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Traveling wave solutions for a neutral reaction–diffusion equation with non-monotone reaction

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Abstract

In the present paper, we firstly improve the results on traveling wave solution that were established in (Liu and Weng in J. Differ. Equ. 258:3688–3741, 2015) for a neutral reaction–diffusion equation with quasi-monotone reaction. Secondly, by constructing two auxiliary equations and using Schauder's Fixed Point Theorem, we further establish the existence and the asymptotic properties of the traveling wave solution for the equation with non-monotone reaction. Two examples are also given as the application of our results.

MSC: 35K57; 35Q99; 34K40

Keywords: Neutral reaction–diffusion equation; Traveling wave solution; Schauder's Fixed Point Theorem; Auxiliary equations

1 Introduction

In the present paper, we consider the following neutral reaction–diffusion equation

$$\frac{\partial}{\partial t}L(u)(t,x) = D\frac{\partial^2}{\partial x^2}L(u)(t,x) + F\big(L(u)(t,x), u(t,x), u(t-r,x)\big), \quad t>0, x\in\mathbb{R}, \tag{1}$$

where $L(u)(t,x) = u(t,x) - bu(t-r,x)$, and $D>0$, $r\geq 0$, $0\leq b<1$ are constants; $F(0,0,0) = F((1-b)K,K,K) = 0$ for some $K>0$.

When $b=0$, then Eq. (1) reduces to the delayed reaction–diffusion equation

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) + F\big(u(t,x), u(t-r,x)\big). \tag{2}$$

A special case of such equations reads as

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) - g\big(u(t,x)\big) + h\big(u(t,x)\big)\int_{\mathbb{R}} f\big(u(t-r,y)\big)J(x-y)\,dy, \tag{3}$$

or its local version (taking formally $J(x)=\delta(x)$, the Dirac delta function)

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) - g\big(u(t,x)\big) + h\big(u(t,x)\big)f\big(u(t-r,x)\big), \tag{4}$$

which has been wildly investigated in the literature (see [22, 26, 27, 31] and references therein).

As a prototype of such equations, we mention the evolution model of the adult population of a single species with two age classes and moving around in an unbounded 1-dimensional spatial domain as follows:

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) - du(t,x) + \epsilon \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{(x-y)^2}{4\alpha}} b\bigl(u(t-r,y)\bigr)\,dy, \tag{5}$$

which was derived by So, Wu and Zou [25], where $D > 0$ and $d > 0$ denote respectively the diffusion and death rates of the adult population, $r \ge 0$ is the maturation time for the species, $b(\cdot)$ is the birth function, and $\epsilon > 0$ and $\alpha \ge 0$ reflect the impact of the death and the dispersal rates of the immature on the matured population, respectively.

There are also other prototypes of Eq. (2) or Eq. (3) which have been studied intensively, such as the model

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) - \beta u^2(t,x) + \alpha e^{-\gamma r} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\, dr}} e^{-\frac{(x-y)^2}{4dr}} u(t-r,y)\,dy, \tag{6}$$

proposed by Gourley and Kuang [8] to describe the evolution of the mature population of a single species with age structure, the well-known diffusive Hutchison's equation

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\bigl(1 - u(t-r,x)\bigr), \tag{7}$$

and Nicholson's blowflies diffusive equation

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) - du(t,x) + pu(t-r,x)e^{u(t-r,x)}, \tag{8}$$

or its non-local version

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) - du(t,x) + p\int_{\mathbb{R}} K(x-y)u(t-r,y)e^{u(t-r,y)}\,dy, \tag{9}$$

and so on. For more details about these models, we refer the readers to [3, 18, 26, 33], and the references therein.

The neutral equations are usually adopted to model the evolution of population [5, 7, 17]. To model a ring array of coupled lossless transmission lines, Wu and Xia [29] proposed a neutral difference–differential system with discrete diffusion. In [30], by taking a limit, they further derived the following partial neutral functional differential equation:

$$\begin{aligned} \frac{\partial}{\partial t}\bigl[u(t,x) - qu(t-r,x)\bigr] &= d\frac{\partial^2}{\partial x^2}\bigl[u(t,x) - qu(t-r,x)\bigr] - au(t,x) \\ &\quad - aqu(t-r,x) - g\bigl[u(t,x) - qu(t-r,x)\bigr], \end{aligned}$$

and established the existence and global continuation of rotating waves.

In the last few decades, neutral equations have been investigated wildly, and a variety of themes have been touched, such as the existence and uniqueness, regularity and stability of solutions [1, 4, 9, 11, 14, 16, 28], periodic solutions [10, 24], controllability [23], and so on. But, to the best of our knowledge, there are few works that treat the traveling wave solution of partial neutral functional equations [12, 13, 20].

In [20], Liu and Weng established the existence of spreading speed and traveling wave solution for Eq. (1) with $0 < b < 1$ and with a quasi-monotone reaction (i.e., $F(s_1, s_2, s_3)$ is monotone in $s_2, s_3 \in [0, K]$), and the equation

$$\frac{\partial}{\partial t}L(u)(t,x) = D\frac{\partial^2}{\partial x^2}L(u)(t,x) - aL(u)(t,x) + \frac{p}{2}\left[u(t,x)e^{-u(t,x)} + u(t-r,x)e^{-u(t-r,x)}\right], \tag{10}$$

where $p > a > 0$, $0 < b < 1$, $1 < \frac{p}{a(1-b)} \le e$.

However, when $0 \le b < 1$, it is still an open problem to study the existence of traveling wave solution for Eq. (1) with non-monotone reaction $F(s_1, s_2, s_3)$, or Eq. (10) with $\frac{p}{a(1-b)} > e$.

Concerning the traveling wave solution for the delayed diffusion equation (not of neutral type) with non-monotone reaction, we firstly cite the paper by Ma [22]. In this paper, the author established the existence of a traveling wave solution for a non-local delayed diffusion equation (3) with non-monotone function f by constructing two auxiliary equations with monotone reaction and by using Schauder's Fixed Point Theorem. This method is powerful when dealing with the traveling wave solution of evolution equations with non-monotone reaction (see [2, 6, 15, 19, 33–35] and the references therein). Another paper we cite here is [36] by Yi and Zou. In it, the authors studied a class of non-monotone discrete-time dynamical systems. By using two properly chosen auxiliary systems with order preserving, they obtained results on the asymptotic behavior, the spreading speed, and the existence/nonexistence of traveling waves which can be applied to many classes of evolution equation with non-monotone nonlinearity in the reaction term.

Motivated by the above paper, we study the traveling wave solution for Eq. (1) with $0 \le b < 1$ and with non-monotone reaction $F(s_1, s_2, s_3)$, and this paper is organized as follows. In Sect. 2, we will relax some conditions which were stated in [20] and investigate the existence and asymptotic properties of traveling wave solution for Eq. (1) with quasi-monotone reaction $F(s_1, s_2, s_3)$, by adopting new super- and subsolutions and using Schauder's Fixed Point Theorem. In Sect. 3, we study the existence and asymptotic properties of the traveling wave solution for Eq. (1) with non-monotone reaction. Two special models are discussed in Sect. 4 as an application of our results.

2 The quasi-monotone case

In this section, we study the existence of traveling wave solution for Eq. (1) with quasi-monotone reaction F.

In [20], the authors have obtained the existence of a traveling wave solution for Eq. (1) with quasi-monotone reaction F and with $0 < b < 1$, but they imposed more stringent conditions on the reaction function F because of the methods they used. Here we relax some restrictions on F and use new methods to establish the existence of a traveling wave solution for Eq. (1) and list basic assumptions as follows:

(H1) $F((1-b)s, s, s) > 0$ for $s \in (0, K)$ and $F(s_1, s_2, s_3)$ is non-decreasing with respect to $s_2, s_3 \in [0, K]$;

(H2) $F_2', F_3' \geq 0$, and $F_1' + \frac{1}{1-b}(F_2' + F_3') > 0$, where $F_j' = \frac{\partial F}{\partial s_j}(0,0,0)$, $j = 1,2,3$;
(H3) There exist $\sigma \in (0,1]$ and $M_1, M_2, M_3 \geq 0$, such that for any $s_1, s_2, s_3 \in [0,K]$,

$$0 \leq F_1' s_1 + F_2' s_2 + F_3' s_3 - F(s_1,s_2,s_3) \leq M_1 s_1^{1+\sigma} + M_2 s_2^{1+\sigma} + M_3 s_3^{1+\sigma};$$

(H4) $F(s_1,s_2,s_3)$ is Lipschitz continuous with respect to $s_1, s_2, s_3 \in [0,K]$.

Let $v(t,x) := L(u)(t,x)$, then we get

$$u(t,x) = \sum_{i=0}^{+\infty} b^i v(t - ir, x), \tag{11}$$

and Eq. (1) is transformed into a partial functional differential equation with infinite number of delays as follows:

$$\frac{\partial}{\partial t} v(t,x) = D\frac{\partial^2}{\partial x^2} v(t,x) + F\left(v(t,x), \sum_{i=0}^{+\infty} b^i v(t-ir,x), \sum_{i=0}^{+\infty} b^i v\bigl(t-(i+1)r,x\bigr)\right). \tag{12}$$

Obviously, $v \equiv 0$ and $v \equiv (1-b)K$ are exactly two equilibria of Eq. (12).

It is clear that the traveling wave solution $v(t,x) = V(\xi)$ ($\xi = x + ct$) of Eq. (12) is a solution of the wave profile equation

$$cV'(\xi) = DV''(\xi) + F\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\bigl(\xi - c(i+1)r\bigr)\right), \quad \xi \in \mathbb{R}. \tag{13}$$

Next, we pursue the solution $V(\xi)$ of Eq. (13), which satisfies

$$V(-\infty) = 0, \qquad V(+\infty) = (1-b)K.$$

Let $q(\lambda, c) = 0$ be the characteristic equation of Eq. (13) at $V = 0$, where

$$\begin{aligned} q(\lambda,c) &= D\lambda^2 - c\lambda + F_1' + F_2' \sum_{i=0}^{+\infty} b^i e^{-\lambda cir} + F_3' \sum_{i=0}^{+\infty} b^i e^{-\lambda c(i+1)r} \\ &= D\lambda^2 - c\lambda + F_1' + \frac{1}{1 - be^{-\lambda cr}}\bigl(F_2' + F_3' e^{-\lambda cr}\bigr). \end{aligned} \tag{14}$$

By a similar argument as in [20], we obtain the following lemma.

Lemma 1 *Assume* (H2) *holds. Then there exists a pair* (λ_*, c_*) *such that*
(i) $q(\lambda_*, c_*) = 0$, $\frac{\partial q}{\partial \lambda}(\lambda_*, c_*) = 0$;
(ii) *If* $c \in (0, c_*)$, *then* $q(\lambda, c) > 0$ *for all* $\lambda > 0$;
(iii) *For any* $c > c_*$, $q(\lambda, c) = 0$ *has two roots* λ_1 *and* λ_2 *with* $0 < \lambda_1 < \lambda_2$ *and* $q(\lambda, c) < 0$ *for all* $\lambda \in (\lambda_1, \lambda_2)$.

Remark 1 It is clear that c_* in Lemma 1 can also be defined by

$$c_* = \inf\bigl\{c > 0 : q(c, \lambda) = 0 \text{ for some } \lambda > 0\bigr\}. \tag{15}$$

Definition 1 $V : \mathbb{R} \to \mathbb{R}$ is called a supersolution (subsolution) of Eq. (13), if V is continuous on $\mathbb{R}$, twice differentiable on $\mathbb{R} \setminus \Gamma$, $V'(\xi+) \le (\ge)\ V'(\xi-)$, $\xi \in \Gamma$ and satisfies

$$DV''(\xi) - cV'(\xi) + F\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\big(\xi - c(i+1)r\big)\right) \le (\ge)\ 0, \quad \xi \in \mathbb{R} \setminus \Gamma, \tag{16}$$

where Γ is a finite set.

For any $c > c_*$, define the functions

$$\overline{V}(\xi) = \min\left\{(1-b)K, \alpha\left(e^{\lambda_1\xi} + \theta e^{\beta\lambda_1\xi}\right)\right\}, \qquad \underline{V}(\xi) = \max\left\{0, \alpha\left(e^{\lambda_1\xi} - \theta e^{\beta\lambda_1\xi}\right)\right\}, \tag{17}$$

where $1 < \beta < \min\{1+\sigma, \frac{\lambda_2}{\lambda_1}\}$.

Let ξ_1, ξ_2 be the roots of equations

$$\alpha\left(e^{\lambda_1\xi} + \theta e^{\beta\lambda_1\xi}\right) = (1-b)K \quad \text{and} \quad \alpha\left(e^{\lambda_1\xi} - \theta e^{\beta\lambda_1\xi}\right) = 0,$$

respectively, then $\xi_2 = -\frac{\ln\theta}{\lambda_1(\beta-1)} \le 0$ whenever $\theta \ge 1$. By [32, Lemma 2.2], we have the following lemma.

Lemma 2 *$\overline{V}(\xi)$ and $\underline{V}(\xi)$ have the following properties*:

(i) $\overline{V}(\xi) \ge \underline{V}(\xi)$, $\xi \in \mathbb{R}$;

(ii) $\overline{V}(\xi)$ *is non-decreasing on* $\mathbb{R}$;

(iii) *For any* $\xi', \xi'' \in \mathbb{R}$, $|\overline{V}(\xi') - \overline{V}(\xi'')| \le \alpha\beta\lambda_1(1-b)K|\xi' - \xi''|$.

Lemma 3 *Assume* (H1)–(H3) *hold, and for any* $c > c_*$, α, θ *satisfy the inequalities*:

$$\alpha \ge 0, \quad \theta \ge 1, \quad \textit{and} \quad \theta q(\beta\lambda_1, c) + \alpha^\sigma\left[M_1 + \frac{1}{(1-b)^{1+\sigma}}(M_2 + M_3)\right] \le 0, \tag{18}$$

then $\overline{V}(\xi)$ *and* $\underline{V}(\xi)$ *are a supersolution and a subsolution of Eq.* (13), *respectively.*

Proof Firstly, we show that $\overline{V}(\xi)$ is a supersolution of Eq. (13). In fact, by the definition of $\overline{V}(\xi)$, $\overline{V}(\xi) = (1-b)K$ for any $\xi > \xi_1$, and $\overline{V}(\xi) = \alpha(e^{\lambda_1\xi} + \theta e^{\beta\lambda_1\xi})$ for any $\xi \le \xi_1$, so $\overline{V}(\xi)$ is continuous on $\mathbb{R}$, twice differentiable on $\mathbb{R} \setminus \{\xi_1\}$, $\overline{V}'(\xi_1+) = 0 \le \overline{V}'(\xi_1-) = \alpha(\lambda_1 e^{\lambda_1\xi_1} + \theta\beta\lambda_1 e^{\beta\lambda_1\xi_1})$.

Furthermore, $\overline{V}(\xi) \le (1-b)K$ for all $\xi \in \mathbb{R}$, so from (H1) it follows that, for any $\xi > \xi_1$,

$$D\overline{V}''(\xi) - c\overline{V}'(\xi) + F\left(\overline{V}(\xi), \sum_{i=0}^{+\infty} b^i \overline{V}(\xi - cir), \sum_{i=0}^{+\infty} b^i \overline{V}\big(\xi - c(i+1)r\big)\right) \le F\big((1-b)K, K, K\big) = 0.$$

On the other hand, by Lemma 1(iii), we have $q(\beta\lambda_1,c)<0$. Again, for any $\xi<\xi_1$, $\overline{V}(\xi-cir)=\alpha(e^{\lambda_1(\xi-cir)}+\theta e^{\beta\lambda_1(\xi-cir)})$ $(i=0,1,2,\ldots)$. So from (H3) we have

$$\begin{aligned}
&D\overline{V}''(\xi)-c\overline{V}'(\xi)+F\left(\overline{V}(\xi),\sum_{i=0}^{+\infty}b^i\overline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\overline{V}\bigl(\xi-c(i+1)r\bigr)\right)\\
&\quad=\bigl(D\lambda_1^2-c\lambda_1\bigr)\alpha e^{\lambda_1\xi}+\bigl[D(\beta\lambda_1)^2-c\beta\lambda_1\bigr]\alpha\theta e^{\beta\lambda_1\xi}\\
&\qquad+F\left(\overline{V}(\xi),\sum_{i=0}^{+\infty}b^i\overline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\overline{V}\bigl(\xi-c(i+1)r\bigr)\right)\\
&\quad=q(\lambda_1,c)\alpha e^{\lambda_1\xi}+q(\beta\lambda_1,c)\alpha\theta e^{\beta\lambda_1\xi}\\
&\qquad-\left[F_1'+F_2'\sum_{i=0}^{+\infty}b^ie^{-\lambda_1cir}+F_3'\sum_{i=0}^{+\infty}b^ie^{-\lambda_1c(i+1)r}\right]\alpha e^{\lambda_1\xi}\\
&\qquad-\left[F_1'+F_2'\sum_{i=0}^{+\infty}b^ie^{-\beta\lambda_1cir}+F_3'\sum_{i=0}^{+\infty}b^ie^{-\beta\lambda_1c(i+1)r}\right]\alpha\theta e^{\beta\lambda_1\xi}\\
&\qquad+F\left(\overline{V}(\xi),\sum_{i=0}^{+\infty}b^i\overline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\overline{V}\bigl(\xi-c(i+1)r\bigr)\right)\\
&\quad<-\left[F_1'\overline{V}(\xi)+F_2'\sum_{i=0}^{+\infty}b^i\overline{V}(\xi-cir)+F_3'\sum_{i=0}^{+\infty}b^i\overline{V}\bigl(\xi-c(i+1)r\bigr)\right]\\
&\qquad+F\left(\overline{V}(\xi),\sum_{i=0}^{+\infty}b^i\overline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\overline{V}\bigl(\xi-c(i+1)r\bigr)\right)\\
&\quad\le 0.
\end{aligned}$$

Next, we prove that $\underline{V}(\xi)$ is a subsolution of Eq. (13). In fact, we have $\underline{V}(\xi)=0$ for $\xi>\xi_2$, and $\underline{V}(\xi)=\alpha(e^{\lambda_1\xi}-\theta e^{\beta\lambda_1\xi})$ for $\xi\le\xi_2$, so $\underline{V}(\xi)$ is continuous on $\mathbb{R}$, twice differentiable on $\mathbb{R}\setminus\{\xi_2\}$, $\underline{V}'(\xi_2+)=0\ge\underline{V}'(\xi_2-)=\alpha\lambda_1(1-\beta)\theta^{\frac{1}{1-\beta}}$. Note that $\underline{V}(\xi)\ge 0$ for all $\xi\in\mathbb{R}$, we get from (H1) that, for $\xi>\xi_2$,

$$\begin{aligned}
&D\underline{V}''(\xi)-c\underline{V}'(\xi)+F\left(\underline{V}(\xi),\sum_{i=0}^{+\infty}b^i\underline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\underline{V}\bigl(\xi-c(i+1)r\bigr)\right)\\
&\quad\ge F(0,0,0)=0.
\end{aligned}$$

On the other hand, for any $\xi<\xi_2$, $\underline{V}(\xi-cir)=\alpha(e^{\lambda_1(\xi-cir)}-\theta e^{\beta\lambda_1(\xi-cir)})$ $(i=0,1,2,\ldots)$, then from (H1), (H3) and Lemma 1(iii), we get

$$\begin{aligned}
&D\underline{V}''(\xi)-c\underline{V}'(\xi)+F\left(\underline{V}(\xi),\sum_{i=0}^{+\infty}b^i\underline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\underline{V}\bigl(\xi-c(i+1)r\bigr)\right)\\
&\quad=\bigl(D\lambda_1^2-c\lambda_1\bigr)\alpha e^{\lambda_1\xi}-\bigl[D(\beta\lambda_1)^2-c\beta\lambda_1\bigr]\alpha\theta e^{\beta\lambda_1\xi}\\
&\qquad+F\left(\underline{V}(\xi),\sum_{i=0}^{+\infty}b^i\underline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\underline{V}\bigl(\xi-c(i+1)r\bigr)\right)\\
&\quad=q(\lambda_1,c)\alpha e^{\lambda_1\xi}-q(\beta\lambda_1,c)\alpha\theta e^{\beta\lambda_1\xi}-\left[F_1'+F_2'\sum_{i=0}^{+\infty}b^ie^{-\lambda_1cir}+F_3'\sum_{i=0}^{+\infty}b^ie^{-\lambda_1c(i+1)r}\right]\alpha e^{\lambda_1\xi}
\end{aligned}$$

$$
\begin{aligned}
&+\left[F_1'+F_2'\sum_{i=0}^{+\infty}b^ie^{-\beta\lambda_1 cir}+F_3'\sum_{i=0}^{+\infty}b^ie^{-\beta\lambda_1 c(i+1)r}\right]\alpha\theta e^{\beta\lambda_1\xi}\\
&+F\left(\underline{V}(\xi),\sum_{i=0}^{+\infty}b^i\underline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\underline{V}\big(\xi-c(i+1)r\big)\right)\\
=&-q(\beta\lambda_1,c)\alpha\theta e^{\beta\lambda_1\xi}-\left[F_1'\underline{V}(\xi)+F_2'\sum_{i=0}^{+\infty}b^i\underline{V}(\xi-cir)+F_3'\sum_{i=0}^{+\infty}b^i\underline{V}\big(\xi-c(i+1)r\big)\right.\\
&\left.-F\left(\underline{V}(\xi),\sum_{i=0}^{+\infty}b^i\underline{V}(\xi-cir),\sum_{i=0}^{+\infty}b^i\underline{V}\big(\xi-c(i+1)r\big)\right)\right]\\
\ge&-q(\beta\lambda_1,c)\alpha\theta e^{\beta\lambda_1\xi}-M_1\underline{V}^{1+\sigma}(\xi)-M_2\left[\sum_{i=0}^{+\infty}b^i\underline{V}(\xi-cir)\right]^{1+\sigma}\\
&-M_3\left[\sum_{i=0}^{+\infty}b^i\underline{V}\big(\xi-c(i+1)r\big)\right]^{1+\sigma}\\
=&-e^{\beta\lambda_1\xi}\left\{\alpha\theta q(\beta\lambda_1,c)+M_1e^{-\beta\lambda_1\xi}\underline{V}^{1+\sigma}(\xi)+M_2e^{-\beta\lambda_1\xi}\left[\sum_{i=0}^{+\infty}b^i\underline{V}(\xi-cir)\right]^{1+\sigma}\right.\\
&\left.+M_3e^{-\beta\lambda_1\xi}\left[\sum_{i=0}^{+\infty}b^i\underline{V}\big(\xi-c(i+1)r\big)\right]^{1+\sigma}\right\}\\
\ge&-e^{\beta\lambda_1\xi}\left\{\alpha\theta q(\beta\lambda_1,c)+M_1\alpha^{1+\sigma}e^{(1+\sigma-\beta)\lambda_1\xi}+M_2\alpha^{1+\sigma}e^{(1+\sigma-\beta)\lambda_1\xi}\left(\sum_{i=0}^{+\infty}b^ie^{-\lambda_1 cir}\right)^{1+\sigma}\right.\\
&\left.+M_3\alpha^{1+\sigma}e^{(1+\sigma-\beta)\lambda_1\xi}\left(\sum_{i=0}^{+\infty}b^ie^{-\lambda_1 c(i+1)r}\right)^{1+\sigma}\right\}\\
\ge&-\alpha e^{\beta\lambda_1\xi}\left\{\theta q(\beta\lambda_1,c)+\alpha^{\sigma}\left[M_1+M_2\left(\sum_{i=0}^{+\infty}b^i\right)^{1+\sigma}+M_3\left(\sum_{i=0}^{+\infty}b^i\right)^{1+\sigma}\right]\right\}\\
=&-\alpha e^{\beta\lambda_1\xi}\left\{\theta q(\beta\lambda_1,c)+\alpha^{\sigma}\left[M_1+\frac{1}{(1-b)^{1+\sigma}}(M_2+M_3)\right]\right\}\\
\ge&\,0.
\end{aligned}
$$
□

From the above lemma, we indeed get a family of pairs of super- and subsolutions which take α, β, θ as parameters. Next, we just adopt a pair of super- and subsolution with $\alpha = 1$, and still denote them by $\overline{V}$, $\underline{V}$.

By (H4), there exist $L_1, L_2, L_3 > 0$, such that, for any $s_i', s_i'' \in [0, K]$, $i = 1, 2, 3$,

$$\left|F(s_1',s_2',s_3')-F(s_1'',s_2'',s_3'')\right|\le L_1|s_1'-s_1''|+L_2|s_2'-s_2''|+L_3|s_3'-s_3''|.$$

Let $\gamma \ge L_1$ and rewrite Eq. (13) as follows:

$$
\begin{aligned}
&cV'(\xi)-DV''(\xi)+\gamma V(\xi)\\
&=\gamma V(\xi)+F\left(V(\xi),\sum_{i=0}^{+\infty}b^iV(\xi-cir),\sum_{i=0}^{+\infty}b^iV\big(\xi-c(i+1)r\big)\right).
\end{aligned}
\tag{19}
$$

Set

$$\Lambda_1 = \frac{c - \sqrt{c^2 + 4D\gamma}}{2D} < 0, \qquad \Lambda_2 = \frac{c + \sqrt{c^2 + 4D\gamma}}{2D} > 0, \tag{20}$$

and let $0 < \mu < \Lambda_2$. Define

$$\|V\|_\mu := \sup_{\xi \in \mathbb{R}} \bigl|V(\xi)\bigr| e^{-\mu\xi}; \qquad \mathbf{E} := \bigl\{V \in C(\mathbb{R};\mathbb{R}) : \|V\|_\mu < +\infty\bigr\},$$

and

$$\Omega := \left\{ V \in \mathbf{E} \,\middle|\, \begin{array}{l} \text{(i)} \quad V(\xi) \text{ is nondecreasing on } \mathbb{R}; \\ \text{(ii)} \ \ \underline{V}(\xi) \le V(\xi) \le \overline{V}(\xi) \text{ for all } \xi \in \mathbb{R}; \\ \text{(iii)} \ \bigl|V(\xi') - V(\xi'')\bigr| \le 2(1-b)K\sqrt{\frac{\gamma}{D}}\bigl|\xi' - \xi''\bigr| \text{ for all } \xi', \xi'' \in \mathbb{R} \end{array} \right\},$$

then $(\mathbf{E}, \|\cdot\|_\mu)$ is a Banach space. Obviously, $\overline{V} \in \Omega$ with $\beta\lambda_1 \le 2\sqrt{\frac{\gamma}{D}}$, i.e., Ω is non-empty. Moreover, it is easy to verify that Ω is convex and compact in $\mathbf{E}$.

Set

$$H(V)(\xi) = \gamma V(\xi) + F\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\bigl(\xi - c(i+1)r\bigr)\right), \tag{21}$$

and define an operator $\mathcal{T}$ on Ω by

$$\mathcal{T}(V)(\xi) = \frac{1}{D(\Lambda_2 - \Lambda_1)}\left[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi - s)} H(V)(s)\,ds + \int_{\xi}^{+\infty} e^{\Lambda_2(\xi - s)} H(V)(s)\,ds\right], \tag{22}$$

then a fixed point $V(\xi)$ of $\mathcal{T}$ is the solution of Eq. (13), and vice versa.

Lemma 4 *Assume* (H1) *and* (H4) *hold, then*

(1) $0 \le \mathcal{T}[V](\xi) \le (1-b)K$ *for all* $V \in C(\mathbb{R}; [0, (1-b)K])$;

(2) *For any* $\xi \in \mathbb{R}, \mathcal{T}[V](\xi)$ *is non-decreasing with respect to* $V \in C(\mathbb{R}; [0, (1-b)K])$;

(3) $\underline{V}(\xi) \le \mathcal{T}[\underline{V}](\xi) \le \mathcal{T}[V](\xi) \le \mathcal{T}[\overline{V}](\xi) \le \overline{V}(\xi)$ *for any* $V \in \Omega$, $\xi \in \mathbb{R}$;

(4) *For any* $V \in \Omega, \mathcal{T}[V](\xi)$ *is non-decreasing with respect to* $\xi \in \mathbb{R}$;

(5) *For any* $V \in \Omega$ *and* $\xi', \xi'' \in \mathbb{R}$, *it follows that*

$$\bigl|\mathcal{T}[V]\bigl(\xi'\bigr) - \mathcal{T}[V]\bigl(\xi''\bigr)\bigr| \le 2(1-b)K\sqrt{\frac{\gamma}{D}}\bigl|\xi' - \xi''\bigr|.$$

Proof (1) Firstly, we claim that, for any $\xi \in \mathbb{R}, H[V](\xi)$ is non-decreasing with respect to $V \in C(\mathbb{R}; [0, (1-b)K])$.

In fact, for any $V_1, V_2 \in C(\mathbb{R}; [0, (1-b)K])$ with $V_1(\xi) \le V_2(\xi)$, $\xi \in \mathbb{R}$, it follows that

$$\begin{aligned} &H[V_2](\xi) - H[V_1](\xi) \\ &\quad = \gamma\bigl[V_2(\xi) - V_1(\xi)\bigr] + F\left(V_2(\xi), \sum_{i=0}^{+\infty} b^i V_2(\xi - cir), \sum_{i=0}^{+\infty} b^i V_2\bigl(\xi - c(i+1)r\bigr)\right) \end{aligned}$$

$$
\begin{aligned}
&- F\left(V_1(\xi), \sum_{i=0}^{+\infty} b^i V_1(\xi - cir), \sum_{i=0}^{+\infty} b^i V_1\big(\xi - c(i+1)r\big)\right) \\
&\geq \gamma\big[V_2(\xi) - V_1(\xi)\big] + F\left(V_2(\xi), \sum_{i=0}^{+\infty} b^i V_1(\xi - cir), \sum_{i=0}^{+\infty} b^i V_1\big(\xi - c(i+1)r\big)\right) \\
&- F\left(V_1(\xi), \sum_{i=0}^{+\infty} b^i V_1(\xi - cir), \sum_{i=0}^{+\infty} b^i V_1\big(\xi - c(i+1)r\big)\right) \\
&\geq (\gamma - L_1)\big[V_2(\xi) - V_1(\xi)\big] \geq 0.
\end{aligned}
$$

The claim holds. So, for any $V \in C(\mathbb{R}; [0, (1-b)K])$,

$$
0 \leq H[V](\xi) \leq H\big[(1-b)K\big] = \gamma(1-b)K + F\big((1-b)K, K, K\big) = \gamma(1-b)K, \quad \xi \in \mathbb{R},
$$

and then, we get that, for any $V \in C(\mathbb{R}; [0, (1-b)K])$, $\xi \in \mathbb{R}$,

$$
0 \leq \mathcal{T}[V](\xi) \leq \frac{\gamma(1-b)K}{D(\Lambda_2 - \Lambda_1)}\left[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi - s)}\, ds + \int_{\xi}^{+\infty} e^{\Lambda_2(\xi - s)}\, ds\right] = (1-b)K.
$$

(2) The monotonicity of $\mathcal{T}[V](\xi)$ with respect to $V \in C(\mathbb{R}; [0, (1-b)K])$ follows from that of $H[V](\xi)$.

(3) For any $V \in \Omega$, from (2), we get that

$$
\mathcal{T}[\underline{V}](\xi) \leq \mathcal{T}[V](\xi) \leq \mathcal{T}[\overline{V}](\xi), \quad \xi \in \mathbb{R},
$$

and a similar argument as in [21] shows that

$$
\underline{V}(\xi) \leq \mathcal{T}[\underline{V}](\xi), \qquad \mathcal{T}[\overline{V}](\xi) \leq \overline{V}(\xi), \quad \xi \in \mathbb{R}.
$$

(4) For any $V \in \Omega$ and $\xi', \xi'' \in \mathbb{R}$ with $\xi' \leq \xi''$, we have $V(\xi') \leq V(\xi'')$, so $H(V)(\xi') \leq H(V)(\xi'')$. Noting the fact that

$$
\begin{aligned}
\mathcal{T}(V)(\xi) &= \frac{1}{D(\Lambda_2 - \Lambda_1)}\left[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi - s)} H(V)(s)\, ds + \int_{\xi}^{+\infty} e^{\Lambda_2(\xi - s)} H(V)(s)\, ds\right] \\
&= \frac{1}{D(\Lambda_2 - \Lambda_1)}\left[\int_{0}^{+\infty} e^{\Lambda_1 s} H(V)(\xi - s)\, ds + \int_{-\infty}^{0} e^{\Lambda_2 s} H(V)(\xi - s)\, ds\right],
\end{aligned}
$$

it is easy to see that $\mathcal{T}(V)(\xi)$ is non-decreasing with respect to $\xi \in \mathbb{R}$.

(5) For any $V \in \Omega$, we have $0 \leq H[V](\xi) \leq \gamma(1-b)K$, $\xi \in \mathbb{R}$, and then, for any $\xi', \xi'' \in \mathbb{R}$ with $\xi' \geq \xi''$,

$$
\begin{aligned}
&D(\Lambda_2 - \Lambda_1)\big|\mathcal{T}(V)\big(\xi'\big) - \mathcal{T}(V)\big(\xi''\big)\big| \\
&\quad \leq \left|\int_{-\infty}^{\xi'} e^{\Lambda_1(\xi' - s)} H(V)(s)\, ds - \int_{-\infty}^{\xi''} e^{\Lambda_1(\xi'' - s)} H(V)(s)\, ds\right| \\
&\qquad + \left|\int_{\xi'}^{+\infty} e^{\Lambda_2(\xi' - s)} H(V)(s)\, ds - \int_{\xi''}^{+\infty} e^{\Lambda_2(\xi'' - s)} H(V)(s)\, ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_{\xi''}^{\xi'} e^{\Lambda_1(\xi'-s)}H(V)(s)\,ds + \int_{-\infty}^{\xi''} \left|e^{\Lambda_1(\xi'-s)} - e^{\Lambda_1(\xi''-s)}\right| H(V)(s)\,ds \\
&\quad + \int_{\xi'}^{+\infty} \left|e^{\Lambda_2(\xi'-s)} - e^{\Lambda_2(\xi''-s)}\right| H(V)(s)\,ds + \int_{\xi''}^{\xi'} e^{\Lambda_2(\xi''-s)}H(V)(s)\,ds \\
&\le \gamma(1-b)K\left\{\int_{\xi''}^{\xi'} e^{\Lambda_1(\xi'-s)}\,ds + \int_{-\infty}^{\xi''}\left[e^{\Lambda_1(\xi''-s)} - e^{\Lambda_1(\xi'-s)}\right]ds\right. \\
&\quad \left. + \int_{\xi'}^{+\infty}\left[e^{\Lambda_2(\xi'-s)} - e^{\Lambda_2(\xi''-s)}\right]ds + \int_{\xi''}^{\xi'} e^{\Lambda_2(\xi''-s)}\,ds\right\} \\
&\le \gamma(1-b)K\left\{2\left|\xi'-\xi''\right| + \frac{1}{\Lambda_1}\left[e^{\Lambda_1(\xi'-\xi'')} - 1\right] + \frac{1}{\Lambda_2}\left[1 - e^{\Lambda_2(\xi''-\xi')}\right]\right\} \\
&\le 4\gamma(1-b)K\left|\xi'-\xi''\right|,
\end{aligned}
$$

i.e.,

$$
\left|\mathcal{T}(V)(\xi') - \mathcal{T}(V)(\xi'')\right| \le \frac{4\gamma(1-b)K}{D(\Lambda_2-\Lambda_1)}\left|\xi'-\xi''\right| \le 2(1-b)K\sqrt{\frac{\gamma}{D}}\left|\xi'-\xi''\right|. \qquad \square
$$

Theorem 1 *Assume that* (H1)–(H4) *hold. Then*

(1) *For any $c \ge c_*$, the traveling wave solution $V(\xi)$ of Eq.* (12) *exists*; *$V(\xi)$ is nondecreasing on $\mathbb{R}$, $V(-\infty) = 0$, and $V(+\infty) = (1-b)K$. In addition, for $c > c_*$,*

$$
\lim_{\xi\to-\infty} V(\xi)e^{-\lambda_1\xi} = 1 \quad \textit{and} \quad \lim_{\xi\to-\infty} V'(\xi)e^{-\lambda_1\xi} = \lambda_1;
$$

(2) *For any $c < c_*$ and $\lambda > 0$, there is no traveling solution $V(\xi)$ of Eq.* (12) *which satisfies*

$$
\lim_{\xi\to-\infty} V(\xi)e^{-\lambda\xi} = 1 \quad \textit{and} \quad \lim_{\xi\to-\infty} V'(\xi)e^{-\lambda\xi} = \lambda.
$$

Proof (1) Firstly, by Lemma 4, we have $\mathcal{T}(\Omega) \subset \Omega$.

Next, we will show that $\mathcal{T}$ is continuous with respect to $\|\cdot\|_\mu$ in Ω.

In fact, for any $V_1, V_2 \in \Omega$, we have

$$
\begin{aligned}
&\left|\mathcal{T}(V_1)(\xi) - \mathcal{T}(V_2)(\xi)\right|e^{-\mu\xi} \\
&= \frac{e^{-\mu\xi}}{D(\Lambda_2-\Lambda_1)}\left|\left[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}H(V_1)(s)\,ds + \int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}H(V_1)(s)\,ds\right]\right. \\
&\quad \left. - \left[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}H(V_2)(s)\,ds + \int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}H(V_2)(s)\,ds\right]\right| \\
&\le \frac{e^{-\mu\xi}}{D(\Lambda_2-\Lambda_1)}\left[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}\left|H(V_1)(s) - H(V_2)(s)\right|ds\right. \\
&\quad \left. + \int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}\left|H(V_1)(s) - H(V_2)(s)\right|ds\right] \\
&\le \frac{e^{-\mu\xi}}{D(\Lambda_2-\Lambda_1)}\left\{(\gamma + L_1)\left[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}\left|V_1(s) - V_2(s)\right|ds\right.\right. \\
&\quad \left. + \int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}\left|V_1(s) - V_2(s)\right|ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ L_2\Bigg[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}\sum_{i=0}^{+\infty} b^i\big|V_1(s-cir)-V_2(s-cir)\big|\,ds \\
&+ \int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}\sum_{i=0}^{+\infty} b^i\big|V_1(s-cir)-V_2(s-cir)\big|\,ds\Bigg] \\
&+ L_3\Bigg[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}\sum_{i=0}^{+\infty} b^i\big|V_1\big(s-c(i+1)r\big)-V_2\big(s-c(i+1)r\big)\big|\,ds \\
&+ \int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}\sum_{i=0}^{+\infty} b^i\big|V_1\big(s-c(i+1)r\big)-V_2\big(s-c(i+1)r\big)\big|\,ds\Bigg]\Bigg\} \\
\le{}& \frac{\|V_1-V_2\|_\mu e^{-\mu\xi}}{D(\Lambda_2-\Lambda_1)}\Bigg\{(\gamma+L_1)\Bigg[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}\cdot e^{\mu s}\,ds+\int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}\cdot e^{\mu s}\,ds\Bigg] \\
&+ L_2\Bigg[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}\sum_{i=0}^{+\infty} b^i e^{\mu(s-cir)}\,ds+\int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}\sum_{i=0}^{+\infty} b^i e^{\mu(s-cir)}\,ds\Bigg] \\
&+ L_3\Bigg[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}\sum_{i=0}^{+\infty} b^i e^{\mu(s-c(i+1)r)}\,ds+\int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}\sum_{i=0}^{+\infty} b^i e^{\mu(s-c(i+1)r)}\,ds\Bigg]\Bigg\} \\
\le{}& \frac{\|V_1-V_2\|_\mu e^{-\mu\xi}}{D(\Lambda_2-\Lambda_1)}\Bigg\{\bigg(\gamma+L_1+\frac{L_2+L_3e^{-cr}}{1-be^{-cr}}\bigg) \\
&\times\Bigg[\int_{-\infty}^{\xi} e^{\Lambda_1(\xi-s)}\cdot e^{\mu s}\,ds+\int_{\xi}^{+\infty} e^{\Lambda_2(\xi-s)}\cdot e^{\mu s}\,ds\Bigg]\Bigg\} \\
={}& \frac{\|V_1-V_2\|_\mu}{D(\Lambda_2-\mu)(\mu-\Lambda_1)}\bigg(\gamma+L_1+\frac{L_2+L_3e^{-cr}}{1-be^{-cr}}\bigg).
\end{aligned}
$$

By virtue of Schauder's Fixed Point Theorem, for any $c > c_*$, there exists a fixed point $V(\xi)$ of $\mathcal{T}(V)$ on Ω, i.e., there exists traveling wave solution $V(\xi)$ of Eq. (12) which is nondecreasing on $\mathbb{R}$, and $V(-\infty) = 0$, $V(+\infty) = (1-b)K$.

Furthermore, by using a similar argument to the proof of [20, Theorem 3.3], we get

$$
\lim_{\xi\to-\infty} V(\xi)e^{-\lambda_1\xi} = 1 \quad \text{and} \quad \lim_{\xi\to-\infty} V'(\xi)e^{-\lambda_1\xi} = \lambda_1.
$$

For the case $c = c_*$, let $\{c_n\}$ be a sequence such that $c_n > c_*$, $n = 1,2,\ldots$, and $\lim_{n\to+\infty} c_n = c_*$, then for $c = c_n$, $n = 1,2,\ldots$, Eq. (12) admits a nondecreasing solution $V_n(\xi)$, which satisfies

$$
V_n(\xi) = \frac{1}{D(\Lambda_{2,n}-\Lambda_{1,n})}\bigg[\int_{-\infty}^{\xi} e^{\Lambda_{1,n}(\xi-s)}H(V_n)(s)\,ds+\int_{\xi}^{+\infty} e^{\Lambda_{2,n}(\xi-s)}H(V_n)(s)\,ds\bigg]
$$

and

$$
V_n(0) = \frac{(1-b)K}{2}, \qquad V_n(-\infty) = 0, \qquad V_n(+\infty) = (1-b)K,
$$

where

$$
\Lambda_{1,n} = \frac{c_n-\sqrt{c_n^2+4D\gamma}}{2D} < 0, \qquad \Lambda_{2,n} = \frac{c_n+\sqrt{c_n^2+4D\gamma}}{2D} > 0, \quad n = 1,2,\ldots.
$$

Adopting a limit argument similar to the proof of [20, Theorem 3.3] or [34, Theorem 2.5], we can obtain a subsequence of $\{V_n(\xi)\}$ which is convergent uniformly for ξ in any bounded subset of $\mathbb{R}$. Without loss of generality, denote it still by $\{V_n(\xi)\}$, then there exists limit $\lim_{n\to+\infty} V_n(\xi) = V_*(\xi)$, $\xi \in \mathbb{R}$, and $V_*(0) = \frac{(1-b)K}{2}$, $V_*(\xi)$ is nondecreasing in $\mathbb{R}$, $V_*(-\infty) = 0$, and $V_*(+\infty) = (1-b)K$.

(2) Suppose that there exist some $c < c_*$ and $\lambda > 0$, such that the traveling wave solution $V(\xi)$ of Eq. (12) exists and satisfies

$$\lim_{\xi\to-\infty} V(\xi)e^{-\lambda\xi} = 1, \qquad \lim_{\xi\to-\infty} V'(\xi)e^{-\lambda\xi} = \lambda,$$

then we have

$$\begin{aligned} &cV'(\xi)e^{-\lambda\xi} \\ &\quad = DV''(\xi)e^{-\lambda\xi} + e^{-\lambda\xi}F\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\bigl(\xi - c(i+1)r\bigr)\right). \end{aligned}$$

Let $\xi \to -\infty$, we get

$$D\lim_{\xi\to-\infty} V''(\xi)e^{-\lambda\xi} = c\lambda - F_1' - F_2'\sum_{i=0}^{+\infty} b^i e^{-\lambda cir} - F_3'\sum_{i=0}^{+\infty} b^i e^{-\lambda c(i+1)r},$$

and then

$$\begin{aligned} &D\lim_{\xi\to-\infty}\bigl[V'(\xi)e^{-\lambda\xi}\bigr]' \\ &\quad = -D\lambda^2 + c\lambda - F_1' - F_2'\sum_{i=0}^{+\infty} b^i e^{-\lambda cir} - F_3'\sum_{i=0}^{+\infty} b^i e^{-\lambda c(i+1)r} \\ &\quad = -q(\lambda, c). \end{aligned}$$

From $\lim_{\xi\to-\infty} V'(\xi)e^{-\lambda\xi} = \lambda$, we get $\lim_{\xi\to-\infty}[V'(\xi)e^{-\lambda\xi}]' = 0$, so $q(\lambda, c) = 0$. This is a contradiction to Lemma 1(ii). □

Theorem 2 *Assume that* (H1)–(H4) *hold. Then*

(1) *For any $c \ge c_*$, the traveling wave solution $U(\xi)$ of Eq.* (1) *exists; $U(\xi)$ is nondecreasing on $\mathbb{R}$, $U(-\infty) = 0$, $U(+\infty) = K$, and for any $c > c_*$,*

$$\lim_{\xi\to-\infty} U(\xi)e^{-\lambda_1\xi} = \frac{1}{1 - be^{-\lambda_1 cr}} \quad \textit{and} \quad \lim_{\xi\to-\infty} U'(\xi)e^{-\lambda_1\xi} = \frac{\lambda_1}{1 - be^{-\lambda_1 cr}};$$

(2) *For any $c < c_*$ and $\lambda > 0$, there is no traveling solution $U(\xi)$ of Eq.* (1) *which satisfies*

$$\lim_{\xi\to-\infty} U(\xi)e^{-\lambda\xi} = \frac{1}{1 - be^{-\lambda cr}} \quad \textit{and} \quad \lim_{\xi\to-\infty} U'(\xi)e^{-\lambda\xi} = \frac{\lambda}{1 - be^{-\lambda cr}}.$$

Proof (1) For any $c \ge c_*$, by the Theorem 1, the traveling wave solution $V(\xi)$ of Eq. (12) exists, and $V(\xi)$ is nondecreasing on $\mathbb{R}$, $V(-\infty) = 0$, $V(+\infty) = (1-b)K$, and for any $c > c_*$,

$$\lim_{\xi\to-\infty} V(\xi)e^{-\lambda_1\xi} = 1 \quad \text{and} \quad \lim_{\xi\to-\infty} V'(\xi)e^{-\lambda_1\xi} = \lambda_1.$$

Firstly, the boundedness of $V(\xi - cir)$ $(i = 0, 1, \ldots)$ implies that $\sum_{i=0}^{+\infty} b^i V(\xi - cir)$ is uniformly convergent on $\xi \in \mathbb{R}$. Let $U(\xi) = \sum_{i=0}^{+\infty} b^i V(\xi - cir)$, then $U(\xi)$ is a traveling wave solution of Eq. (1) and is non-decreasing on $\mathbb{R}$, $U(-\infty) = 0$, $U(+\infty) = K$ and $U'(\xi) = \sum_{i=0}^{+\infty} b^i V'(\xi - cir)$.

Furthermore, for any $c > c_*$, $\sum_{i=0}^{+\infty} b^i V(\xi - cir)e^{-\lambda_1 \xi}$ and $\sum_{i=0}^{+\infty} b^i V'(\xi - cir)e^{-\lambda_1 \xi}$ are uniformly convergent for $\xi \in \mathbb{R}$, respectively, since

$$\lim_{\xi \to -\infty} V(\xi - cir)e^{-\lambda_1(\xi - cir)} = 1, \qquad \lim_{\xi \to -\infty} V'(\xi - cir)e^{-\lambda_1(\xi - cir)} = \lambda_1$$

for all $i = 0, 1, 2 \ldots$, and

$$\sum_{i=0}^{+\infty} b^i e^{-\lambda_1 cir} = \frac{1}{1 - be^{-\lambda_1 cr}}.$$

So,

$$\lim_{\xi \to -\infty} U(\xi)e^{-\lambda_1 \xi} = \lim_{\xi \to -\infty} \sum_{i=0}^{+\infty} b^i V(\xi - cir)e^{-\lambda_1(\xi - cir)} e^{-\lambda_1 cir} = \frac{1}{1 - be^{-\lambda_1 cr}};$$

and

$$\lim_{\xi \to -\infty} U'(\xi)e^{-\lambda_1 \xi} = \lim_{\xi \to -\infty} \sum_{i=0}^{+\infty} b^i V'(\xi - cir)e^{-\lambda_1(\xi - cir)} e^{-\lambda_1 cir} = \frac{\lambda_1}{1 - be^{-\lambda_1 cr}}.$$

(2) Assume that there exist $c < c_*$ and $\lambda > 0$, such that Eq. (1) has traveling wave solution $U(\xi)$ which satisfies

$$\lim_{\xi \to -\infty} U(\xi)e^{-\lambda \xi} = \frac{1}{1 - be^{-\lambda cr}} \quad \text{and} \quad \lim_{\xi \to -\infty} U'(\xi)e^{-\lambda \xi} = \frac{\lambda}{1 - be^{-\lambda cr}}.$$

Let $V(\xi) = U(\xi) - bU(\xi - cr)$, then $V(\xi)$ is a traveling wave solution of Eq. (12) and

$$\lim_{\xi \to -\infty} V(\xi)e^{-\lambda \xi} = \lim_{\xi \to -\infty} \left[U(\xi)e^{-\lambda \xi} - bU(\xi - cr)e^{-\lambda(\xi - cr)} e^{-\lambda cr}\right] = 1;$$

and

$$\lim_{\xi \to -\infty} V'(\xi)e^{-\lambda \xi} = \lim_{\xi \to -\infty} \left[U'(\xi)e^{-\lambda \xi} - bU'(\xi - cr)e^{-\lambda(\xi - cr)} e^{-\lambda cr}\right] = \lambda.$$

This is a contradiction to Theorem 1(2). □

3 The non-monotone case

In this section, we will discuss the traveling wave solution of Eq. (1) with the non-monotone reaction $F(s_1, s_2, s_3)$, and we need the following assumptions: there exist continuous functions $F_\pm(s_1, s_2, s_3)$ and constants $K_1, K_2 > 0$, such that $K_1 \le K \le K_2$ and

(A1) $F_\pm(0, 0, 0) = 0$, $F_+((1 - b)K_2, K_2, K_2) = F_-((1 - b)K_1, K_1, K_1) = 0$, $F_+((1 - b)s, s, s) > 0$ for $s \in (0, K_2)$ and $F_-((1 - b)s, s, s) > 0$ for $s \in (0, K_1)$;

(A2) $F_-(s_1, s_2, s_3) \le F(s_1, s_2, s_3) \le F_+(s_1, s_2, s_3)$ for $s_1, s_2, s_3 \in [0, K_2]$;

(A3) $F'_{\pm,j} = F'_j$, where $F'_{\pm,j}$ are defined similarly to F'_j in (H4), $j = 1,2,3$;

(A4) There exist $\sigma \in (0,1]$ and $M_{\pm,1}, M_{\pm,2}, M_{\pm,3} \geq 0$, such that

$$0 \leq F'_{-,1}s_1 + F'_{-,2}s_2 + F'_{-,3}s_3 - F_-(s_1,s_2,s_3) \leq M_{-,1}s_1^{1+\sigma} + M_{-,2}s_2^{1+\sigma} + M_{-,3}s_3^{1+\sigma};$$

for any $s_1, s_2, s_3 \in [0, K_1]$, and

$$0 \leq F'_{+,1}s_1 + F'_{+,2}s_2 + F'_{+,3}s_3 - F_+(s_1,s_2,s_3) \leq M_{+,1}s_1^{1+\sigma} + M_{+,2}s_2^{1+\sigma} + M_{+,3}s_3^{1+\sigma};$$

for any $s_1, s_2, s_3 \in [0, K_2]$;

(A5) $F_\pm(s_1,s_2,s_3)$ is non-decreasing with respect to $s_2, s_3 \in [0, K_2]$ and Lipschitz continuous with respect to $s_1, s_2, s_3 \in [0, K_2]$.

Consider the following equations:

$$\frac{\partial}{\partial t}L(u)(t,x) = D\frac{\partial^2}{\partial x^2}L(u)(t,x) + F_+\big(L(u)(t,x), u(t,x), u(t-r,x)\big) \tag{23}$$

and

$$\frac{\partial}{\partial t}L(u)(t,x) = D\frac{\partial^2}{\partial x^2}L(u)(t,x) + F_-\big(L(u)(t,x), u(t,x), u(t-r,x)\big). \tag{24}$$

From (A1), it is obvious that $u \equiv 0$ and $u \equiv K_1$ are equilibria of Eq. (23) and $u \equiv 0$ and $u \equiv K_2$ are those of Eq. (24).

By the transformation $v(t,x) = L(u)(t,x)$, Eqs. (23) and (24) reduce to

$$\frac{\partial v}{\partial t}(t,x) = D\frac{\partial^2 v}{\partial x^2}(t,x) + F_+\left(v(t,x), \sum_{i=0}^{+\infty} b^i v(t-ir,x), \sum_{i=0}^{+\infty} b^i v\big(t-(i+1)r,x\big)\right) \tag{25}$$

and

$$\frac{\partial v}{\partial t}(t,x) = D\frac{\partial^2 v}{\partial x^2}(t,x) + F_-\left(v(t,x), \sum_{i=0}^{+\infty} b^i v(t-ir,x), \sum_{i=0}^{+\infty} b^i v\big(t-(i+1)r,x\big)\right), \tag{26}$$

respectively, and the wave profile equations are given as follows:

$$cV'(\xi) = DV'' + F_+\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\big(\xi - c(i+1)r\big)\right) \tag{27}$$

and

$$cV'(\xi) = DV'' + F_-\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\big(\xi - c(i+1)r\big)\right), \tag{28}$$

respectively.

It is obviously that, Eqs. (13), (27), and (28) have the same characteristic equation, $q(\lambda, c) = 0$ at $V = 0$, where $q(\lambda, c)$ is defined in (14). Let c_* and λ_1 are defined as in Lemma 1, then we have the following lemma.

Lemma 5 *Assume* (H2), (A1)–(A5) *hold. Then*:

(1) *For any* $c \geq c_*$, *the traveling wave solutions* $V_+(\xi)$ *of Eq.* (25) *and* $V_-(\xi)$ *of Eq.* (26) *exist*; $V_\pm(\xi)$ *is nondecreasing on* $\mathbb{R}$, $V_\pm(-\infty) = 0$, $V_-(+\infty) = (1-b)K_1$, $V_+(+\infty) = (1-b)K_2$, *and for any* $c > c_*$,

$$\lim_{\xi\to-\infty} V_\pm(\xi)e^{-\lambda_1\xi} = 1 \quad \textit{and} \quad \lim_{\xi\to-\infty} V'_\pm(\xi)e^{-\lambda_1\xi} = \lambda_1;$$

(2) *For any* $c < c_*$ *and* $\lambda > 0$, *there is no traveling solution* $V_+(\xi)$ *of Eq.* (25) *and* $V_-(\xi)$ *of Eq.* (26) *such that*

$$\lim_{\xi\to-\infty} V_\pm(\xi)e^{-\lambda\xi} = 1 \quad \textit{and} \quad \lim_{\xi\to-\infty} V'_\pm(\xi)e^{-\lambda\xi} = \lambda.$$

Lemma 6 *For any* $c \geq c_*$, *let* $V_+(\xi)$ *and* $V_-(\xi)$ *be the traveling wave solution of Eqs.* (25) *and* (26), *respectively, then there exists* $\overline{\xi} \in \mathbb{R}$ *such that* $V_-(\xi) \leq V_+(\xi + \overline{\xi})$, $\xi \in \mathbb{R}$.

The proof is similar to that of [34, Lemma 3.5] and we omit it here.

Assume $L_{+,j}, L_{-,j} > 0$ $(j = 1,2,3)$ are Lipschitz constants of F_+ and F_-, respectively, and L_1 is as defined in Sect. 2. Let $\overline{\gamma} \geq \max\{L_{+,1}, L_{-,1}, L_1\}$ and rewrite Eqs. (27) and (28), respectively, as follows:

$$\begin{aligned} &cV'(\xi) - DV''(\xi) + \overline{\gamma}V(\xi) \\ &\quad = \overline{\gamma}V(\xi) + F_+\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\bigl(\xi - c(i+1)r\bigr)\right) \end{aligned}$$

and

$$\begin{aligned} &cV'(\xi) - DV''(\xi) + \overline{\gamma}V(\xi) \\ &\quad = \overline{\gamma}V(\xi) + F_-\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\bigl(\xi - c(i+1)r\bigr)\right). \end{aligned}$$

Set

$$\overline{\varLambda}_1 = \frac{c - \sqrt{c^2 + 4D\overline{\gamma}}}{2D} < 0, \qquad \overline{\varLambda}_2 = \frac{c + \sqrt{c^2 + 4D\overline{\gamma}}}{2D} > 0,$$

and let $0 < \overline{\mu} < \overline{\varLambda}_2$. Define

$$\|V\|_{\overline{\mu}} := \sup_{\xi\in\mathbb{R}} \bigl|V(\xi)\bigr| e^{-\overline{\mu}\xi}; \qquad \overline{\mathbf{E}} := \bigl\{V \in C(\mathbb{R};\mathbb{R}) : \|V\|_{\overline{\mu}} < +\infty\bigr\},$$

and

$$\overline{\varOmega} := \left\{ V \in C\bigl(\mathbb{R};\bigl[0,(1-b)K_2\bigr]\bigr) \;\middle|\; \begin{array}{l} \text{(i)}\ \ V_-(\xi) \leq V(\xi) \leq V_+(\xi + \overline{\xi}), \xi \in \mathbb{R}; \\ \text{(ii)}\ \bigl|V(\xi') - V(\xi'')\bigr| \leq 2(1-b)K_2\sqrt{\dfrac{\overline{\gamma}}{D}}\bigl|\xi' - \xi''\bigr| \\ \qquad \text{for all } \xi', \xi'' \in \mathbb{R} \end{array} \right\}.$$

Then $(\overline{\mathbf{E}}, \|\cdot\|_{\overline{\mu}})$ is a Banach space and $\overline{\Omega}$ is non-empty since $V_\pm \in \overline{\Omega}$. Moreover, it is easy to verify that $\overline{\Omega}$ is convex and compact in $\overline{\mathbf{E}}$.

Set

$$H_\pm(V)(\xi) = \overline{\gamma} V(\xi) + F_\pm\left(V(\xi), \sum_{i=0}^{+\infty} b^i V(\xi - cir), \sum_{i=0}^{+\infty} b^i V\big(\xi - c(i+1)r\big)\right),$$

and define an operator $\overline{\mathcal{T}}$ on Ω by

$$\overline{\mathcal{T}}(V)(\xi) = \frac{1}{D(\overline{\Lambda}_2 - \overline{\Lambda}_1)}\left[\int_{-\infty}^{\xi} e^{\overline{\Lambda}_1(\xi-s)} H(V)(s)\,ds + \int_{\xi}^{+\infty} e^{\overline{\Lambda}_2(\xi-s)} H(V)(s)\,ds\right],$$

where H is defined as in (21) with γ replaced by $\overline{\gamma}$.

Theorem 3 *Assume* (H2), (H4), (A1)–(A5) *hold. Then for any* $c \geq c_*$, *there exists a traveling wave solution* $V(\xi)$ *of Eq.* (12) *with* $V(-\infty) = 0$,

$$(1-b)K_1 \leq \liminf_{\xi\to+\infty} V(\xi) \leq \limsup_{\xi\to+\infty} V(\xi) \leq (1-b)K_2,$$

and for any $c > c_*$,

$$1 \leq \liminf_{\xi\to-\infty} V(\xi)e^{-\lambda_1\xi} \leq \limsup_{\xi\to-\infty} V(\xi)e^{-\lambda_1\xi} \leq e^{\lambda_1\overline{\xi}}.$$

Proof For any $V \in \overline{\Omega}$, we have $0 \leq V_-(\xi) \leq V(\xi) \leq V_+(\xi + \overline{\xi}) \leq (1-b)K_2$, and then

$$0 \leq H_-[V](\xi) \leq H[V](\xi) \leq H_+[V](\xi) \leq \overline{\gamma}(1-b)K_2, \quad \xi \in \mathbb{R}.$$

Furthermore,

$$0 \leq \overline{\mathcal{T}}(V)(\xi) \leq \frac{\overline{\gamma}(1-b)K_2}{D(\overline{\Lambda}_2 - \overline{\Lambda}_1)}\left[\int_{-\infty}^{\xi} e^{\overline{\Lambda}_1(\xi-s)}\,ds + \int_{\xi}^{+\infty} e^{\overline{\Lambda}_2(\xi-s)}\,ds\right] \leq (1-b)K_2, \quad \xi \in \mathbb{R}.$$

Again, since $V_-(\xi)$ is a solution of Eq. (28) and $H_-(V)(\xi)$ is non-decreasing on $V \in C(\mathbb{R}, [0, (1-b)K_2])$ for any $\xi \in \mathbb{R}$, we have that, for any $V \in \overline{\Omega}$, $\xi \in \mathbb{R}$,

$$\begin{aligned}
V_-(\xi) &= \frac{1}{D(\overline{\Lambda}_2 - \overline{\Lambda}_1)}\left[\int_{-\infty}^{\xi} e^{\overline{\Lambda}_1(\xi-s)} H_-(V_-)(s)\,ds + \int_{\xi}^{+\infty} e^{\overline{\Lambda}_2(\xi-s)} H_-(V_-)(s)\,ds\right] \\
&\leq \frac{1}{D(\overline{\Lambda}_2 - \overline{\Lambda}_1)}\left[\int_{-\infty}^{\xi} e^{\overline{\Lambda}_1(\xi-s)} H_-(V)(s)\,ds + \int_{\xi}^{+\infty} e^{\overline{\Lambda}_2(\xi-s)} H_-(V)(s)\,ds\right] \\
&\leq \frac{1}{D(\overline{\Lambda}_2 - \overline{\Lambda}_1)}\left[\int_{-\infty}^{\xi} e^{\overline{\Lambda}_1(\xi-s)} H(V)(s)\,ds + \int_{\xi}^{+\infty} e^{\overline{\Lambda}_2(\xi-s)} H(V)(s)\,ds\right] \\
&= \overline{\mathcal{T}}(V)(\xi).
\end{aligned}$$

Similarly, we have $\overline{\mathcal{T}}(V)(\xi) \leq V_+(\xi + \overline{\xi})$ for any $V \in \overline{\Omega}$, $\xi \in \mathbb{R}$.

By a similar proof to that of Lemma 4(5), we get that, for any $V \in \overline{\Omega}$,

$$\big|\overline{\mathcal{T}}(V)(\xi') - \overline{\mathcal{T}}(V)(\xi'')\big| \leq 2(1-b)K_2\sqrt{\frac{\overline{\gamma}}{D}}\,|\xi' - \xi''| \quad \text{for all } \xi', \xi'' \in \mathbb{R}.$$

So, $\overline{\mathcal{T}}(\overline{\Omega}) \subset \overline{\Omega}$. It is easy to verify that $\overline{\mathcal{T}}$ is continuous with respect to the norm $\|\cdot\|_{\overline{\mu}}$ on $\overline{\Omega}$ by making use of a similar argument as in the proof of Theorem 1. By virtue of Schauder's Fixed Point Theorem, operator $\overline{\mathcal{T}}$ has a fixed point $V(\xi)$ on $\overline{\Omega}$, which is the traveling wave solution of Eq. (12) and satisfies

$$V_-(\xi) \le V(\xi) \le V_+(\xi + \overline{\xi}), \quad \xi \in \mathbb{R}.$$

Furthermore, by Lemma 5, we have $V(-\infty) = 0$,

$$\begin{aligned}(1-b)K_1 &= \lim_{\xi\to+\infty} V_-(\xi) \le \liminf_{\xi\to+\infty} V(\xi) \le \limsup_{\xi\to+\infty} V(\xi) \\ &\le \lim_{\xi\to+\infty} V_+(\xi+\overline{\xi}) \le (1-b)K_2,\end{aligned}$$

and for any $c > c_*$,

$$\begin{aligned}1 &= \lim_{\xi\to-\infty} V_-(\xi)e^{-\lambda_1\xi} \le \liminf_{\xi\to-\infty} V(\xi)e^{-\lambda_1\xi} \le \limsup_{\xi\to-\infty} V(\xi)e^{-\lambda_1\xi} \\ &\le \lim_{\xi\to-\infty} V_+(\xi+\overline{\xi})e^{-\lambda_1(\xi+\overline{\xi})}e^{\lambda_1\overline{\xi}} = e^{\lambda_1\overline{\xi}}.\end{aligned}$$

□

By a similar proof as for Theorem 2, we obtain a theorem as follows.

Theorem 4 *Assume that* (H2), (H4), (A1)–(A5) *hold. Then for any* $c \ge c_*$, *there exists a traveling wave solution* $U(\xi)$ *of Eq.* (1) *with* $U(-\infty) = 0$,

$$K_1 \le \liminf_{\xi\to+\infty} U(\xi) \le \limsup_{\xi\to+\infty} U(\xi) \le K_2,$$

and for any $c > c_*$,

$$\frac{1}{1-be^{-\lambda_1 cr}} \le \liminf_{\xi\to-\infty} U(\xi)e^{-\lambda_1\xi} \le \limsup_{\xi\to-\infty} U(\xi)e^{-\lambda_1\xi} \le \frac{e^{\lambda_1\overline{\xi}}}{1-be^{-\lambda_1 cr}}.$$

4 Applications and discussions

In this section, as an application of Theorem 4, we firstly consider the following partial functional differential equation of neutral type:

$$\frac{\partial}{\partial t}L(u)(t,x) = D\frac{\partial^2}{\partial x^2}L(u)(t,x) - g\big(L(u)(t,x)\big) + h\big(u(t,x)\big)f\big(u(t-r,x)\big). \tag{29}$$

We need the following assumptions:

(a1) $g(0) = f(0) = 0$; $g((1-b)K) = h(K)f(K)$ for some $K > 0$;

(a2) There exists $K_2 \ge K$ such that $0 < f(u) \le g((1-b)K_2)/h(K_2)$, $u \in (0, K_2]$; $g((1-b)u) \le g((1-b)K_2)/h(K_2)$, $u \in (0, K_2)$;

(a3) $h(0)f'(0) > (1-b)g'(0) > 0$; $f'(0)u \ge f(u)$, $g(u) \ge g'(0)u$, $h(0) \ge h(u) > 0$ and $f'(0)u > g((1-b)u)/h(u)$, $u \in (0, K_2)$;

(a4) There exists $\sigma \in (0, 1]$ such that

$$\limsup_{u\to0^+}\big[f'(0) - f(u)/u\big]u^{-\sigma} < +\infty; \qquad \limsup_{u\to0^+}\big[g(u)/u - g'(0)\big]u^{-\sigma} < +\infty;$$

and

$$\limsup_{u\to 0^+}\big[h(0)-h(u)\big]u^{-(1+\sigma)}<+\infty;$$

(a5) g, h and f are Lipschitz continuous on $[0,K_2]$.

Let

$$K_1=\inf\Big\{u\Big|g\big((1-b)u\big)/h(u)=\inf_{s\in(0,K_2]}\big\{f(s)|f(s)\le g\big((1-b)s\big)/h(s)\big\}\Big\}.$$

From the conditions $f'(0)>(1-b)g'(0)/h(0)$ and $f(u)>0$, $u\in(0,K_2]$ in (a3), K_1 is well defined and $0<K_1\le K$ and $f(u)\ge g((1-b)u)$, $u\in(0,K_1)$.

Define the following functions:

$$f_1(u)=\begin{cases}\min\{\min_{s\in[u,K_2]}f(s),g((1-b)K_1)/h(K_1)\}, & u\in[u,K_1],\\ \min\{f(u),g((1-b)K_1)/h(K_1)\}, & u\in[K_1,+\infty);\end{cases}$$

and

$$f_2(u)=\begin{cases}\min\{f'(0)u,g((1-b)K_2)/h(K_2)\}, & u\in[u,K_2],\\ \max\{f(u),g((1-b)K_2)/h(K_2)\}, & u\in[K_2,+\infty),\end{cases}$$

then f_1 and f_2 have the following properties:

Lemma 7 *Assume* (a1)–(a5) *hold. Then*

(1) f_n $(n=1,2)$ *is Lipschitz continuous on* $[0,+\infty)$ *and non-decreasing on* $[0,K_2]$;

(2) $f_1(u)\le f(u)\le f_2(u)$, $u\in[0,+\infty)$;

(3) $f_n(0)=0, f_n(K_n)=g((1-b)K_n)/h(K_n)$ *and* $f_n(u)>g((1-b)u)/h(u)$, $u\in(0,K_n)$, $n=1,2$;

(4) $f_n'(0)=f'(0)$, $n=1,2$;

(5) $\limsup_{u\to0^+}[f_n'(0)-f_n(u)/u]u^{-\sigma}<+\infty$.

Set

$$F(s_1,s_2,s_3)=-g(s_1)+h(s_2)f(s_3),$$

and

$$F_+(s_1,s_2,s_3)=-g(s_1)+h(s_2)f_2(s_3),\qquad F_-(s_1,s_2,s_3)=-g(s_1)+h(s_2)f_1(s_3).$$

From assumptions (a3)–(a5) and Lemma 7, it is easy to verify that F and $F_\pm$ satisfy assumptions (H2), (H4), and (A1)–(A5).

Let

$$q(\lambda,c)=D\lambda^2-c\lambda-g'(0)+\frac{h(0)f'(0)e^{-\lambda cr}}{1-be^{-\lambda cr}},$$

and assume c_* is defined as in (15). Then from the Theorem 4, we have

Theorem 5 *Assume that* (a1)–(a5) *hold. Then for any* $c \geq c_*$, *there exists a traveling wave solution* $U(\xi)$ *of* (29) *with* $U(-\infty) = 0$,

$$K_1 \leq \liminf_{\xi \to +\infty} U(\xi) \leq \limsup_{\xi \to +\infty} U(\xi) \leq K_2,$$

and there exists some $\overline{\xi} \in \mathbb{R}$ *such that, for any* $c > c_*$,

$$\frac{1}{1 - be^{-\lambda_1 cr}} \leq \liminf_{\xi \to -\infty} U(\xi)e^{-\lambda_1 \xi} \leq \limsup_{\xi \to -\infty} U(\xi)e^{-\lambda_1 \xi} \leq \frac{e^{\lambda_1 \overline{\xi}}}{1 - be^{-\lambda_1 cr}},$$

where λ_1 *is the smaller root of* $q(\lambda, c) = 0$.

As another application example of Theorem 4, we consider Eq. (10) and state it again as follows for convenience of reading:

$$\begin{aligned} \frac{\partial}{\partial t} L(u)(t,x) = {} & D \frac{\partial^2}{\partial x^2} L(u)(t,x) - aL(u)(t,x) \\ & + \frac{p}{2} \bigl[u(t,x)e^{-u(t,x)} + u(t-r,x)e^{-u(t-r,x)} \bigr], \end{aligned} \tag{30}$$

where $p > a > 0$, $0 \leq b < 1$, $\frac{p}{a(1-b)} > e$. It is obvious that Eq. (30) has exactly two equilibria $u = 0$ and $u = K = \ln \frac{p}{a(1-b)} > 0$.

Let $F(s_1, s_2, s_3) = -as_1 + \frac{p}{2}[s_2 e^{-s_2} + s_3 e^{-s_3}]$, $K_2 = \frac{p}{a(1-b)}$, then $K_2 > K$, and $F(s_1, s_2, s_3)$ is non-monotone on $s_2, s_3 \in [0, K_2]$.

Set $f(s) = \frac{p}{2} se^{-s}$, and choose $\overline{K} \in (0, K)$, such that $f(\overline{K}) = \frac{p}{2} K_2 e^{-K_2} = \frac{p^2}{2a(1-b)} e^{-\frac{p}{a(1-b)}}$. Let $K_1 = \frac{p^2}{a^2(1-b)^2} e^{-\frac{p}{a(1-b)}}$, then $\overline{K} < K_1 < K$.

Define

$$F_+(s_1, s_2, s_3) = \begin{cases} -as_1 + \frac{p}{2}[s_2 + s_3], & s_1, s_2, s_3 \in [0, 1], \\ -as_1 + p, & s_1, s_2, s_3 \in (1, +\infty), \end{cases}$$

and

$$F_-(s_1, s_2, s_3) = \begin{cases} -as_1 + \frac{p}{2}[s_2 e^{-s_2} + s_3 e^{-s_3}], & s_1, s_2, s_3 \in [0, \overline{K}], \\ -as_1 + \frac{p^2}{a(1-b)} e^{-\frac{p}{a(1-b)}}, & s_1, s_2, s_3 \in (\overline{K}, +\infty). \end{cases}$$

Then it is easy to verify that F and $F_\pm$ satisfy assumptions (H2), (H4), and (A1)–(A5).

Let

$$q(\lambda, c) = D\lambda^2 - c\lambda - a + \frac{p}{2(1 - be^{-\lambda cr})} \bigl(1 + e^{-\lambda cr} \bigr),$$

and suppose c_* is defined as in (15). Then from Theorem 4, we have

Theorem 6 *For any* $c \geq c_*$, *there exists a traveling wave solution* $U(\xi)$ *of* (30) *with* $U(-\infty) = 0$,

$$\frac{p^2}{a^2(1-b)^2} e^{-\frac{p}{a(1-b)}} \leq \liminf_{\xi \to +\infty} U(\xi) \leq \limsup_{\xi \to +\infty} U(\xi) \leq \frac{p}{a(1-b)},$$

and there exists some $\overline{\xi} \in \mathbb{R}$ *such that, for any* $c > c_*$,

$$\frac{1}{1 - be^{-\lambda_1 cr}} \le \liminf_{\xi \to -\infty} U(\xi)e^{-\lambda_1 \xi} \le \limsup_{\xi \to -\infty} U(\xi)e^{-\lambda_1 \xi} \le \frac{e^{\lambda_1 \overline{\xi}}}{1 - be^{-\lambda_1 cr}},$$

where λ_1 *is the smaller root of* $q(\lambda, c) = 0$.

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