_x - pu - qh - \alpha q_{xx})(2tuu_x + 2\beta tu_{xx} + 2th_x + u_{xx} + u)\\
&\quad + (\alpha q_x - \beta p)D_x(2tuu_x + 2\beta tu_{xx} + 2th_x + u_{xx} + u)\\
&\quad - \alpha q D_x^2(2tuu_x + 2\beta tu_{xx} + 2th_x + u_{xx} + u)\\
&\quad - (p + qu + \beta q_x)(2thu_x + 2tuh_x + 2\alpha tu_{xxx} - 2\beta th_{xx} + h_{xx} + 2h)\\
&\quad + \beta q D_x(2thu_x + 2tuh_x + 2\alpha tu_{xxx} - 2\beta th_{xx} + h_{xx} + 2h).
\end{aligned}
$$

For X_5, we have

$$
\begin{aligned}
C^1 &= p\bigl(3u^2 u_x + 6\beta uu_{xx} + 6\beta u_x^2 + 4\beta^2 u_{xxx} + 6hu_x + 6uh_x + 4\alpha u_{xxx}\bigr)\\
&\quad + q\bigl(6huu_x + 3u^2 h_x - 6\beta u_x h_x + 6\alpha uu_{xxx} - 6\beta uh_{xx} + 12\alpha u_x h_{xx}\\
&\quad + 4\beta^2 h_{xxx} + 6hh_x + 4\alpha h_{xxx}\bigr),
\end{aligned}
$$

$$\begin{aligned}
C^2 &= (\beta p_x - pu - qh - \alpha q_{xx}) + (\alpha q_x - \beta p)D_x\big(3u^2u_x + 6\beta uu_{xx} + 6\beta u_x^2 + 4\beta^2 u_{xxx} \\
&\quad + 6hu_x + 6uh_x + 4\alpha u_{xxx}\big) - \alpha q D_x^2\big(3u^2u_x + 6\beta uu_{xx} + 6\beta u_x^2 + 4\beta^2 u_{xxx} + 6hu_x \\
&\quad + 6uh_x + 4\alpha u_{xxx}\big) - (p + qu + \beta q_x)\big(6huu_x + 3u^2h_x - 6\beta u_x h_x + 6\alpha uu_{xxx} - 6\beta uh_{xx} \\
&\quad + 12\alpha u_x h_{xx} + 4\beta^2 h_{xxx} + 6hh_x + 4\alpha h_{xxx}\big) + \beta q D_x\big(6huu_x + 3u^2h_x - 6\beta u_x h_x \\
&\quad + 6\alpha uu_{xxx} - 6\beta uh_{xx} + 12\alpha u_x h_{xx} + 4\beta^2 h_{xxx} + 6hh_x + 4\alpha h_{xxx}\big).
\end{aligned}$$

For X_6, we infer that

$$\begin{aligned}
C^1 &= p\big(3tu^2u_x + 6\beta tuu_{xx} + 6\beta th_x^2 + 4\beta^2 tu_{xxx} + 6thu_x + 6tuh_x + 4\alpha tu_{xxx} \\
&\quad + 2xuu_x + 2\beta xu_{xx} + 2xh_x + u^2 + 4h\big) + q\big(6tuhu_x + 3tu^2h_x - 6\beta th_x u_x \\
&\quad + 6\alpha tuu_{xxx} - 6\beta tuh_{xx} + 12\alpha tu_x u_{xx} + 4\beta^2 th_{xxx} + 6thh_x + 2xhu_x + 2xuh_x \\
&\quad + 4\alpha th_{xxx} + 2\alpha xu_{xxx} - 2\beta xh_{xx} + 4hu - 10\beta h_x + 6\alpha u_{xx}\big), \\
C^2 &= (\beta p_x - pu - qh - \alpha q_{xx})\big(3tu^2u_x + 6\beta tuu_{xx} + 6\beta th_x^2 + 4\beta^2 tu_{xxx} + 6thu_x + 6tuh_x \\
&\quad + 4\alpha tu_{xxx} + 2xuu_x + 2\beta xu_{xx} + 2xh_x + u^2 + 4h\big) + (\alpha q_x - \beta p)D_x\big(3tu^2u_x + 6\beta tuu_{xx} \\
&\quad + 6\beta th_x^2 + 4\beta^2 tu_{xxx} + 6thu_x + 6tuh_x + 4\alpha tu_{xxx} + 2xuu_x + 2\beta xu_{xx} + 2xh_x \\
&\quad + u^2 + 4h\big) - \alpha q D_x^2\big(3tu^2u_x + 6\beta tuu_{xx} + 6\beta th_x^2 + 4\beta^2 tu_{xxx} + 6thu_x + 6tuh_x \\
&\quad + 4\alpha tu_{xxx} + 2xuu_x + 2\beta xu_{xx} + 2xh_x + u^2 + 4h\big) - (p + qu + \beta q_x)\big(6tuhu_x + 3tu^2h_x \\
&\quad - 6\beta th_x u_x + 6\alpha tuu_{xxx} - 6\beta tuh_{xx} + 12\alpha tu_x u_{xx} + 4\beta^2 th_{xxx} + 6thh_x \\
&\quad + 2xhu_x + 2xuh_x + 4\alpha th_{xxx} + 2\alpha xu_{xxx} - 2\beta xh_{xx} + 4hu - 10\beta h_x + 6\alpha u_{xx}\big) \\
&\quad + \beta q D_x\big(6tuhu_x + 3tu^2h_x - 6\beta th_x u_x + 6\alpha tuu_{xxx} - 6\beta tuh_{xx} + 12\alpha tu_x u_{xx} \\
&\quad + 4\beta^2 th_{xxx} + 6thh_x + 2xhu_x + 2xuh_x + 4\alpha th_{xxx} + 2\alpha xu_{xxx} - 2\beta xh_{xx} \\
&\quad + 4hu - 10\beta h_x + 6\alpha u_{xx}\big),
\end{aligned}$$

where $p = h$ and $q = u$, which can substituted into the above conserved densities so that we obtain more explicit formulas for c^1 and C^2. Here we omit the computations.

3 Lie symmetry groups and similarity solutions

In this section, we apply the point symmetries (13) to consider the Lie symmetry groups and some similarity solutions to system (2). Denote the Lie symmetry groups generated by V_1, V_2, V_3, V_4 by g_1, g_2, g_3, g_4, respectively. It is easy to see that

$$\begin{aligned}
&g_1 : (x,t,u,h) \to (x, t+\epsilon, u, h), \\
&g_2 : (x,t,u,h) \to (x+\epsilon, t, u, h), \\
&g_3 : (x,t,u,h) \to \big(x, e^{\epsilon}t, u-\epsilon, h\big), \\
&g_4 : (x,t,u,h) \to \big(e^{\frac{1}{2}\epsilon}x, e^{\epsilon}t, e^{-\frac{1}{2}\epsilon}u, e^{-\epsilon}h\big).
\end{aligned}$$

If $u = f(x,t)$, $h = g(x,t)$ are solutions to e system (2), then we can get the following new solutions based on these symmetry groups:

$$
\begin{aligned}
&u^{(1)} = u(x,t-\epsilon) = f(x,t-\epsilon), \qquad h^{(1)} = g(x,t-\epsilon),\\
&u^{(2)} = f(x-\epsilon,t), \qquad h^{(2)} = g(x-\epsilon,t),\\
&u^{(3)} = f\left(x,e^{-\epsilon}t\right) - \epsilon, \qquad h^{(3)} = g\left(x,e^{-\epsilon}t\right),\\
&u^{(4)} = e^{-\frac{1}{2}\epsilon} f\left(e^{-\frac{1}{2}\epsilon}x,e^{-\epsilon}t\right), \qquad h^{(4)} = e^{-\epsilon} g\left(e^{-\frac{1}{2}\epsilon}x,e^{-\epsilon}t\right).
\end{aligned}
$$

Of course, we can go on getting some iteration solutions following the work [8]:

$$
\begin{aligned}
&\begin{cases} u^{(3,1)} = f(x,e^{-\epsilon}t) - 2\epsilon, \\ h^{(3,1)} = g(x,e^{-\epsilon}t), \end{cases}\\
&\begin{cases} u^{(4,1)} = e^{-\epsilon} f(e^{-\frac{1}{2}\epsilon}x,e^{-\epsilon}t), \\ h^{(4,1)} = e^{-2\epsilon} g(e^{-\frac{1}{2}\epsilon}x,e^{-\epsilon}t), \end{cases}\\
&\ldots,\\
&\begin{cases} u^{(4,n)} = e^{-\frac{2n-1}{n}\epsilon} f - \frac{1}{2}\epsilon x,e^{-\epsilon}t), \\ h^{(4,n)} = e^{-n\epsilon} g(e^{-\frac{1}{2}\epsilon}x,e^{-\epsilon}t), \end{cases}
\end{aligned}
$$

where $n \in N^{+}$.

For the transformation groups g_1 and g_2, the invariant solutions are traveling wave solutions. Indeed, set

$$u = U(\xi), \qquad h = H(\xi), \qquad \xi = x - ct. \tag{23}$$

Substituting (23) into system (2), we have

$$
\begin{cases} -cU' = UU' + H' + \beta U'', \\ -cH' = (UH)' + \alpha U''' - \beta H''. \end{cases}
$$

Integrating gives rise to

$$
\begin{cases} -cU = \frac{1}{2}U^2 + H + \beta U', \\ -cH = UH + \alpha U'' - \beta H'. \end{cases} \tag{24}
$$

From (24) we find that

$$\left(\alpha + \beta^2\right)U'' - \frac{1}{2}U^3 - \frac{3}{2}cU^2 - c^2 U = 0. \tag{25}$$

Let $U' = y(\xi)$. Then (25) turns to

$$\left(U'\right)^2 = \frac{1}{\alpha + \beta^2}\left(\frac{1}{4}U^u + cU^3 + c^2 U^2\right). \tag{26}$$

Suppose $U' = aU + U^2$. Then inserting this into (26), we have that

$$\alpha^2 + \beta^2 = \frac{1}{4}, \qquad a = 2 \pm c.$$

Thus we get

$$U' = 2 \pm cU + U^2 =: 2\epsilon cU + U^2. \tag{27}$$

Solving (27) yields

$$U = \frac{2\epsilon c\bar{c}e^{2\epsilon c\xi}}{1 - \bar{c}e^{2\epsilon c\xi}}, \tag{28}$$

and hence

$$H = -\frac{2\epsilon c\bar{c}(\epsilon c - 2\epsilon c\beta)e^{2\epsilon c\xi}}{1 - \bar{c}e^{2\epsilon c\xi}} - \left(\frac{1}{2} + \beta\right)\frac{4c^2\bar{c}^2 e^{4\epsilon c\xi}}{(1 - \bar{c}e^{2\epsilon\xi})^2}, \tag{29}$$

where $\bar{c}$ is an integral constant, which does not vanish. Again applying (23) and (28)–(29), we have

$$\begin{cases} u(x,t) = \frac{2\epsilon c\bar{c}e^{2\epsilon c(x-ct)}}{1-\bar{c}e^{2\epsilon c(x-ct)}}, \\ h(x,t) = -\frac{2\epsilon c\bar{c}(\epsilon c-2\epsilon c\beta)e^{2\epsilon c(x-ct)}}{1-\bar{c}e^{2\epsilon c(x-ct)}} - (\frac{1}{2} + \beta)\frac{4c^2\bar{c}^2 e^{4\epsilon c(x-ct)}}{(1-\bar{c}e^{2\epsilon(x-ct)})^2}. \end{cases}$$

The characteristic equation of the vector field V_3 presents

$$\frac{dt}{0} = \frac{dx}{t} = \frac{du}{-1},$$

which gives

$$w = \xi x + u, \qquad x\xi = t.$$

The resulting group-invariant solution reads

$$u = f(t) - tx, \qquad h = g(t). \tag{30}$$

Substituting (30) into system (2), we infer

$$\begin{cases} \frac{df}{dt} + tf = x + t^2 x, \\ \frac{dg}{dt} = -tg(t), \end{cases}$$

which has the following set of solutions:s

$$\begin{cases} f(t) = \exp(-\frac{1}{2}t^2)[x\int^t \exp(\frac{1}{2}t^2)\,dt + \frac{x}{3}t^3 e^{\frac{t^2}{2}} - \frac{2}{3}xe^{\frac{t^2}{2}} + \tilde{c}], \\ g(t) = c_0 \exp(-\frac{1}{2}t^2), \end{cases} \tag{31}$$

where $\tilde{c}$, c_0 are constants. Substituting (31) into (30), we obtain the similarity solutions of system (2):

$$\begin{cases} u(x,t) = xe^{-\frac{1}{2}t^2} \int^t e^{\frac{1}{2}t^2}\,dt + \frac{1}{3}x(t^3-2) + \tilde{c}e^{-\frac{1}{2}t^2} - tx, \\ h(x,t) = c_0 e^{-\frac{1}{2}t^2}. \end{cases}$$

Similarly, for g_4, the characteristic equation reads

$$\frac{dt}{t} = \frac{dx}{\frac{1}{2}x} = \frac{du}{-\frac{1}{2}u} = \frac{dh}{-h},$$

from which we get the invariants

$$\xi = x^2 t^{-1}, \qquad u = x^{-1} f(\xi), \qquad h = x^{-\frac{1}{2}} g(\xi), \tag{32}$$

where $f(\xi)$, $g(\xi)$ are arbitrary invariant functions with respect to the variable ξ. Inserting (32) into the system, we get the following ordinary differential equations with variable coefficients:

$$\begin{cases} -f'(\xi) = (2\beta-1)\xi^{-2}f^2(\xi) + 2(1-\beta)\xi^{-1}f'(\xi) - 2\xi^{-2}g(\xi) \\ \qquad\qquad + 2\xi^{-1}g'(\xi) + 2\beta f''(\xi), \\ -g'(\xi) = -3\xi^2 f(\xi)g(\xi) + 2\xi^{-1}f'(\xi)g(\xi) + 2\xi^{-1}f(\xi)g'(\xi) + 6\alpha\xi^{-2}f(\xi) \\ \qquad\qquad - 2\alpha\xi^{-1}f'(\xi) - 2\alpha f''(\xi) + 4\alpha\xi f'''(\xi) - 6\beta\xi^{-2}g(\xi) \\ \qquad\qquad + 6\beta\xi^{-1}g'(\xi) - 4\beta g''(\xi). \end{cases} \tag{33}$$

Now we look for the formal series solutions to (33), so we assume that

$$f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \qquad g(\xi) = \sum_{n=0}^{\infty} b_n \xi^n. \tag{34}$$

Substituting (34) into system (2) and comparing the coefficients of both sides in the system (33), one infers that

$$2b_0 = (2\beta-1)a_0^2,$$

$$(4\beta-2)a_0a_1 + (2-2\beta)a_1 = 0,$$

$$(4\beta-2)a_0a_2 + 4a_2 + 2b_2 + (2\beta-1)a_1^2 + a_1 = 0,$$

$$-2a_2 + (4\beta-2)a_0a_3 + (4\beta-2)a_1a_2 + (6\beta+6)a_3 + 4b_3 + 2a_2 = 0,$$

$$\ldots,$$

$$-\sum_{k=0}^{4} a_k a_{4-k} + 8(4-\beta)a_4 + 8b_4 + 2\beta \sum_{k=0}^{4} a_k a_{4-k} = 0,$$

$$(1-n)a_{n-1} = 2nb_n + (2\beta-1)\sum_{k=0}^{n} a_k a_{n-k} + \bigl(2n - 4\beta n + 2\beta n^2\bigr)a_n, \quad n \geq 5,$$

$$a_0b_0 - 2\alpha a_0 + 2\beta b_0 = 0,$$

$$
\begin{aligned}
&a_0b_1 + a_1b_0 - 4\alpha a_1 = 0,\\
&a_2b_0 - a_1b_1 + a_0b_2 + 2a_0b_1 - 2\alpha a_2 - 14\beta b_2 + b_1 + 6\beta b_1 = 0,\\
&-3a_3b_0 + 2a_1b_2 + 3a_0b_3 + 2b_2 + 12\alpha a_3 + 12\beta b_3 = 0,\\
&\ldots,\\
&-3a_3 = -3\sum_{k=0}^{4} a_kb_{4-k} + 2\sum_{k=0}^{3}(3-k)a_{3-k} + 2\sum_{k=0}^{4}(4-k)a_{4-k} + 2\sum_{k=0}^{3}(3-k)a_kb_{3-k}\\
&\qquad + 2\sum_{k=0}^{4}(4-k)a_kb_{4-k} - 8\alpha a_4 - 24a_4 - 30\beta b_4,\\
&-4a_4 = -3\sum_{k=0}^{5} a_kb_{5-k} + 2\sum_{k=0}^{5}(5-k)a_{5-k}b_k + 2\sum_{k=0}^{5}(5-k)a_kb_{5-k} + 190\alpha a_5 - 56\beta b_5,\\
&\ldots,\\
&(1-n)a_{n-1} = -3\sum_{k=0}^{n} a_kb_{n-k} + 2\sum_{k=0}^{n}(n-k)a_{n-k}b_k + 2\sum_{k=0}^{n}(n-k)a_kb_{n-k} + 6\alpha a_{n-2}\\
&\qquad + \bigl[-2\alpha n - 2\alpha n(n-1) + 4\alpha n(n-1)(n-2)\bigr]a_n\\
&\qquad + \bigl[6\beta n - 6\beta - 4\beta n(n-1)\bigr]b_n, \quad n \geq 6.
\end{aligned}
$$

When $a_1 \neq 0$, there exists a constraint between α and β:

$$5\beta^2 - 8\alpha\beta - 4\beta + 4\alpha + 1 = 0.$$

In terms of the above relations among a_i, b_i $(i = 0, 1, 2, \ldots, n)$, we can write out the formal series solutions where a_1 is a free parameter. As for the convergence of the series, we skip the discussion. However, we can follow the way presented in [8] to proceed.

4 A new scheme for seeking conservation laws of system (2)

In this section, we adopt the approach given by Göktas and Hereman to investigate the conserved densities and the fluxes of the system for the given rank of ever term in the equations. We rewrite the system (2) as follows

$$
\begin{cases}
u_{1,t} = u_1u_{1,x} + u_{2,x} + \beta u_{1,2x},\\
u_{2,t} = (u_1u_2)_x + \alpha u_{1,3x} - \beta u_{2,2x},
\end{cases}
\tag{35}
$$

where $u_{i,nx}$ denotes the nth derivative with respect to x for the variables u_i $(i = 1, 2)$. We denote the weight of the variables u_i and derivative ∂_t by $w(u_i)$ and $w(\partial_t)$, respectively. We easily find that

$$
\begin{aligned}
&r_{1,1} = w(u_1) + w(\partial_t), \qquad r_{1,2} = 1 + w(u_1) + w(u_2),\\
&r_{1,3} = 3 + w(u_1), \qquad r_{1,4} = 2 + w(u_2),\\
&r_{2,1} = w(u_2) + w(\partial_t), \qquad r_{2,2} = 1 + w(u_1) + w(u_2),\\
&r_{2,3} = 3 + w(u_1), \qquad r_{2,4} = 2 + w(u_2).
\end{aligned}
$$

Solving

$$\begin{cases} r_{1,1} = r_{1,2} = r_{1,3} = r_{1,4}, \\ r_{2,1} = r_{2,2} = r_{2,3} = r_{2,4}, \end{cases}$$

we get

$$w(u_1) = 1, \qquad w(u_2) = 2, \qquad w(\partial_t) = 2.$$

List of the variables in system (2):

$$\mathcal{V} = \{v_1, v_2\}, \qquad v_1 = u_2, \qquad v_2 = u_1.$$

In what follows, we consider the case where rank = 4 and

$$\mathcal{B}_0 = \{(1;0)\}.$$

For $q = 1$ and $m = 0$, a direct calculation gives

$$T_{1,s} = v_1^s = u_2^s, \qquad W_{1,s} = sw(v_1) = 2s, \qquad p_{1,0} = \left[\!\left[\frac{4-0}{w(v_1)} \right]\!\right] = 2.$$

Hence $s = 0, 1, 2$, and

$$\mathcal{B}_1 = B_{1,0} = \{(1;0), (u_2;2), (u_2^2;4)\}.$$

For $q = 2$ and $m = 0$, we have

$$T_{2,s} = u_1^s, \qquad W_{2,s} = W_{1,0} + sw(v_2) = s, \qquad p_{2,0} = \left[\!\left[\frac{4-0}{1} \right]\!\right] = 4.$$

Hence $s = 0, 1, 2, 3, 4$, and

$$B_{2,0} = \{(1;0), (u_1;1), (u_1^2;2), (u_1^3;3), (u_1^4;4)\}.$$

For $q = 2$ and $m = 1$, we have that

$$T_{2,s} = u_2 u_1^s, \qquad W_{2,s} = 2 + s, \qquad p_{2,1} = \left[\!\left[\frac{4-2}{1} \right]\!\right] = 2.$$

Thus $s = 0, 1, 2$. It follows that

$$B_{2,1} = \{(u_2;2), (u_1 u_2;3), (u_2 u_1^2;4)\}.$$

For $q = 2$ and $m = 2$, we find that

$$T_{2,s} = u_2 u_1^s, \qquad W_{2,s} = 4 + s, \qquad p_{2,2} = \left[\!\left[\frac{4-W_{1,2}}{w(v_2)} \right]\!\right] = \left[\!\left[\frac{4-4}{1} \right]\!\right] = 0.$$

Hence $s = 0$. Thus we get

$$\begin{aligned} B_{2,2} &= \{(u_2^2;4)\}, \\ \mathcal{B}_2 &= B_{2,0} \cup B_{2,1} \cup B_{2,2} \\ &= \{(1;0),(u_1;1),(u_1^2;2),(u_1^3;3),(u_1^4;4),(u_2;2),(u_1u_2;3),(u_2i_1^2;4),(u_2^2;4)\}. \end{aligned}$$

Computation of l for each pair of $\mathcal{B}_2$ reads

$$l = 4,3,2,1,0,2,1,0,0.$$

Gathering the terms by applying the number l of partial derivatives with respect to x, we get that

$$\mathcal{H} = \{0, u_{1,3x}, u_{1,x}^2, u_1u_{1,2x}, u_1^2u_{1,x}, u_1^4, u_{2,2x}, u_{1,x}u_2, u_1u_{2,x}, u_2u_1^2, u_2^2\}.$$

Removing from $\mathcal{H}$ the constant terms and the terms that can be written as an x-derivative, or an x-derivative up to terms retained earlier in the set $\mathcal{H}$, we have that

$$\mathcal{I} = \{u_{1,x}^2, u_1^4, u_1u_{2,x}, u_2u_1^2, u_2^2\}.$$

Let

$$\rho = c_1u_{1,x}^2 + c_2u_1^4 + c_3u_1u_{2,x} + c_4u_2u_1^2 + c_5u_2^2,$$

where c_i $(i = 1,2,3,4)$ are constants to be determined. Then we have

$$\begin{aligned} D_t(\rho) &= 2c_1u_{1,x}u_{1,xt} + 4c_2u_1^3u_{1,t} + c_3(u_{1,t}u_{2,x} + u_1u_{2,xt}) \\ &\quad + c_4u_{2,t}u_1^2 + 2c_4u_1u_2u_{1,t} + 2c_5u_2u_{2,t}. \end{aligned} \tag{36}$$

Substituting (35) into (36) to cancel the terms with t-derivatives, we get

$$\begin{aligned} D_t(\rho)|_{(35)} &= 2c_1u_{1,x}(u_{1,x}^2 + u_1u_{1,2x} + u_{2,2x} + \beta u_{1,3x}) + 4c_2u_1^3(u_1u_{1,x} + u_{2,x} + \beta u_{1,2x}) \\ &\quad + c_3[u_{2,x}(u_1u_{1,x} + u_{2,x} + \beta u_{1,2x}) + u_1(u_2u_{1,2x} + 2u_{1,x}u_{2,x} \\ &\quad + u_1u_{2,2x} + \alpha u_{1,3x} - \beta u_{2,3c})] + 2c_4u_1u_2(u_1u_{1,x} + u_{2,x} + \beta u_{1,2x}) \\ &\quad + c_4u_1^2(u_{1,x}u_2 + u_1u_{2,x} + \alpha u_{1,4x} - \beta u_{2,3x}) \\ &\quad + 2c_5u_2(u_{1,x}u_2 + u_1u_{2,x} + \alpha u_{1,3x} - \beta u_{2,2x}). \end{aligned}$$

Acting on the variational derivative to $D_t(\rho)|_{(35)}$ and comparing the coefficients of the same terms, we get the following relations:

$$\begin{aligned} S = \{&\alpha c_4 + c_1 = 0, 2\beta c_4 + c_3 = 0, 2\beta c_4 + c_3 = 0, 2\alpha c_3 - 4\beta c_1 = 0, \alpha c_5 + c_1 = 0, \\ &2\beta c_5 + c_3 = 0, c_4 - c_5 = 0\}. \end{aligned}$$

Solving S gives that

$$c_1 = -\alpha c_4, \qquad c_3 = -2\beta c_4, \qquad c_5 = c_4, \tag{37}$$

where c_4 is a free parameter, and c_2 can be an arbitrary constant.

Suppose $c_4 = 1$ and $c_2 = 0$. Then $c_1 = -\alpha$, $c_3 = -2\beta$, $c_5 = 1$. Thus, we have the conserved density:

$$\rho = -\alpha u - 1, x^2 - 2\beta u_1 u_{2,x} + u_2 u_1^2 + u_2^2.$$

The resulting flux of system (2) when rank = 4 is given by

$$\begin{aligned} J &= \alpha\left(2u_1 u_{1,x}^2 - u_1^2 u_{1,2x}\right) + 2\alpha(u_{1,x}u_{2,x} - u_2 u_{1,2x}) + 2\alpha\beta u_1 u_{1,3x} \\ &\quad - 2\beta^2 u_1 u_{2,2x} + 2\beta u_2 u_{2,x} + 3\beta u_1^2 u_{2,x} + u_1^3 u_2 + 2u_2^2 u_1. \end{aligned}$$

If $c_1 = c_3 = c_4 = c_5 = 0$ and $c_2 \neq 0$, then the conserved density is of the form

$$\rho = c_2 u_1^4,$$

whereas

$$\begin{aligned} D_t(\rho) &= 4c_2 u_1^3(u_1 u_{1,x} + u_{2,x} + \beta u_{1,2x}) \\ &= D_x\left[\frac{1}{5}c_2 u_1^5 + 4c_2 u_1^3 u_{1,x} + 4c_2\beta u_1^3 u_2 - 12c_2 D_x^{-1}\left(u_1^2 u_{1,x}^2\right) - 12c_2\beta D_x^{-1}\left(u_1^2 u_2 u_{1,x}\right)\right]. \end{aligned}$$

Hence the flux has nonlocal differential form

$$J = c_2\left[-\frac{1}{5}u_1^5 - 4u_1^3 u_{1,x} - 4\beta u_1^3 u_2 + 12D_x^{-1}\left(u_1^2 u_{1,x}^2\right) + 12\beta D_x^{-1}\left(u_1^2 u_2 u_{1,x}\right)\right].$$

When rank $= 5, 6, \ldots,$ we can obtain the resulting conserved densities and fluxes of system (2), and we omit the complicated computations.

5 Special solutions of the stationary system (2)

An important application of the conservation laws of evolution equations is the study of nonvariant solutions of symmetry groups. In this section, we want to apply the conservation laws of system (2) to investigating some noninvariant solutions of the symmetries. For simplicity, we only choose simpler conservation laws and only consider the solutions of the stationary system (2). System (2) can be written as

$$u_t = \left(\frac{1}{2}u^2 + h + \beta u_x\right)_x, \qquad h_t = (uh + \alpha u_{xx} - \beta h_x)_x, \tag{38}$$

where the conserved densities are u and h, and the corresponding fluxes are $\frac{1}{2}u^2 + h + \beta u_x$ and $uh + \alpha u_{xx} - \beta h_x$, which are the simplest conservation laws. We further want to use them to deduce some stationary solutions of system (2). Taking

$$u = g(x), \qquad \frac{1}{2}u^2 + h + \beta u_x = f(t),$$

and substituting into the second conserved equation in (38), we have that

$$f'(t) = g'(x)f(t) - \frac{3}{2}g^2g'(x) - 2\beta g'(x)^2 - 2\beta gg'' + \alpha g''' + \beta^2 gg'''. \tag{39}$$

Assume that $f(t) = 0$ and then integrate (39):

$$\left(\beta^2 g + \alpha\right)g'' - \frac{1}{2}\beta^2\left(g'\right)^2 - 2\beta gg' - \frac{1}{2}g^3 = 0. \tag{40}$$

Case 1: When $\alpha > 0$, $\beta = 0$, Eq. (39) reduces to

$$\alpha g'' - \frac{1}{2}g^3 = 0,$$

which has a special solution

$$g = -\frac{2\sqrt{\alpha}}{\epsilon x}, \quad \epsilon = \pm 1.$$

Thus we obtain a set of special solutions to system (2):

$$u = -\frac{2\sqrt{\alpha}}{\epsilon x}, \qquad h = -\frac{2\alpha}{x^2}. \tag{41}$$

Case 2: When $\beta \neq 0$, assume a formal solution of (40)

$$g = \frac{A}{B + Cx}. \tag{42}$$

Then we get

$$g' = \frac{AC}{(B + Cx)^2}, \qquad g'' = \frac{2A^2C}{(B + Cx)^3}. \tag{43}$$

Inserting (42) and (43) into (40) and comparing the coefficients at the powers of x, we infer that

$$A = -\frac{\alpha}{\beta}C, \quad \beta = 4. \tag{44}$$

Taking $B = 1$, we have the special solution to system (2)

$$\begin{cases} u = -\frac{\alpha C}{4(1+Cx)}, \\ h = -\frac{\alpha^2 C^2}{32(1+Cx)^2} + \frac{\alpha C}{(1+Cx)^2}, \end{cases} \tag{45}$$

where $C \neq 0$ is a parameter.

In the case $f(t) = 0$, we find that $u_t = h_t = 0$. Hence the system becomes the following ordinary differential equations:

$$\begin{cases} D_x(\frac{1}{2}u^2 + h + \beta u_x) = 0, \\ D_x(hu + \alpha u_{xx} - \beta h_x) = 0, \end{cases}$$

which can be written as

$$\frac{1}{2}u^2 + h + \beta u_x = c_1, \qquad hu + \alpha u_{xx} - \beta h_x = c_2, \tag{46}$$

where c_1, c_2 are integral constants. It is easy to rewrite (46) as the following differential equation with respect to the variable u:

$$\frac{1}{2}u^3 - (\alpha + \beta^2)u_{xx} = c_1 u - c_2. \tag{47}$$

When $c_1 = c_2 = 0$, solving (47) yields

$$u = -\frac{2\sqrt{\alpha + \beta^2}}{\epsilon x}. \tag{48}$$

Substituting (48) into the first equation in (46), we get

$$h = -\beta u_x - \frac{1}{2}u^2 = \frac{2\sqrt{\alpha + \beta^2}}{\epsilon x^2} - \frac{2(\alpha + \beta^2)}{x^2}. \tag{49}$$

When $c_1 \neq 0$ and $c_2 \neq 0$, (47) becomes

$$u_{xx} + \frac{1}{2(\alpha + \beta^2)}u^3 - \frac{c_1 u}{\alpha + \beta^2} + \frac{c_2}{\alpha + \beta^2} = 0. \tag{50}$$

Let $u_x = y$. Then (50) can be written as

$$y\frac{dy}{du} + \frac{u^3}{2(\alpha + \beta^2)} - \frac{c_1 u}{\alpha + \beta^2} + \frac{c_2}{\alpha + \beta^2} = 0. \tag{51}$$

Integrating (5) with respect to u, we get

$$\frac{1}{2}y^2 + \frac{1}{8(\alpha + \beta^2)}u^4 - \frac{c_1 u^2}{2(\alpha + \beta^2)} + \frac{c_2 u}{\alpha + \beta^2} + c_0 = 0, \tag{52}$$

where c_0 is an integral constant. Thus (52) can be written as

$$u' = \sqrt{\frac{c_1 u^2}{\alpha + \beta^2} - \frac{1}{4(\alpha + \beta^2)}u^4 - \frac{2c_2 u}{\alpha + \beta^2} - 2c_0}. \tag{53}$$

Fan [26] studied the solutions to the formal ODE (53) taking different parameters. Now we follow his method to give some exact solutions to (53).

(i) When $c_0 = 0$, (53) has the following solutions:

$$u = \sqrt{4c_1}\,\mathrm{sech}\left(\sqrt{\frac{c_1}{\alpha + \beta^2}}x\right), \quad c_1 > 0, \alpha + \beta_2 > 0,$$

$$u = \sqrt{4c_1}\,\sec\left(\sqrt{-\frac{c_1}{\alpha + \beta^2}}x\right), \quad c_1 > 0, \alpha + \beta_2 < 0,$$

$$u = -\frac{1}{\sqrt{-4(\alpha + \beta^2)}x}, \quad c_1 = 0, \alpha + \beta^2 < 0.$$

(ii) Equation (53) still has the following three Jacobi elliptic function solutions: when $\frac{c_1}{\alpha+\beta^2} > 0$, $c_0 = \frac{2c_1^2 m^2(m^2-1)}{(\alpha+\beta^2)(2m^2-1)^2}$,

$$u = \sqrt{\frac{4c_1 m^2}{2m^2-1}}\,\mathrm{cn}\left(\sqrt{\frac{c_1}{(\alpha+\beta^2)(2m^2-1)}}x\right);$$

when $\frac{c_1}{\alpha+\beta^2} < 0$, $c_0 = \frac{2c_1^2 m^2}{(\alpha+\beta^2)(m^2+1)}$,

$$u = \sqrt{\frac{4c_1 m^2}{m^2+1}}\,\mathrm{sn}\left(\sqrt{\frac{-c_1}{(\alpha+\beta^2)(m^2+1)}}x\right);$$

when $\frac{c_1}{\alpha+\beta^2} > 0$, $c_0 = \frac{2c_1^2(1-m^2)}{(\alpha+\beta^2)(2-m^2)^2}$,

$$u = \sqrt{\frac{4c_1}{2-m^2}}\,\mathrm{dn}\left(\sqrt{\frac{c_1}{(\alpha+\beta^2)(2-m^2)}}x\right),$$

where m denotes the module of Jacobi elliptic functions. Substituting all these u into (46), we can obtain the resulting h.

From the above discussion we find that we indeed obtain some new special solutions of system (2). However, there are two questions to further consider. One is that when $f(t) \neq 0$, can we obtain solutions dependent on the variable t? Of course, we can. Because we may investigate the traveling-wave solutions of system (2) by setting $\xi = x - ct$, which can transform system (2) to the ODEs with respect to the variable ξ. As for this, we do not further discuss them. Another one is that when $f(t) = 0$, can we obtain solutions only dependent on the variable x and not on the variable t? We do not think so. In fact, if we replace system (2) with the system

$$\begin{cases} u_t = (\frac{1}{2}u^2 + h + \beta h_x)_x, \\ h_t = (uh + \alpha u_{xx} - \beta h_x)_x, \end{cases} \tag{54}$$

then we can get a set of solutions dependent on the variable t. In fact, taking $u_t = g(x)$, $\frac{1}{2}u^2 + h + \beta h_x = 0$, we have

$$h_x + \frac{1}{\beta}h = -\frac{1}{2\beta}g^2(x).$$

Substituting this into the second equation in (54) gives rise to

$$G_x + \frac{1}{\beta}G = 0,$$

where $G(x,t) = hg(x) + \alpha g_{xx} - \beta h_x$, which has the solution

$$G = \sigma(t)e^{\frac{-1}{\beta}x},$$

from which we have

$$hg(x) + \alpha g_{xx} - \beta h_x = \sigma(t)e^{\frac{-1}{\beta}x},$$

where $\sigma(t)$ is a function to be determined. A direct verification indicates that

$$g(x) = -1, \qquad h = -\frac{1}{2} + e^{(-\frac{1}{\beta}x + \frac{1}{2}t)}$$

is a set of solutions of (54). We see that the function h is dependent on the variable t.

Acknowledgements
Not applicable.

Funding
This work is supported by the Fundamental Research Funds for the Central University (No. 2017XKZD11).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The idea to deduce point symmetry and Lie–Bäcklund transformations of the system similarity solutions was finished by YZ, and conservation laws were done by NB and YZ. Some exact solutions of the stationary system (2) belong to HG. All authors read and approved the final manuscript.

Author details
[1]College of Mathematics, China University of Mining and Technology, Xuzhou, P.R. China. [2]School of Computer and Technology, China University of Mining and Technology, Xuzhou, P.R. China. [3]Maths Department, College of Information Engineering, Liaoning University of Traditional Chinese Medicine, Shenyang, P.R. China.

Publisher's Note

Received: 7 June 2019 Accepted: 9 October 2019 Published online: 16 October 2019

References
1. Broer, L.J.F.: Approximate equations for long water waves. Appl. Sci. Res. **31**, 377–395 (1975)
2. Kupershmidt, B.A.: Mathematics of dispersive water waves. Commun. Math. Phys. **99**, 51–73 (1985)
3. Kaup, D.J.: A higher-order water-wave equation and the method for solving it. Prog. Theor. Phys. **54**, 396–408 (1975)
4. Matveev, V.B., Yavor, M.I.: Solutions presque périodiques et a *N*-solitons de l'équation hydrodynamique non linéaire de Kaup. Ann. Inst. Henri Poincaré A, Phys. Théor. **31**(1), 25–41 (1979)
5. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1993)
6. Zhao, Z.L., Han, B.: Residual symmetry, Backlund transformation and CRE solvability of a (2 + 1)-dimensional nonlinear system. Nonlinear Dyn. **94**(1), 461–474 (2018)
7. Zhao, Z.L., Han, B., Zhao, Z., He, L.: Multiple lump solutions of the (3 + 1)-dimensional potential Yu–Toda–Sasa–Fukuyama equation. Appl. Math. Lett. **95**, 114–121 (2019)
8. Zhang, Y.F., et al.: Symmetry properties and explicit solutions of some nonlinear differential and fractional equations. Appl. Math. Comput. **337**, 408–418 (2018)
9. Liu, H.Z., Li, J.B., Zhang, Q.X.: Lie symmetry analysis and exact explicit solutions for general Burger's equation. J. Comput. Appl. Math. **228**, 1–9 (2009)
10. Zhang, X.Z., Zhang, Y.F.: Some similarity solutions and numerical solutions to the time-fractional Burgers system. Symmetry **11**, 112 (2019). https://doi.org/10.3390/sym11010112
11. Lou, S.Y., Huang, F.: Alice–Bob physics: coherent solutions of nonlocal KdV systems. Sci. Rep. **7**, 1–13 (2016)
12. Lou, S.Y., Yao, R.X.: Invariant functions, symmetries and primary branch solutions of first order autonomous systems. Commun. Theor. Phys. **68**, 21–28 (2017)
13. Ma, W.X.: Conservation laws of discrete evolution equations by symmetries and adjoint symmetries. Symmetry **7**, 714–725 (2015)
14. Qu, C.Z., Ji, L.N.: Invariant subspaces and conditional Lie–Backlund symmetries of inhomogeneous nonlinear diffusion equations. Sci. China Math. **56**, 2187–2203 (2013)
15. Ji, L.N., Qu, C.Z.: Conditional Lie–Backlund symmetries and invariant subspaces to nonlinear diffusion equations with convection and source. Stud. Appl. Math. **131**, 266–301 (2013)
16. Ma, W.X.: Conservation laws by symmetries and adjoint symmetries. Discrete Contin. Dyn. Syst., Ser. S **11**, 707–721 (2018)
17. Ma, W.X.: A generating scheme for conservation laws of discrete zero curvature equations and its application. Comput. Math. Appl. (to be published)
18. Ma, W.X., Li, J., Khalique, C.M.: A study on lump solutions to a generalized Hirota–Satsuma–Ito equation in (2 + 1)-dimensions. Complexity **2018**, Artical ID 9059858 (2018)
19. Ma, W.X.: Interaction solutions to Hirota–Satsuma–Ito equation in (2 + 1)-dimensions. Front. Math. China **14**, 619–929 (2019)
20. Ma, W.X.: A search for lump solutions to a combined fourth-order nonlinear PDE in (2 + 1)-dimensions. J. Appl. Anal. Comput. **9**(4), 1319–1332 (2019)

21. Ma, W.X., Zhou, Y.: Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. J. Differ. Equ. **264**, 2633–2659 (2018)
22. Zhang, Y., Ma, W.-X.: A study on rational solutions to a KP-like equation. Z. Naturforsch. A **70**(4), 263–268 (2015)
23. Ma, W.X.: A refined invariant subspace method and applications to evolution equations. Sci. China **55**(9), 1769–1778 (2012)
24. Ibragimov, N.H., Avdonina, E.D.: Nonlinear self-adjointness, conservation laws, and the construction of solutions of partial differential equations using conservation laws. Russ. Math. Surv. **68**, 889–921 (2013)
25. Göktas, Ü., Hereman, W.: Symbolic computation of conserved densitie of systems of nonlinear evolution equations. J. Symb. Comput. **11**, 1–31 (2008)
26. Fan, E.G.: Integrable Systems and Computer Algebra. Science Press, Beijing (2004) (in Chinese)