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Oscillation theorems for three classes of conformable fractional differential equations

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Abstract

In this paper, we consider the oscillation theory for fractional differential equations. We obtain oscillation criteria for three classes of fractional differential equations of the forms

$$T_\alpha^{t_0} x(t) + \sum_{i=1}^{m} p_i(t) x(\tau_i(t)) = 0, \quad t \ge t_0,$$

$$T_\alpha^{t_0}\big(r(t)(T_\alpha^{t_0}(x(t) + p(t)x(\tau(t))))^\beta\big) + q(t)x^\beta(\sigma(t)) = 0, \quad t \ge t_0,$$

and

$$T_\alpha^{t_0}\big(r_2 T_\alpha^{t_0}\big(r_1 (T_\alpha^{t_0} y)^\beta\big)\big)(t) + p(t)\big(T_\alpha^{t_0} y(t)\big)^\beta + q(t) f\big(y(g(t))\big) = 0, \quad t \ge t_0,$$

where T_α denotes the conformable differential operator of order α, $0 < \alpha \le 1$.

MSC: 34C10; 26A33; 65Q10

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1 Introduction

Fractional differential equations have been of great interest recently. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in self-similar and porous structures, fluid flows, electrical networks, chemical physics, and many other branches of science.

The oscillation of fractional differential equations as a new research field has received significant attention, and some interesting results have already been obtained. We refer to [1–11] and the references therein. The definition of the fractional-order derivative used is either the Caputo or the Riemann–Liouville fractional-order derivative involving an integral expression and the gamma function. Because of the definition, the oscillation of these types of fractional equations cannot be studied by regular methods, for example, by the Riccati transformation. It can only be studied by transforming it into an integer-order equation. In 2012, Chen et al. [4] studied the oscillation behavior of the following fractional differential equation:

$$\big[r(t)\big(D_-^\alpha\big)\eta(t)\big]' - q(t) f\left(\int_t^\infty (v-t)^{-\alpha} y(v)\,dv\right) = 0 \quad \text{for } t > 0,$$

where $D_-^\alpha y$ denotes the Liouville right-sided fractional derivative of order α,

$$\bigl(D_-^\alpha y\bigr)(t) := -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty (\nu - t)^{-\alpha} y(\nu)\,d\nu \quad \text{for } t \in \mathbb{R}_+ := (0,\infty).$$

By the Riccati transformation the authors obtained some sufficient conditions.

Recently, Khalil et al. [12] introduced a new well-behaved definition of local fractional derivative, called the conformable fractional derivative, depending just on the basic limit definition of the derivative. This new theory is improved by Abdeljawad [13]. For recent results on conformable fractional derivatives, we refer the reader to [14–23]. This new definition satisfies formulas of the derivatives of the product and quotient of two functions and has a simpler chain rule. In addition to the definition of conformable fractional derivative, a definition of conformable fractional integral, the Rolle theorem, and the mean value theorem for conformable fractional differentiable functions were given. These properties are more conducive to the study of the oscillation of fractional-order equations.

In fact, some works in this field have shown the significance of conformable fractional derivative. For example, [24] discusses the potential conformable quantum mechanics, [25] discusses the conformable Maxwell equations, and [26, 27] show that the conformable fractional derivative models present good agreements with experimental data, but there are less oscillation results.

In the paper, we study oscillation criteria of conformable fractional differential equations. Our main goal is to generalize the oscillatory criteria in [28–37] to the conformable fractional derivative. The three equations represent three classes of equations of different orders. For example, in 2016, Akca et al. [33] studied the equation

$$x'(t) + \sum_{i=1}^m p_i(t)x\bigl(\tau_i(t)\bigr) = 0, \quad t \geq 0,$$

and obtained the following:

Theorem 1.1 *Assume that* $0 < \varsigma := \liminf_{t\to\infty}\int_{\tau(t)}^t \sum_{i=1}^m p_i(s)\,ds \leq \frac{1}{e}$ *and for some* $r \in \mathbb{N}$, *we have*

$$\limsup_{t\to\infty}\int_{h(t)}^t (\zeta - t_0)^{\alpha-1}\sum_{i=1}^m p_i(\zeta)a_r\bigl(h(t),\tau_i(\zeta)\bigr)\,d\zeta > \frac{1+\ln\lambda_0}{\lambda_0},$$

where $h(t) = \max_{1\leq i\leq m} h_i(t)$, $h_i(t) = \sup_{0\leq s\leq t}\tau_i(s)$, $a_1(t,s) := \exp\{\int_s^t \sum_{i=1}^m p_i(\zeta)\,d\zeta\}$, $a_{r+1}(t,s) := \exp\{\int_s^t \sum_{i=1}^m p_i(\zeta)a_r(\zeta,\tau_i(\zeta))\,d\zeta\}$, *and* λ_0 *is the smaller root of the equation* $e^{\varsigma\lambda} = \lambda$. *Then the above equation oscillates.*

From this we can unify the oscillation theory of integral-order and fractional-order differential equations. Through the inequality principle, iterative sequences, and the Riccati transformation method this can be extended to the conformable fractional derivatives by Lemma 2.2.

A solution x is called oscillatory if it is eventually neither positive nor negative. Otherwise, the solution is said to be nonoscillatory. An equation is oscillatory if all its solutions oscillate. In this paper, x is differentiable on $[t_0,\infty)$. This paper is organized as follows. In

Sect. 2, we introduce some notation and definitions on conformable fractional integrals. In Sect. 3, we present the main theorems on α-order equations. Section 4 is devoted to the oscillatory results on 2α-order equation. In Sect. 5, we demonstrate the oscillatory results for 3α-order equations. In each section, we give examples to illustrate the significance of the results.

2 Conformable fractional calculus

For the convenience of the reader, we give some background from fractional calculus theory. These materials can be found in the recent literature, see [12, 13, 23].

Definition 2.1 ([13]) The (left) fractional derivative of a function $f : [a,\infty) \to R$ of order $\alpha \in (0,1]$ starting from a is defined by

$$\bigl(T_\alpha^a f\bigr)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}.$$

When $a = 0$, we write T_α.

Note that if f is differentiable, then $(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(t)$.

Definition 2.2 ([13]) The left fractional integral of order $\alpha \in (0,1]$ starting at a is defined by

$$\bigl(I_\alpha^a f\bigr)(t) = \int_a^t f(x)\,d\alpha(x,a) = \int_a^t (x-a)^{\alpha-1} f(x)\,dx.$$

Definition 2.3 ([13]) Let $f : [a,\infty) \to \mathbb{R}$ be a continuous function, and let $\alpha \in (0,1]$. Then, for all $t > a$, we have

$$T_\alpha^a I_\alpha^a f(t) = f(t).$$

Definition 2.4 ([13]) Let $f : (a,b) \to \mathbb{R}$ be let a differentiable function, and let $\alpha \in (0,1]$. Then, for all $t > a$, we have

$$I_\alpha^a T_\alpha^a (f)(t) = f(t) - f(a).$$

Proposition 2.1 ([13]) *Let $f : (a,\infty) \to \infty \to \mathbb{R}$ be a twice differentiable function, and let $0 < \alpha, \beta \le 1$ be such that $1 < \alpha + \beta \le 2$. Then*

$$\bigl(T_\alpha^a T_\beta^a\bigr)(t) = T_{\alpha+\beta}^a f(t) + (1-\beta)(t-a)^{-\beta} T_\alpha^a f(t).$$

Proposition 2.2 ([23]) *Let $\alpha \in (0,1]$, and let f and g be α-differentiable at a point $t > 0$ on $[a,\infty)$. Then*

(1) $T_\alpha^a(af + bg) = aT_\alpha^a(f) + bT_\alpha^a(g)$ *for all* $a, b \in \mathbb{R}$,
(2) $T_\alpha^a(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$,
(3) $T_\alpha^a(fg) = fT_\alpha^a(g) + gT_\alpha^a(f)$,
(4) $T_\alpha^a(\frac{f}{g}) = \frac{gT_\alpha^a(f) - fT_\alpha^a(g)}{g^2}$,

(5) $T_\alpha^a(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$, *and*
(6) $T_\alpha^a(f \circ g)(t) = f'(g(t))T_\alpha^a(g)(t)$ *for* f *differentiable at* $g(t)$.

Lemma 2.1 ([13]) *Let* $f, g : [a,b] \to \mathbb{R}$ *be two functions such that* fg *is differentiable, and let* $\alpha \in (0,1]$. *Then*

$$\int_a^b f(x)T_\alpha^a(g)(x)\,d\alpha(x,a) = fg\Big|_a^b - \int_a^b g(x)T_\alpha^a(f)(x)\,d\alpha(x,a).$$

Lemma 2.2 *Let* $f : (t_0,\infty) \to \mathbb{R}$ *be differentiable, and let* $\alpha \in (0,1]$. *If* $T_\alpha^{t_0} f(t) = M(t)$, *then for all* $t > s > t_0$, *we have*

$$f(t) - f(s) = I_\alpha^{t_0} M(t).$$

Proof We can conclude from $T_\alpha^{t_0} f(t) = M(t)$ that

$$\left(\frac{t-t_0}{t-s}\right)^{1-\alpha} T_\alpha^s f(t) = M(t),$$

that is,

$$T_\alpha^s f(t) = \left(\frac{t-t_0}{t-s}\right)^{\alpha-1} M(t).$$

Then applying I_α to the latter from s to t, we have

$$I_\alpha^s T_\alpha^s f(t) = I_\alpha^s\left[\left(\frac{t-t_0}{t-s}\right)^{\alpha-1} M(t)\right],$$

that is,

$$f(t) - f(s) = I_\alpha^{t_0} M(t).$$

The proof of Lemma 2.2 is complete. □

3 α-Order conformable fractional differential equations with finite nonmonotone delay arguments

In this section, we deal with the differential equations of the form

$$T_\alpha^{t_0} x(t) + \sum_{i=1}^m p_i(t)x\bigl(\tau_i(t)\bigr) = 0, \quad t \ge t_0, \tag{3.1}$$

where T_α denotes the conformable differential operator of order $\alpha \in (0,1]$, $p_i(t)$, $1 \le i \le m$, are nonnegative functions, $\tau_i(t)$, $1 \le i \le m$, are nonmonotone functions of positive real numbers such that

$$\tau_i(t) \le t, \qquad t \ge t_0, \qquad \lim_{t\to\infty} \tau_i(t) = \infty, \qquad 1 \le i \le m.$$

To prove our main results, we establish some fundamental results in this section.

Lemma 3.1 *Assume that $x(t)$ is an eventually positive solution of* (3.1) *and $a_r(t,s)$, $r \in \mathbb{N}^+$, is defined as*

$$a_1(t,s) = \exp\left\{\int_s^t (\zeta - t_0)^{\alpha-1} \sum_{i=1}^m p_i(\zeta)\,d\zeta\right\},$$
$$a_{r+1}(t,s) = \exp\left\{\int_s^t (\zeta - t_0)^{\alpha-1} \sum_{i=1}^m p_i(\zeta) a_r\bigl(\zeta, \tau_i(\zeta)\bigr)\right\}. \tag{3.2}$$

Then

$$x(t)a_r(t,s) \le x(s), \quad 0 \le s \le t, r \in \mathbb{N}^+. \tag{3.3}$$

Proof Let $x(t)$ be an eventually positive solution of equation (3.1). Then there exists $t_1 > t_0$ such that $x(t) > 0$ and $x(\tau_i(t)) > 0$, $1 \le i \le m$, for all $t \ge t_1$, so

$$T_\alpha^{t_0} x(t) = -\sum_{i=1}^m p_i(t) x\bigl(\tau_i(t)\bigr) \le 0, \quad t \ge t_1.$$

This means that $x(t)$ is monotonically decreasing, that is, $x(\tau_i(t)) \ge x(t)$, $1 \le i \le m$, and it is easy to put it into the original equation:

$$T_\alpha^{t_0} x(t) + x(t) \sum_{i=1}^m p_i(t) \le 0, \quad t \ge t_1.$$

Dividing this equation by $x(t)$, we get

$$\frac{T_\alpha^{t_0} x(t)}{x(t)} \le -\sum_{i=1}^m p_i(t), \quad t \ge t_1,$$

that is,

$$(t - t_0)^{1-\alpha} \frac{x'(t)}{x(t)} \le -\sum_{i=1}^m p_i(t), \quad t \ge t_1.$$

Integrating the last inequality from s to t, $0 \le s \le t$, we get

$$\ln x(\zeta)\Big|_s^t \le \int_s^t \left((\zeta - t_0)^{\alpha-1}\left(-\sum_{i=1}^m p_i(\zeta)\right)\right) d\zeta,$$

that is,

$$\ln x(t) \le \ln x(s) - \int_s^t (\zeta - t_0)^{\alpha-1} \sum_{i=1}^m p_i(\zeta)\,d\zeta.$$

So

$$x(s) \ge x(t) \exp\left\{\int_s^t (\zeta - t_0)^{\alpha-1} \sum_{i=1}^m p_i(\zeta)\,d\zeta\right\},$$

that is, estimate (3.3) is valid for $r = 1$. Supposing that (3.3) is established for $r = n$, we obtain

$$x(t)a_n(t,s) \le x(s),$$

so

$$T_\alpha^{t_0} x(t) + \sum_{i=1}^{m} p_i(t)x(t)a_n\bigl(t,\tau_i(t)\bigr) \le 0.$$

Repeating these steps can, we obtain

$$x(s) \ge x(t)\exp\left\{\int_s^t (\zeta - t_0)^{\alpha-1}\sum_{i=1}^{m} p_i(\zeta)a_n\bigl(\zeta,\tau_i(\zeta)\bigr)\,d\zeta\right\},$$

that is, $x(t)a_{n+1}(t,s) \le x(s)$. So Lemma 3.1 is proved by mathematical induction. □

Lemma 3.2 *Assume that $x(t)$ is an eventually positive solution of* (3.1) *and*

$$0 < \beta := \liminf_{t\to\infty}\int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1}\sum_{i=1}^{m} p_i(\zeta)\,d\zeta \le \frac{1}{e}, \tag{3.4}$$

where

$$h(t) = \max_{1\le i\le m} h_i(t), \qquad h_i(t) = \max_{0\le s\le t}\tau_i(s), \quad t \ge 0. \tag{3.5}$$

Then

$$\gamma = \liminf_{t\to\infty}\frac{x(h(t))}{x(t)} \ge \lambda_0, \tag{3.6}$$

where λ_0 is the smaller root of the equation $\lambda = e^{\beta\lambda}$.

Proof Let $x(t)$ be an eventually positive solution of equation (3.1). Then there exists $t_1 > t_0$ such that $x(t) > 0$ and $x(\tau_i(t)) > 0$, $1 \le i \le m$, for all $t \ge t_1$. Thus we can conclude from (3.1) that

$$T_\alpha^{t_0} x(t) = -\sum_{i=1}^{m} p_i(t)x\bigl(\tau_i(t)\bigr) \le 0, \quad t \ge t_1.$$

This means that $x(t)$ is monotonically decreasing and positive.

By (3.4), for any $\varepsilon \in (0,\beta)$, there is t_ε such that

$$\int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1}\sum_{i=1}^{m} p_i(\zeta)\,d\zeta \ge \beta - \varepsilon, \quad t \ge t_\varepsilon \ge t_1.$$

We will show that

$$\liminf_{t\to\infty}\frac{x(h(t))}{x(t)} \ge \lambda_1, \tag{3.7}$$

where λ_1 is the smaller root of the equation

$$e^{(\beta-\varepsilon)\lambda} = \lambda.$$

For contradiction, we assume that

$$\gamma = \liminf_{t\to\infty} \frac{x(h(t))}{x(t)} < \lambda_1.$$

Therefore

$$e^{(\beta-\varepsilon)\gamma} > \gamma. \tag{3.8}$$

Then for any $\delta \in (0,\gamma)$, there exists t_δ such that $\frac{x(h(t))}{x(t)} \geq \gamma - \delta$ for $t \geq t_\delta$. Dividing both sides of (3.1) by $x(t)$, we have

$$-\frac{T_\alpha^{t_0} x(t)}{x(t)} = \sum_{i=1}^{m} p_i(t) \frac{x(\tau_i(t))}{x(t)} \geq \sum_{i=1}^{m} p_i(t) \frac{x(h(t))}{x(t)} \geq (\gamma - \delta) \sum_{i=1}^{m} p_i(t).$$

Integrating the latter from $h(t)$ to t, we obtain

$$-\int_{h(t)}^{t} \frac{x'(s)}{x(s)}\,ds \geq \int_{h(t)}^{t} (s-t_0)^{\alpha-1} \left((\gamma-\delta) \sum_{i=1}^{m} p_i(s) \right) ds,$$

or

$$-\int_{h(t)}^{t} \frac{x'(s)}{x(s)}\,ds \geq (\gamma-\delta)(\beta-\varepsilon),$$

so

$$\frac{x(h(t))}{x(t)} \geq e^{(\gamma-\delta)(\beta-\varepsilon)}.$$

Therefore

$$\gamma = \liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \geq e^{(\gamma-\delta)(\beta-\varepsilon)},$$

which implies

$$\gamma \geq e^{(\beta-\varepsilon)\gamma},$$

which is a contradiction to hypothesis (3.8). So (3.7) is true. Since (3.7) implies (3.6), the proof of Lemma 3.2 is complete. □

Theorem 3.1 *Assume that* (3.4) *holds and for some* r, *we have*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} (\zeta-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(h(t),\tau_i(\zeta)\bigr)\,d\zeta > \frac{1+\ln\lambda_0}{\lambda_0}, \tag{3.9}$$

where $h(t)$ is defined by (3.5), $a_r(t,s)$ is defined by (3.2), and λ_0 is the smaller root of the equation $e^{\beta\lambda} = \lambda$. Then equation (3.1) oscillates.

Proof If equation (3.1) has a solution $x(t)$, then $-x(t)$ is also a solution of equation (3.1), so we only consider the situation where a solution of (3.1) is eventually positive, that is, there is an integer $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau_i(t)) > 0$, $1 \leq i \leq m$, for all $t \geq t_1$. By (3.1) we have

$$T_\alpha^{t_0} x(t) = -\sum_{i=1}^{m} p_i(t) x\bigl(\tau_i(t)\bigr) \leq 0, \quad t \geq t_1.$$

It is shown that $x(t)$ is an eventually decreasing function.

By Lemma 3.2 inequality (3.6) holds. It can be easily seen that $\lambda_0 > 1$, so for any real number $0 < \varepsilon \leq \lambda_0 - 1$, we have

$$\frac{x(h(t))}{x(t)} \geq \lambda_0 - \varepsilon, \quad t \geq t_2 \geq t_1.$$

Then there is $t^* \in (h(t), t)$ satisfying

$$\frac{x(h(t))}{x(t^*)} = \lambda_0 - \varepsilon, \quad t \geq t_2. \tag{3.10}$$

Then integrating from t^* to t equation (3.1) and substituting into (3.3), we have

$$x(t) - x\bigl(t^*\bigr) + x\bigl(h(t)\bigr) \int_{t^*}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(h(t), \tau_i(\zeta)\bigr)\, d\zeta \leq 0.$$

Combining this with (3.10), we have

$$\int_{t^*}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(h(t), \tau_i(\zeta)\bigr)\, d\zeta \leq \frac{x(t^*)}{x(h(t))} = \frac{1}{\lambda_0 - \varepsilon}. \tag{3.11}$$

Dividing (3.1) by $x(t)$, substituting into (3.3), and then integrating from $h(t)$ to t^*, we have

$$-\int_{h(t)}^{t^*} \frac{x'(\zeta)}{x(\zeta)}\, d\zeta \geq \int_{h(t)}^{t^*} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) \frac{x(h(t))}{x(\zeta)} a_r\bigl(h(t), \tau_i(\zeta)\bigr)\, d\zeta,$$

and because of $T_\alpha^{t_0} x(t) < 0$, we have

$$\begin{aligned}
&(\lambda_0 - \varepsilon) \int_{h(t)}^{t^*} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(h(t), \tau_i(\zeta)\bigr)\, d\zeta \\
&\quad \leq \int_{h(t)}^{t^*} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) \frac{x(h(t))}{x(t)} a_r\bigl(h(t), \tau_i(\zeta)\bigr)\, d\zeta \\
&\quad \leq -\int_{h(t)}^{t^*} \frac{x'(\zeta)}{x(\zeta)}\, d\zeta,
\end{aligned}$$

that is,

$$\int_{h(t)}^{t^*} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\big(h(t), \tau_i(\zeta)\big)\, d\zeta \le \frac{1}{\lambda_0 - \varepsilon} \ln \frac{x(h(t))}{x(t^*)}. \tag{3.12}$$

Adding (3.12) to (3.11), we get

$$\int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\big(h(t), \tau_i(\zeta)\big)\, d\zeta \le \frac{1 + \ln(\lambda_0 - \varepsilon)}{\lambda_0 - \varepsilon}.$$

This inequality holds for all $0 < \varepsilon \le \lambda_0 - 1$, so as $\varepsilon \to 0$, we obtain

$$\limsup_{t\to\infty} \int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\big(h(t), \tau_i(\zeta)\big)\, d\zeta \le \frac{1 + \ln \lambda_0}{\lambda_0}.$$

This is a contradiction to (3.9). The proof of Theorem 3.1 is complete. □

Lemma 3.3 *Assume that $x(t)$ is an eventually positive solution of (3.1) and that β and $h(t)$ are defined by (3.4) and (3.5). Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \ge \frac{1}{2}\big(1 - \beta - \sqrt{1 - 2\beta - \beta^2}\big) := A(\beta). \tag{3.13}$$

Proof Assume that $x(t) > 0$ for $t > T_1 \ge t_0$. Then there exists $T_2 \ge T_1$ such that $x(\tau_i(t)) > 0$, $i = 1, 2, \ldots, m$. In view of (3.1), $T_\alpha^{t_0} x(t) \le 0$ on $[T_2, \infty)$. Clearly, (3.13) holds for $\beta = 0$. If $0 < \beta \le \frac{1}{e}$, then for any $\varepsilon \in (0, \beta)$, there exists N_ε such that

$$\int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta)\, d\zeta > \beta - \varepsilon, \quad t > N_\varepsilon. \tag{3.14}$$

For fixed ε, we will show that for each $t > N_\varepsilon$, there exists λ_t such that $h(\lambda_t) < t < \lambda_t$ and

$$\int_{t}^{\lambda_t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta)\, d\zeta = \beta - \varepsilon. \tag{3.15}$$

In fact, for a given $t > N_\varepsilon$, $f(\lambda) := \int_t^\lambda (\zeta - t_0)^{\alpha-1} \sum_{i=1}^m p_i(\zeta)\, d\zeta$ is continuous. Because of $\lim_{t\to\infty} h(t) = \infty$ and (3.14), we have $\lim_{\lambda\to\infty} f(\lambda) > \beta - \varepsilon > 0$. Hence there exists $\lambda_t > t$ such that $f(\lambda) = \beta - \varepsilon$, that is, (3.15) holds. From (3.14) we have

$$\int_{h(\lambda_t)}^{\lambda_t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta)\, d\zeta > \beta - \varepsilon = \int_{t}^{\lambda_t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta)\, d\zeta,$$

and therefore $h(\lambda_t) < t$.

Integrating (3.1) from t $(> T_3 = \max\{T_2, N_\varepsilon\})$ to λ_t, we have

$$x(t) - x(\lambda_t) \ge \int_{t}^{\lambda_t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) x\big(\tau_i(\zeta)\big)\, d\zeta. \tag{3.16}$$

We see that $h(t) \le h(\zeta) \le h(\lambda_t) < t$ for $t \le y \le \lambda_t$. Integrating (3.1) from $\tau_i(\zeta)$ to t, we have that for $t \le \zeta \le \lambda_t$,

$$\begin{aligned} &x\bigl(\tau_i(\zeta)\bigr) - x(t) \\ &\quad\ge \int_{\tau_i(\zeta)}^{t} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u) x\bigl(\tau_i(u)\bigr)\,du \\ &\quad\ge x\bigl(h(t)\bigr) \int_{\tau_i(\zeta)}^{t} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du \\ &\quad> x\bigl(h(t)\bigr) \biggl(\int_{h(\zeta)}^{\zeta} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du - \int_{t}^{\zeta} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du \biggr) \\ &\quad> x\bigl(h(t)\bigr) \biggl((\beta-\varepsilon) - \int_{t}^{\zeta} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du \biggr). \end{aligned} \tag{3.17}$$

From (3.16) and (3.17) we have

$$\begin{aligned} x(t) &\ge x(\lambda_t) + \int_{t}^{\lambda_t} (\zeta-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) x\bigl(\tau_i(\zeta)\bigr)\,d\zeta \\ &> x(\lambda_t) + \int_{t}^{\lambda_t} (\zeta-t_0)^{\alpha-1} \\ &\quad\times \sum_{i=1}^{m} p_i(\zeta) \biggl[x(t) + x\bigl(h(t)\bigr) \biggl((\beta-\varepsilon) - \int_{t}^{\zeta} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du \biggr) \biggr]\,d\zeta \\ &= x(\lambda_t) + x(t)(\beta-\varepsilon) \\ &\quad+ x\bigl(h(t)\bigr) \biggl[(\beta-\varepsilon)^2 \\ &\quad- \int_{t}^{\lambda_t} (\zeta-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) \int_{t}^{\zeta} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du \biggr]\,d\zeta. \end{aligned} \tag{3.18}$$

Noting the known formula

$$\begin{aligned} &\int_{t}^{\lambda_t} (\zeta-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) \int_{t}^{\zeta} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du\,d\zeta \\ &\quad= \int_{t}^{\lambda_t} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u) \int_{u}^{\lambda_t} (\zeta-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta)\,d\zeta\,du, \end{aligned}$$

or

$$\begin{aligned} &\int_{t}^{\lambda_t} (\zeta-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) \int_{t}^{\zeta} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du\,d\zeta \\ &\quad= \int_{t}^{\lambda_t} (\zeta-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) \int_{\zeta}^{\lambda_t} (u-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(u)\,du\,d\zeta, \end{aligned}$$

we have

$$\begin{aligned}
&\int_t^{\lambda_t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^m p_i(\zeta) \int_t^{\zeta} (u - t_0)^{\alpha-1} \sum_{i=1}^m p_i(u)\, du\, d\zeta \\
&= \frac{1}{2} \int_t^{\lambda_t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^m p_i(\zeta) \int_t^{\lambda_t} (u - t_0)^{\alpha-1} \sum_{i=1}^m p_i(u)\, du\, d\zeta \\
&= \frac{1}{2} \left[\int_t^{\lambda_t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^m p_i(\zeta)\, d\zeta \right]^2 \\
&= \frac{1}{2} (\beta - \varepsilon)^2.
\end{aligned}$$

Substituting this into (3.18), we have

$$x(t) > x(\lambda_t) + x(t)(\beta - \varepsilon) + \frac{1}{2}(\beta - \varepsilon)^2 x\bigl(h(t)\bigr). \tag{3.19}$$

Hence

$$\frac{x(t)}{x(h(t))} > \frac{(\beta - \varepsilon)^2}{2(1 - \beta + \varepsilon)} := d_1,$$

and then

$$x(\lambda_t) > \frac{(\beta - \varepsilon)^2}{2(1 - \beta + \varepsilon)} x\bigl(h(\lambda_t)\bigr) = d_1 x\bigl(h(\lambda_t)\bigr) \geq d_1 x(t).$$

Substituting this into (3.19), we obtain

$$x(t) > x(t)(m + d_1 - \varepsilon) + \frac{1}{2}(m - \varepsilon)^2 x\bigl(h(t)\bigr),$$

and hence

$$\frac{x(t)}{x(h(t))} > \frac{(\beta - \varepsilon)^2}{2(1 - \beta - d_1 + \varepsilon)} := d_2.$$

In general, we have

$$\frac{x(t)}{x(h(t))} > \frac{(\beta - \varepsilon)^2}{2(1 - \beta - d_n + \varepsilon)} := d_{n+1}, \quad n = 1, 2, \ldots.$$

It is not difficult to see that if ε is small enough, then $1 \geq d_n > d_{n-1}$, $n = 2, 3, \ldots$. Hence $\lim_{n\to\infty} d_n = d$ exists and satisfies

$$-2d^2 + 2d(1 - \beta + \varepsilon) = (\beta - \varepsilon)^2,$$

that is,

$$d = \frac{1 - \beta + \varepsilon \pm \sqrt{1 - 2(\beta - \varepsilon) - (\beta - \varepsilon)^2}}{2}.$$

Because of $T_\alpha^{t_0} \leq 0$, we have $d < 1$. Therefore, for all large t,

$$\frac{x(t)}{x(h(t))} > \frac{1-\beta+\varepsilon-\sqrt{1-2(\beta-\varepsilon)-(\beta-\varepsilon)^2}}{2}.$$

Letting $\varepsilon \to 0$, we obtain that

$$\frac{x(t)}{x(h(t))} > \frac{1-\beta-\sqrt{1-2\beta-\beta^2}}{2} = A(\beta).$$

This shows that (3.13) holds. □

Theorem 3.2 *Assume* (3.4) *holds and that for some* r, *we have*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\big(h(t), \tau_i(\zeta)\big)\, d\zeta > 1 - \frac{1}{2}\big(1-\beta-\sqrt{1-2\beta-\beta^2}\big), \tag{3.20}$$

where $h(t)$ *is defined by* (3.5), $a_r(t,s)$ *is defined by* (3.2), *and* λ_0 *is the smaller root of the equation* $e^{\beta\lambda} = \lambda$. *Then equation* (3.1) *oscillates.*

Proof If equation (3.1) has a solution $x(t)$, then $-x(t)$ is also a solution of equation (3.1), so we only consider the situation where a solution of (3.1) is eventually positive, that is, $x(t) > 0$ and $x(\tau_i(t)) > 0$, $1 \leq i \leq m$, for all $t \geq T_3$. By (3.1) we have

$$x'(t) - x\big(h(t)\big)(t-t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(t) \leq 0, \quad t \geq T_3.$$

Integrating from $h(t)$ to t the latter and substituting into (3.3), we have

$$x(t) - x\big(h(t)\big) + x\big(h(t)\big) \int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\big(h(t), \tau_i(\zeta)\big)\, d\zeta \leq 0.$$

Consequently,

$$\int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\big(h(t), \tau_i(\zeta)\big)\, d\zeta \leq 1 - \frac{x(t)}{x(h(t))},$$

which gives

$$\limsup_{t\to\infty} \int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\big(h(t), \tau_i(\zeta)\big)\, d\zeta \leq 1 - \liminf_{t\to\infty} \frac{x(t)}{x(h(t))},$$

and by (3.13) the last inequality leads to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} (\zeta - t_0)^{\alpha-1} \sum_{i=1}^{m} p_i(\zeta) a_r\big(h(t), \tau_i(\zeta)\big)\, d\zeta \leq 1 - \frac{1}{2}\big(1-\beta-\sqrt{1-2\beta-\beta^2}\big),$$

which contradicts (3.20). The proof of the theorem is complete. □

Example 3.1 We consider the delay differential equation

$$T_{\frac{1}{2}}x(t)+p_1(t)x\big(\tau_1(t)\big)+p_2(t)x\big(\tau_2(t)\big)=0,\quad t\geq 0, \tag{3.21}$$

where

$$\tau_1(t)=\begin{cases} t-1, & t\in[3k,3k+1],\\ -3t+12k+3, & t\in[3k+1,3k+2],\\ 5t-12k-3, & t\in[3k+2,3k+3],\end{cases}$$

$$k\in\mathbb{N},\quad\text{and}\quad \tau_2(t)=\tau_1(t)-1,$$

$$p_i(t)=\frac{1}{8}t^{\frac{1}{2}},\quad i=1,2.$$

By (3.5) we obtain

$$h_1(t)=\max_{0\leq s\leq t}\tau_1(s)=\begin{cases} t-1, & t\in[3k,3k+1],\\ 3k, & t\in[3k+1,3k+2],\\ 5t-12k-13, & t\in[3k+2,3k+3],\end{cases}$$

$$k\in\mathbb{N},\quad\text{and}\quad h_2(t)=h_1(t)-1.$$

So $h(t)=\max_{1\leq i\leq 2}\{h_i(t)\}=h_1(t)$.

The functions $F_r:\mathbb{N}\to\mathbb{R}^+$ are defined as $F_r(t)=\int_{h(t)}^t\sum_{i=1}^m p_i(\zeta)a_r(h(t),\tau_i(\zeta))\,d\zeta$. When $t=3k+2.6$, $t\in\mathbb{N}$, for any $r\in\mathbb{N}^+$, the function $F_r(t)$ attains its maximum. In particular,

$$F_1(t=3k+2.6)=\int_{3k}^{3k+2.6}\zeta^{-\frac{1}{2}}\sum_{i=1}^{2}p_i(\zeta)a_r\big(h(t),\tau_i(\zeta)\big)\,d\zeta,$$

where

$$\begin{aligned} a_r\big(h(t),\tau_i(\zeta)\big)&=\exp\left\{\int_{\tau_i(\zeta)}^{h(t)}(\xi-t_0)^{\alpha-1}\sum_{i=1}^{m}p_i(\xi)\,d\xi\right\}=\exp\left\{\int_{\tau_i(\zeta)}^{h(t)}\xi^{-\frac{1}{2}}\frac{1}{4}\xi^{\frac{1}{2}}\,d\xi\right\}\\ &=\exp\left\{\frac{1}{4}\big(h(t)-\tau_i(\zeta)\big)\right\},\end{aligned}$$

so

$$\begin{aligned} F_1(t=3k+2.6)&=\int_{3k}^{3k+2.6}\zeta^{-\frac{1}{2}}\sum_{i=1}^{2}\frac{1}{8}\zeta^{\frac{1}{2}}\exp\left\{\frac{1}{4}\big(h(t)-\tau_i(\zeta)\big)\right\}d\zeta\\ &=\frac{1}{4}\int_{3k}^{3k+2.6}\left(\exp\left\{\frac{1}{4}\big(h(t)-\tau_1(\zeta)\big)\right\}+\exp\left\{\frac{1}{4}\big(h(t)-\tau_2(\zeta)\big)\right\}\right)d\zeta\\ &=\frac{1}{4}\int_{3k}^{3k+1}\left(\exp\left\{\frac{1}{4}\big(h(t)-\tau_1(\zeta)\big)\right\}+\exp\left\{\frac{1}{4}\big(h(t)-\tau_2(\zeta)\big)\right\}\right)d\zeta\\ &\quad+\frac{1}{4}\int_{3k+1}^{3k+2}\left(\exp\left\{\frac{1}{4}\big(h(t)-\tau_1(\zeta)\big)\right\}+\exp\left\{\frac{1}{4}\big(h(t)-\tau_2(\zeta)\big)\right\}\right)d\zeta\end{aligned}$$

$$+\frac{1}{4}\int_{3k+2}^{3k+2.6}\left(\exp\left\{\frac{1}{4}\big(h(t)-\tau_1(\zeta)\big)\right\}+\exp\left\{\frac{1}{4}\big(h(t)-\tau_2(\zeta)\big)\right\}\right)d\zeta$$
$$\approx 1.5052,$$

and therefore

$$\limsup_{t\to\infty} F_1(t) \geq 1.5052.$$

Now we see that

$$\beta = \liminf_{t\to\infty}\int_{h(t)}^{t}\zeta^{-\frac{1}{2}}\sum_{i=1}^{m}p_i(\zeta)\,d\zeta = \frac{1}{4}\big(t-h(t)\big) = \frac{1}{4} \leq \frac{1}{e}.$$

The solution of $\lambda = e^{\beta\lambda}$ is $\lambda_0 = 1.435$, so we get

$$1.5052 > \frac{1+\ln\lambda_0}{\lambda_0} \approx 0.9485,$$
$$1.5052 > 1 > A(\beta).$$

Therefore equation (3.21) satisfies the conditions of Theorems 3.1 and 3.2, and thus equation (3.21) oscillates.

4 Oscillation of 2α-order neutral conformable fractional differential equation

In this section, we deal with differential equations of the form

$$T_\alpha^{t_0}\Big(r(t)\big(T_\alpha^{t_0}\big(x(t)+p(t)x\big(\tau(t)\big)\big)\big)^\beta\Big)+q(t)x^\beta\big(\sigma(t)\big)=0,\quad t\geq t_0, \tag{4.1}$$

where T_α denotes the conformable differential operator of order $\alpha\in(0,1]$, $\beta\geq 1$ is a quotient of odd positive integers, and the functions r, p, q, τ, σ are such that $r,p,q,\tau,\sigma\in C^1([t_0,\infty),(0,\infty))$. We also assume that, for all $t\geq t_0$, $\tau(t)\leq t$, $\sigma(t)\leq t$, $T_\alpha^{t_0}\sigma(t)>0$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$, $0\leq p(t)<1$, $q(t)\geq 0$, and q does not vanish eventually.

We further use the following notation:

$$\varepsilon := \big(\beta/(\beta+1)\big)^{\beta+1},\qquad Q(t):=q(t)\big(1-p\big(\sigma(t)\big)\big)^\beta,$$
$$z(t)=x(t)+p(t)x\big(\tau(t)\big)<\infty,\qquad \pi(t):=\int_t^\infty(s-t_0)^{\alpha-1}r(s)^{-1/\beta}\,ds.$$

Lemma 4.1 *Let $\beta\geq 1$ be a ratio of two odd numbers. Then*

$$\begin{aligned}&A^{(\beta+1)/\beta}-(A-B)^{(\beta+1)/\beta}\leq\frac{4}{2}B^{1/\beta}\beta\big[(1+\beta)A-B\big],\quad AB\geq 0.\\&-Cv^{(\beta+1)/\beta}+Dv\leq\frac{\beta^\beta}{(\beta+1)^{\beta+1}}\frac{D^{\beta+1}}{C^\beta},\quad C>0.\end{aligned}\tag{4.2}$$

Theorem 4.1 *Assume that $\pi(t)=\int_t^\infty(s-t_0)^{\alpha-1}r(s)^{-1/\beta}\,ds<\infty$ and there exists a function $\rho\in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}I_\alpha^{t_0}\left(\rho(t)Q(t)-\big(\sigma(t)-t_0\big)^{(1-\alpha)\beta}\frac{(T_\alpha^{t_0}\rho_+(t))^{\beta+1}r(\sigma(t))}{(\beta+1)^{\beta+1}\rho^{\beta(t)}(T_\alpha^{t_0}\sigma(t))^\beta}\right)=\infty. \tag{4.3}$$

Suppose that there exists a function $\delta \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} I_\alpha^{t_0}\left[\psi(t) - \frac{\delta(t)r(t)((\varphi(t))_+)^{\beta+1}}{(\beta+1)^{\beta+1}}\right] = \infty, \tag{4.4}$$

where

$$\psi(t) := \delta(t)\left[q(t)\left(1 - p\bigl(\sigma(t)\bigr)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t)}\right)^{\beta} + \frac{1-\beta}{r^{1/\beta}(t)\pi^{\beta+1}(t)}\right],$$
$$p(t) < \pi(t)/\pi\bigl(\tau(t)\bigr), \qquad \varphi(t) := \frac{T_\alpha^{t_0}\delta(t)}{\delta(t)} + \frac{1+\beta}{r^{1/\beta}(t)\pi(t)},$$

and $(\varphi(t))_+ := \max\{0,\varphi(t)\}$. *Then equation* (4.1) *oscillates.*

Proof Let $x(t)$ be a nonoscillating solution of (4.1) on $[t_0,\infty)$. Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) \geq x(t) > 0$, and since

$$T_\alpha^{t_0}\left\{r(t)\bigl[T_\alpha^{t_0}\bigl(z(t)\bigr)\bigr]^{\beta}\right\} = -q(t)x^{\beta}\bigl(\sigma(t)\bigr) \leq 0, \tag{4.5}$$

the function $[r(t)T_\alpha^{t_0}z(t)]^{\beta}$ is nonincreasing for all $t \geq t_1$. Therefore $T_\alpha^{t_0}z(t)$ does not change sign eventually, that is, there exists $t_2 \geq t_1$ such that either $T_\alpha^{t_0}z(t) > 0$ or $T_\alpha^{t_0}z(t) < 0$ for all $t \geq t_2$.

Case I. Assume first that $T_\alpha^{t_0}z(t) > 0$ for all $t \geq t_2$. Note that $T_\alpha^{t_0}z(t)|_{t=\sigma(t)} = T_\alpha^{t_0}(z(\sigma(t)))$. Then

$$r(t)\bigl(T_\alpha^{t_0}\bigl(z(t)\bigr)\bigr)^{\beta} \leq r\bigl(\sigma(t)\bigr)\bigl(T_\alpha^{t_0}z\bigl(\sigma(t)\bigr)\bigr)^{\beta},$$

from which it follows that

$$T_\alpha^{t_0}\bigl(z\bigl(\sigma(t)\bigr)\bigr) \geq \bigl(T_\alpha^{t_0}\bigl(z(t)\bigr)\bigr)\left(\frac{r(t)}{r(\sigma(t))}\right)^{1/\beta}. \tag{4.6}$$

Since $x(t) \leq z(t)$, we see that

$$x(t) \geq \bigl[1-p(t)\bigr]z(t), \quad t \geq t_2. \tag{4.7}$$

In view of (4.7) and (4.1),

$$T_\alpha^{t_0}\bigl(r(t)\bigl(T_\alpha^{t_0}\bigl(x(t)+p(t)x\bigl(\tau(t)\bigr)\bigr)\bigr)^{\beta}\bigr) + Q(t)z^{\beta}\bigl(\sigma(t)\bigr) \leq 0, \quad t \geq t_2. \tag{4.8}$$

Put

$$w(t) = \rho(t)\frac{r(t)(T_\alpha^{t_0}z(t))^{\beta}}{z^{\beta}(\sigma(t))}, \quad t \geq t_2. \tag{4.9}$$

Clearly, $w(t) > 0$. Applying $T_\alpha^{t_0}$ to (4.9) and using (4.6) and (4.8), we obtain

$$\begin{aligned} &T_\alpha^{t_0}\bigl(w(t)\bigr) \\ &\quad= \frac{T_\alpha^{t_0}\rho_+(t)}{\rho(t)}w(t) + \rho(t)\frac{T_\alpha^{t_0}(r(t)(T_\alpha^{t_0}z(t))^\beta)}{z^\beta(\sigma(t))} - \rho(t)\frac{r(t)(T_\alpha^{t_0}z(t))^\beta \beta z'(\sigma(t))T_\alpha^{t_0}\sigma(t)}{z^{\beta+1}(\sigma(t))} \\ &\quad\le \frac{T_\alpha^{t_0}\rho_+(t)}{\rho(t)}w(t) - \rho(t)Q(t) - \frac{\beta T_\alpha^{t_0}\sigma(t)}{(\rho(t)r(\sigma(t)))^{\frac{1}{\beta}}(\sigma(t)-t_0)^{1-\alpha}}w^{\frac{\beta+1}{\beta}}(t), \end{aligned}$$

where $T_\alpha^{t_0}\rho_+(t) = \max\{T_\alpha^{t_0}\rho(t), 0\}$. Set

$$F(v) = \frac{T_\alpha^{t_0}\rho_+(t)}{\rho(t)}v - \frac{\beta T_\alpha^{t_0}\sigma(t)}{(\rho(t)r(\sigma(t)))^{\frac{1}{\beta}}(\sigma(t)-t_0)^{1-\alpha}}v^{\frac{\beta+1}{\beta}}, \quad v > 0.$$

By calculation letting $v_0 = (\sigma(t)-t_0)^{(1-\alpha)\beta}\frac{1}{(\beta+1)^\beta}\frac{(T_\alpha^{t_0}\rho_+(t))^\beta}{\rho^{\beta-1}(t)}\frac{r(\sigma(t))}{(T_\alpha^{t_0}\sigma(t))^\beta}$, we have that when

$$v = v_0,$$

the function $F(v)$ attains its maximum $F(v_0)$. So

$$F(v) \le F(v_0) = \bigl(\sigma(t)-t_0\bigr)^{(1-\alpha)\beta}\frac{(T_\alpha^{t_0}\rho_+(t))^{\beta+1}r(\sigma(t))}{(\beta+1)^{\beta+1}\rho^\beta(t)(T_\alpha^{t_0}\sigma(t))^\beta}.$$

Therefore

$$T_\alpha^{t_0}\bigl(w(t)\bigr) \le -\rho(t)Q(t) + \bigl(\sigma(t)-t_0\bigr)^{(1-\alpha)\beta}\frac{(T_\alpha^{t_0}\rho_+(t))^{\beta+1}r(\sigma(t))}{(\beta+1)^{\beta+1}\rho^\beta(t)(T_\alpha^{t_2}\sigma(t))^\beta}.$$

Applying I_α to the last inequality from t_0 to t, we have

$$0 < w(t) \le w(t_0) - I_\alpha^{t_0}\left(\rho(t)Q(t) - \bigl(\sigma(t)-t_0\bigr)^{(1-\alpha)\beta}\frac{(T_\alpha^{t_0}\rho_+(t))^{\beta+1}r(\sigma(t))}{(\beta+1)^{\beta+1}\rho^{\beta(t)}(T_\alpha^{t_0}\sigma(t))^\beta}\right).$$

Letting $t \to \infty$ in this inequality, we get a contradiction to (4.3).

Case II. Assume now that $T_\alpha^{t_0}z(t) < 0$ for all $t \ge t_0$. It follows from (4.1) that $T_\alpha^{t_0}(r(T_\alpha^{t_0}z)^\beta) < 0$ for all $s \ge t \ge t_2$, and thus

$$T_\alpha^{t_0}z(s) \le \left(\frac{r(t)}{r(s)}\right)^{1/\beta}T_\alpha^{t_0}z(t). \tag{4.10}$$

Dividing (4.10) by $(s-t_0)^{1-\alpha}$ and then integrating from t to l, $l \ge t \ge t_2$, we have

$$\begin{aligned} z(l) - z(t) &\le \int_t^l (s-t_0)^{\alpha-1}\left\{\left(\frac{r(t)}{r(s)}\right)^{1/\beta}T_\alpha^{t_0}\bigl(z(t)\bigr)\right\}ds \\ &= r(t)^{1/\beta}T_\alpha^{t_0}\bigl(z(t)\bigr)\int_t^l (s-t_0)^{\alpha-1}r(s)^{-1/\beta}\,ds. \end{aligned}$$

Letting $l \to \infty$, we get

$$z(t) \ge -\pi(t)r^{1/\beta}(t)T_\alpha^{t_0}\bigl(z(t)\bigr), \tag{4.11}$$

which implies that

$$T_\alpha^{t_0}\left(\frac{z(t)}{\pi(t)}\right) = \frac{\pi(t)T_\alpha^{t_0}z(t) - z(t)T_\alpha^{t_0}\pi(t)}{\pi^2(t)} = \frac{\pi(t)T_\alpha^{t_0}z(t) + z(t)r^{-1/\beta}(t))}{\pi^2(t)}$$
$$\geq \frac{\pi(t)T_\alpha^{t_0}z(t) - \pi(t)T_\alpha^{t_0}z(t)}{\pi^2(t)} = 0.$$

Hence we conclude that

$$x(t) = z(t) - p(t)x\bigl(\tau(t)\bigr) \geq z(t) - p(t)z\bigl(\tau(t)\bigr) \geq \left(1 - p(t)\frac{\pi(\tau(t))}{\pi(t)}\right)z(t). \tag{4.12}$$

Using (4.12) in (4.5), we have

$$T_\alpha^{t_0}\bigl\{r(t)\bigl[T_\alpha^{t_0}\bigl(z(t)\bigr)\bigr]^\beta\bigr\} \leq -q(t)\left(1 - p\bigl(\sigma(t)\bigr)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^\beta z^\beta\bigl(\sigma(t)\bigr) \leq 0. \tag{4.13}$$

Define a generalized Riccati substitution by

$$w(t) := \delta(t)\left[\frac{r(t)(T_\alpha^{t_0}z(t))^\beta}{z^\beta(t)} + \frac{1}{\pi^\beta(t)}\right]. \tag{4.14}$$

By (4.11), $w(t) \geq 0$ for all $t \geq t_2$. Applying $T_\alpha^{t_0}$ to (4.14), we have

$$\begin{aligned}
T_\alpha^{t_0}w(t) = {}& \frac{T_\alpha^{t_0}\delta(t)}{\delta(t)}w(t) \\
& + \delta(t)\left(\frac{T_\alpha^{t_0}(r(t)(T_\alpha^{t_0}z(t))^\beta)}{z^\beta} - \frac{\beta r(t)(T_\alpha^{t_0}z(t))^{\beta+1}}{z^{\beta+1}(t)} - \beta\pi^{-(\beta+1)}T_\alpha^{t_0}\pi(t)\right) \\
= {}& \frac{T_\alpha^{t_0}\delta(t)}{\delta(t)}w(t) \\
& + \delta(t)\frac{T_\alpha^{t_0}(r(t)(T_\alpha^{t_0}z(t))^\beta)}{z^\beta} - \beta\delta(t)r(t)\left(\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^\beta(t)}\right)^{(\beta+1)/\beta} \\
& + \frac{\beta\delta(t)}{r^{1/\beta}(t)\pi^{1+\beta}(t)}.
\end{aligned} \tag{4.15}$$

Let $A := w(t)/(\delta(t)r(t))$ and $B = 1/(r(t)\pi^\beta(t))$. Using Lemma 4.1, we conclude that

$$\begin{aligned}
&\left(\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^\beta(t)}\right)^{\frac{\beta+1}{\beta}} \\
&\quad \geq \left(\frac{w(t)}{\delta(t)r(t)}\right)^{\frac{\beta+1}{\beta}} - \frac{1}{\beta r^{1/\beta}(t)\pi(t)}\left[(1+\beta)\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^\beta(t)}\right].
\end{aligned}$$

On the other hand, we get by (4.13) that $T_\alpha^{t_0}z < 0$ and from $\sigma(t) \leq t$ that

$$\frac{T_\alpha^{t_0}\{r(t)[T_\alpha^{t_0}(z(t))]^\beta\}}{z^\beta(t)} \leq -q(t)\left(1 - p\bigl(\sigma(t)\bigr)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^\beta.$$

Thus (4.15) yields

$$\begin{aligned}
T_\alpha^{t_0} w(t) &= \frac{T_\alpha^{t_0}\delta(t)}{\delta(t)} w(t) + \delta(t)\frac{T_\alpha^{t_0}(r(t)(T_\alpha^{t_0}z(t))^\beta)}{z^\beta} \\
&\quad - \beta\delta(t)r(t)\left(\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^\beta(t)}\right)^{(\beta+1)/\beta} + \frac{\beta\delta(t)}{r^{1/\beta}(t)\pi^{1+\beta}(t)} \\
&\le \frac{T_\alpha^{t_0}\delta(t)}{\delta(t)} w(t) - \delta(t)q(t)\left(1 - p\big(\sigma(t)\big)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^\beta + \frac{\beta\delta(t)}{r^{1/\beta}(t)\pi^{1+\beta}(t)} \\
&\quad - \beta\delta(t)r(t)\left(\left(\frac{w(t)}{\delta(t)r(t)}\right)^{\frac{\beta+1}{\beta}} - \frac{1}{\beta r^{1/\beta}(t)\pi(t)}\left[(1+\beta)\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)\pi^\beta(t)}\right]\right) \\
&= -\delta(t)\left[q(t)\left(1 - p\big(\sigma(t)\big)\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^\beta + \frac{1-\beta}{r^{1/\beta}(t)\pi^{\beta+1}(t)}\right] \\
&\quad + \left[\frac{T_\alpha^{t_0}\delta(t)}{\delta(t)} + \frac{1+\beta}{r^{1/\beta}(t)\pi(t)}\right] w(t) - \frac{\beta}{(\delta(t)r(t))^{1/\beta}} w^{(\beta+1)/\beta}(t),
\end{aligned}$$

that is,

$$T_\alpha^{t_0} w(t) \le -\psi(t) + \big(\varphi(t)\big)_+ w(t) - \frac{\beta}{(\delta(t)r(t))^{1/\beta}} w^{(\beta+1)/\beta}(t). \tag{4.16}$$

Denote $C := \beta/(\delta(t)r(t))^{1/\beta}$, $D := (\varphi(t))_+$, and $v := w(t)$. Applying inequality (4.2), we obtain

$$\big(\varphi(t)\big)_+ w(t) - \frac{\beta}{(\delta(t)r(t))^{1/\beta}} w^{(\beta+1)/\beta}(t) \le \frac{\delta(t)r(t)((\varphi(t))_+)^{\beta+1}}{(\beta+1)^{\beta+1}}. \tag{4.17}$$

By (4.16) and (4.17) we have

$$T_\alpha^{t_0} w(t) \le -\psi(t) + \frac{\delta(t)r(t)((\varphi(t))_+)^{\beta+1}}{(\beta+1)^{\beta+1}}.$$

Applying I_α to the latter inequality from t_0 to t, we have

$$I_\alpha^{t_0}\left[\psi(t) - \frac{\delta(t)r(t)((\varphi(t))_+)^{\beta+1}}{(\beta+1)^{\beta+1}}\right] \le -w(t) + w(t_0),$$

which contradicts (4.4). Therefore (4.1) oscillates. □

Example 4.1 We consider the equation

$$T_{\frac{1}{2}}^1\left(t^2 T_{\frac{1}{2}}^1\left(x(t) + p(t)x\left(\frac{t}{2}\right)\right)\right) + q(t)x(t) = 0, \quad t \ge 0, \tag{4.18}$$

where $p(t) = \frac{1}{5}$ and $q(t) = (2 + \frac{4\sqrt{2}}{5})t$. Let $\rho(t) = 1$ and $\delta(t) = 1/t$. Then we have

$$\begin{aligned}
&I_\alpha^{t_0}\left(\rho(t)Q(t) - \big(\sigma(t) - t_0\big)^{(1-\alpha)\beta}\frac{(T_\alpha^{t_0}\rho_+(t))^{\beta+1}r(\sigma(t))}{(\beta+1)^{\beta+1}\rho^{\beta(t)}(T_\alpha^{t_0}\sigma(t))^\beta}\right) \\
&\quad = I_\alpha^{t_0}Q(t) = I_\alpha^{t_0}\left(\frac{4}{5}\left(2 + \frac{4\sqrt{2}}{5}\right)t\right),
\end{aligned}$$

and it is obvious that (4.3) holds. Because of $\varphi(t) = 2/\sqrt{t}$, $\psi(t) = (q_0(1-2\sqrt{2}p_0))/t = \frac{34}{25}$. So

$$I_\alpha^{t_0}\left[\psi(t) - \frac{\delta(t)r(t)((\varphi(t))_+)^{\beta+1}}{(\beta+1)^{\beta+1}}\right] = I_\alpha^{t_0}\left[\frac{34}{25} - \frac{\frac{1}{t}t^2(\frac{2}{\sqrt{t}})^2}{2^2}\right] = I_\alpha^{t_0}\frac{9}{25},$$

and we can conclude that condition (4.4) is satisfied. Hence by Theorem 4.1 we deduce that (4.18) oscillates.

5 Oscillation of 3α-order damped conformable fractional differential equation

This section deals with oscillatory behavior of all solutions of the 3α-order nonlinear delay damped equation of the form

$$T_\alpha^{t_0}\left(r_2 T_\alpha^{t_0}\left(r_1\left(T_\alpha^{t_0}y\right)^\beta\right)\right)(t) + p(t)\left(T_\alpha^{t_0}y(t)\right)^\beta + q(t)f\left(y\left(g(t)\right)\right) = 0, \quad t \geq t_0, \tag{5.1}$$

where $0 < \alpha \leq 1$, and $\beta \geq 1$ is the ratio of positive odd integers. We further assume that the following conditions are satisfied:

(H1) $r_1, r_2, p, q \in C(I, \mathbb{R}^+)$, where $I = [t_0, \infty)$, $\mathbb{R}^+ = (0, \infty)$;

(H2) $g \in C^1(I, \mathbb{R})$, $T_\alpha^{t_0}g(t) \geq 0$ and $g(t) \to \infty$ as $t \to \infty$;

(H3) $f \in C(\mathbb{R}, \mathbb{R})$ is such that $xf(x) > 0$ for $x \neq 0$, and $f(x)/x^\gamma \geq k > 0$, where γ is the ratio of positive odd integers.

We define

$$R_1(t, t_0) = I_\alpha^{t_0}\frac{1}{r_1^{1/\beta}(t)}, \qquad R_2(t, t_0) = I_\alpha^{t_0}\frac{1}{r_2(t)}, \quad \text{and} \quad R^*(t, t_0) = I_\alpha^{t_0}\left(\frac{R_2(t, t_0)}{r_1(t)}\right)^{1/\beta}$$

for $t_0 \leq t_1 \leq t \leq \infty$ and assume that

$$R_1(t, t_0) \to \infty, \quad t \to \infty, \tag{5.2}$$

and

$$R_2(t, t_0) \to \infty, \quad t \to \infty. \tag{5.3}$$

A function y is called a solution of (5.1) if $y, r_1(T_\alpha^{t_0}y)^\beta, r_2(r_1(T_\alpha^{t_0}y)^\beta) \in C^1([t_y, \infty), \mathbb{R})$ and y satisfies (5.1) for $[t_y, \infty)$ for some $t_y \geq t_0$.

For brevity, we define

$$L_0y(t) = y(t), \qquad L_1y(t) = r_1(t)\left(T_\alpha^{t_0}(L_0y)\right)^\beta(t),$$
$$L_2y(t) = r_2(t)T_\alpha^{t_0}(L_1y)(t), \qquad L_3y(t) = T_\alpha^{t_0}(L_2y)(t)$$

on I. Then (5.1) can be written as

$$L_3y(t) + \frac{p(t)}{r_1(t)}L_1y(t) + q(t)f\left(y\left(g(t)\right)\right) = 0.$$

The purpose of this section is to ensure that any solution of (5.1) oscillates when the related second-order linear ordinary fractional differential equation without delay

$$T_\alpha^{t_0}\{r_2(t)T_\alpha^{t_0}z(t)\} + \frac{p(t)}{r_1(t)}z(t) = 0 \tag{5.4}$$

is nonoscillatory.

Next, we state and prove the following lemmas.

Lemma 5.1 *Let y be a nonoscillatory solution of (5.1) on I. Suppose (5.4) is nonoscillatory. Then there exists $t_2 \in [t_1, \infty)$ such that $y(t)L_1y(t) > 0$ or $y(t)L_1y(t) < 0$, $t \geq t_2$.*

Proof Let y be a nonoscillatory solution of (5.1) on $[t_1, \infty)$, say $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1 \geq t_0$. Let $x = -L_1y(t)$. By (5.1) we have

$$T_\alpha^{t_0}(r_2T_\alpha^{t_0}x)(t) + \frac{p(t)}{r_1(t)}x(t) = q(t)f(y(g(t))) > 0, \quad t \geq t_1.$$

Let $u(t)$ be a positive solution of (5.4), say $u(t) > 0$ for $t \geq t_1 \geq t_0$. If x is oscillatory, then x has consecutive zeros at a and b ($t_1 < a < b$) such that $T_\alpha^{t_0}x(a) \geq 0$, $T_\alpha^{t_0}x(b) \leq 0$, and $x(t) > 0$ for $t \in (a, b)$. Then we obtain

$$\begin{aligned}
0 &< \int_a^b \left[T_\alpha^{t_0}(r_2T_\alpha^{t_0}x)(t) + \frac{p(t)}{r_1(t)}x(t)\right]u(t)\,d\alpha(t,a) \\
&= \int_a^b (t-a)^{1-\alpha}(r_2T_\alpha^{t_0}x)'(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\,d\alpha(t,a) + \int_a^b \frac{p(t)}{r_1(t)}x(t)u(t)\,d\alpha(t,a) \\
&= r_2(t)T_\alpha^{t_0}x(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\Big|_a^b - \int_a^b (r_2T_\alpha^{t_0}x)(t)T_\alpha^a\left[\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\right]d\alpha(t,a) \\
&\quad + \int_a^b \frac{p(t)}{r_1(t)}x(t)u(t)\,d\alpha(t,a) \\
&= r_2(t)T_\alpha^{t_0}x(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\Big|_a^b + \int_a^b \frac{p(t)}{r_1(t)}x(t)u(t)\,d\alpha(t,a) \\
&\quad - \int_a^b r_2(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}T_\alpha^a\left[\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\right](t-a)^{1-\alpha}x'(t)\,d\alpha(t,a) \\
&= r_2(t)T_\alpha^{t_0}x(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\Big|_a^b - \left\{r_2(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}T_\alpha^a\left[\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\right]\right\}x(t)\Big|_a^b \\
&\quad + \int_a^b T_\alpha^a\left\{r_2(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}T_\alpha^a\left[\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\right]\right\}x(t)\,d\alpha(t,a) \\
&\quad + \int_a^b \frac{p(t)}{r_1(t)}u(t)x(t)\,d\alpha(t,a) \\
&= r_2(t)T_\alpha^{t_0}x(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\Big|_a^b + \int_a^b \left\{T_\alpha^{t_0}\{r_2(t)T_\alpha^{t_0}u(t)\} + \frac{p(t)}{r_1(t)}u(t)\right\}x(t)\,d\alpha(t,a) \\
&= r_2T_\alpha^{t_0}x(t)\left(\frac{t-t_0}{t-a}\right)^{1-\alpha}u(t)\Big|_a^b \leq 0,
\end{aligned}$$

which yields a contradiction. This completes the proof. □

Lemma 5.2 *If y is a nonoscillatory solution of (5.1) and $y(t)L_1y(t) > 0$, $t \geq t_1 \geq t_0$, then*

$$L_1y(t) \geq R_2(t,t_0)L_2y(t) \quad \text{for all } t \geq t_1 \tag{5.5}$$

and

$$y(t) \geq R^*(t,t_0)(L_2y)^{1/\beta}(t) \quad \text{for all } t \geq t_1. \tag{5.6}$$

Proof If y is a nonoscillatory solution of (5.1), then $y(t) > 0$, $y(g(t)) > 0$, and $L_1y(t) > 0$ for $t \geq t_1 \geq t_0$. It is easy to see that

$$L_3y(t) = -\frac{p(t)}{r_1(t)}L_1y(t) - q(t)f\bigl(y(g(t))\bigr) \leq 0,$$

which implies that $L_2y(t)$ is nonincreasing on $[t_1,\infty)$. Applying I_α to $T_\alpha^{t_0}L_1y(t) = \frac{L_2y(t)}{r_2(t)}$ from t_1 to t and Lemma 2.2, we get

$$L_1y(t) = L_1y(t_1) + I_\alpha^{t_0}\left[\frac{L_2y(t)}{r_2(t)}\right] \geq L_2y(t)I_\alpha^{t_0}\frac{1}{r_2(t)} = L_2y(t)R_2(t,t_0) \quad \text{for any } t \geq t_1.$$

Then

$$T_\alpha^{t_0}y(t) \geq \left(\frac{R_2(t,t_0)}{r_1(t)}\right)^{1/\beta}(L_2y)^{1/\beta}(t).$$

Now, applying I_α to the last inequality from t_1 to t, we can obtain from Lemma 2.2 that

$$\begin{aligned} y(t) &\geq y(t_1) + I_\alpha^{t_0}\left[\left(\frac{R_2(t,t_0)}{r_1(t)}\right)^{1/\beta}(L_2y)^{1/\beta}(t)\right] \\ &\geq (L_2y)^{1/\beta}(t)I_\alpha^{t_0}\left(\frac{R_2(t,t_0)}{r_1(t)}\right)^{1/\beta} = R^*(t,t_0)(L_2y)^{1/\beta}(t) \quad \text{for } t \geq t_1. \end{aligned}$$

This completes the proof. □

In the following two lemmas, we consider the second-order delay differential inequality

$$T_\alpha^{t_0}\bigl(r_2T_\alpha^{t_0}x(t)\bigr) \geq Q(t)x\bigl(h(t)\bigr), \quad t > t_0, \tag{5.7}$$

where the function r_2 is as in (5.1), $Q(t) \in C(I,\mathbb{R}^+)$, and $h(t) \in C^1(I,\mathbb{R})$ is such that $h(t) \leq t$, $T_\alpha^{t_0}h(t) \geq 0$ for $t \geq t_0$, and $h(t) \to \infty$ as $t \to \infty$.

Lemma 5.3 *If*

$$\limsup_{t\to\infty} R_2\bigl(h(t),t_0\bigr)I_\alpha^{t_0}Q(t) > 1, \tag{5.8}$$

then all bounded solutions of (5.7) are oscillatory.

Proof Let $x(t)$ be a bounded nonoscillatory solution of (5.7), say $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. By (5.7), $r_2 T_\alpha^{t_0} x(t)$ is strictly increasing on $[t_1, \infty)$. Hence, for any $t_2 \geq t_1$, applying I_α from t_2 to t in $T_\alpha^{t_0} x(t) = \frac{r_2(t) T_\alpha^{t_0} x(t)}{r_2(t)}$ and Lemma 2.2 yield

$$\begin{aligned} x(t) &= x(t_2) + I_\alpha^{t_0}\left[\frac{r_2(t) T_\alpha^{t_0} x(t)}{r_2(t)}\right] > x(t_2) + r_2(t_2) T_\alpha^{t_0} x(t_2) I_\alpha^{t_0} \frac{1}{r_2(t)} \\ &= x(t_2) + r_2(t_2) T_\alpha^{t_0} x(t_2) R_2(t, t_0), \end{aligned}$$

so $T_\alpha^{t_0} x(t_2) < 0$, as otherwise (5.3) would imply $x(t) \to \infty$ as $t \to \infty$, a contradiction to the boundedness of x. Altogether,

$$x > 0, T_\alpha^{t_0} x < 0, \quad \text{and} \quad T_\alpha^{t_0}\left(r_2 T_\alpha^{t_0} x\right) > 0 \quad \text{on } [t_1, \infty).$$

Now, for $v \geq u \geq t_1$, repeating the previous steps, we have

$$\begin{aligned} x(u) &> x(u) - x(v) = -I_\alpha^{t_0}\left[\frac{r_2(v) T_\alpha^{t_0} x(v)}{r_2(v)}\right] \geq -r_2(v) T_\alpha^{t_0} x(v) I_\alpha^{t_0} \frac{1}{r_2(v)} \\ &= -r_2(v) T_\alpha^{t_0} x(v) R_2(v, t_0). \end{aligned} \tag{5.9}$$

For $t \geq s \geq t_1$, setting $u = h(s)$ and $v = h(t)$ in (5.9), we get

$$x\big(h(s)\big) > -r_2\big(h(t)\big) T_\alpha^{t_0} x\big(h(t)\big) R_2\big(h(t), t_0\big).$$

Applying I_α to (5.7) from $h(t) \geq t_1$ to t, we obtain from Lemma 2.2 that

$$\begin{aligned} -r_2\big(h(t)\big) T_\alpha^{t_0} x\big(h(t)\big) &> r_2(t) T_\alpha^{t_0} x(t) - r_2\big(h(t)\big) T_\alpha^{t_0} x\big(h(t)\big) \\ &\geq I_\alpha^{t_0}\big(Q(t) x\big(h(t)\big)\big) \\ &> -r_2\big(h(t)\big) T_\alpha^{t_0} x\big(h(t)\big) R_2\big(h(t), t_0\big) I_\alpha^{t_0} Q(t), \end{aligned}$$

that is,

$$1 > R_2\big(h(t), t_0\big) I_\alpha^{t_0} Q(t).$$

Taking lim sup as $t \to \infty$ on both sides of this inequality yields a contradiction to (5.8). This completes the proof. □

Lemma 5.4 *If*

$$\limsup_{t \to \infty, u \to \infty} R_2(u, t_0) I_\alpha^{t_0} Q(t) > 1, \tag{5.10}$$

then all bounded solutions of (5.7) *are oscillatory.*

Proof Let x be a bounded nonoscillatory solution of (5.7), say $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. As in Lemma 5.1, we obtain

$$x > 0, \qquad T_\alpha^{t_0} x < 0, \quad \text{and} \quad T_\alpha^{t_0}\left(r_2 T_\alpha^{t_0} x\right) > 0 \quad \text{on } [t_1, \infty).$$

Applying I_α to(5.7) from $u \geq t_1$ to t, we obtain from the previous forms that

$$-r_2(u)T_\alpha^{t_0}x(u) > r_2(t)T_\alpha^{t_0}x(t) - r_2(u)T_\alpha^{t_0}x(u) \geq I_\alpha^{t_0}\big(Q(t)x\big(h(t)\big)\big) \geq x\big(h(t)\big)I_\alpha^{t_0}Q(t),$$

so

$$-T_\alpha^{t_0}x(u) > \left(\frac{1}{r_2(u)}I_\alpha^{t_0}Q(t)\right)x\big(h(t)\big). \tag{5.11}$$

We obtain from (5.11) that

$$x\big(h(t)\big) > x\big(h(t)\big) - x(u) \geq x\big(h(t)\big)I_\alpha^{t_0}\left[\left(\frac{1}{r_2(u)}I_\alpha^{t_0}Q(t)\right)\right],$$

that is,

$$1 > R_2(u,t_0)I_\alpha^{t_0}Q(t).$$

Taking lim sup as $u, t \to \infty$ on both sides of this inequality yields a contradiction to (5.10). This completes the proof. □

Theorem 5.1 *Assume that* (5.2) *and* (5.3) *hold and* $\beta \geq \gamma$. *Suppose that there exist two functions* $m, h \in C^1(I, \mathbb{R})$ *such that*

$$g(t) \leq h(t) \leq t, \qquad T_\alpha^{t_0}h(t) \geq 0, \quad \text{and} \quad m(t) > 0, \quad t \in I,$$

satisfying

$$\limsup_{t\to\infty} I_\alpha^{t_0}\left[km(t)q(t) - \frac{A^2(t)}{4B(t)}\right] = \infty, \tag{5.12}$$

and for $t \geq t_1$,

$$\begin{cases} A(t) = \frac{T_\alpha^{t_0}m(t)}{m(t)} - \frac{p(t)}{r_1(t)}R_2(t,t_0), \\ B(t) = c^* m^{-1}(t)T_\alpha^{t_0}g(t)(R^*(g(t),t_0))^{\gamma-1}(\frac{R_2(g(t),t_0)}{r_1(g(t))})^{1/\beta}(t-t_0)^{\alpha-1}, \end{cases} \tag{5.13}$$

and that (5.8) *or* (5.10) *holds with*

$$Q(t) = ckq(t)\big(R_1\big(h(t),t_0\big)\big)^\gamma - \frac{p(t)}{r_1(t)} \geq 0, \quad t \geq t_1,$$

with $c, c^* > 0$. *Then every solution* y *of* (5.1) *and* $L_2y(t)$ *are oscillatory.*

Proof Let y be a nonoscillatory solution of (5.1) on $[t_1,\infty)$, $t_1 \geq t_0$. We assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$. From Lemma 5.1 we have $L_1y(t) < 0$ or $L_1y(t) > 0$ for $t \geq t_1$.

Step 1. We assume that $L_1y(t) > 0$ on $[t_1,\infty)$. By (5.1) L_2y is strictly decreasing. Hence, for any $t_2 \geq t_1$, we have from Lemma 2.2 that

$$L_1y(t) = L_1y(t_2) + I_\alpha^{t_0}\left[\frac{L_2y(t)}{r_2(t)}\right] \leq L_1y(t_2) + L_2y(t_2)I_\alpha^{t_0}\frac{1}{r_2(t)} = L_1y(t_2) + L_2y(t_2)R_2(t,t_2).$$

So $L_2y(t_2) > 0$ as otherwise (5.3) would imply $L_1y(t) \to -\infty$ as $t \to \infty$, a contradiction to the positivity of L_1y. Altogether, $L_2y > 0$ on $[t_1, \infty)$.

Define the following generalized Riccati transformation:

$$w(t) = m(t)\frac{L_2y(t)}{y^{\gamma}(g(t))}, \quad t \in [t_1, \infty). \tag{5.14}$$

By the product and quotient rules, α-differentiating w, we obtain

$$\begin{aligned}
T_{\alpha}^{t_0}w(t) &= T_{\alpha}^{t_0}\left[m(t)\frac{L_2y(t)}{y^{\gamma}(g(t))}\right] \\
&= T_{\alpha}^{t_0}m(t)\frac{L_2y(t)}{y^{\gamma}(g(t))} \\
&\quad + m(t)\frac{T_{\alpha}^{t_0}(L_2y(t))y^{\gamma}(g(t)) - \gamma[y^{\gamma-1}(g(t))]y'(g(t))T_{\alpha}^{t_0}g(t)L_2y(t)}{y^{2\gamma}(g(t))} \\
&= \frac{T_{\alpha}^{t_0}m(t)}{m(t)}w(t) + m(t)\frac{T_{\alpha}^{t_0}(L_2y(t))}{y^{\gamma}(g(t))} - m(t)\frac{\gamma y'(g(t))T_{\alpha}^{t_0}g(t)L_2y(t)}{y^{\gamma+1}(g(t))} \\
&= \frac{T_{\alpha}^{t_0}m(t)}{m(t)}w(t) + \frac{T_{\alpha}^{t_0}(L_2y)(t)}{L_2y(t)}w(t) - \gamma T_{\alpha}^{t_0}g(t)\frac{y'(g(t))}{y(g(t))}w(t).
\end{aligned}$$

Using (5.1), (5.5), and assumption (H3) on f, we obtain

$$\begin{aligned}
&\frac{T_{\alpha}^{t_0}m(t)}{m(t)}w(t) + \frac{T_{\alpha}^{t_0}(L_2y)(t)}{L_2y(t)}w(t) \\
&\quad= \frac{T_{\alpha}^{t_0}m(t)}{m(t)}w(t) - \frac{\frac{p(t)}{r_1(t)}L_1y(t) + q(t)f(y(g(t)))}{L_2y(t)}w(t) \\
&\quad= \frac{T_{\alpha}^{t_0}m(t)}{m(t)}w(t) - \frac{\frac{p(t)}{r_1(t)}L_1y(t)}{L_2y(t)}w(t) - \frac{q(t)f(y(g(t)))}{L_2y(t)}w(t) \\
&\quad\le \frac{T_{\alpha}^{t_0}m(t)}{m(t)}w(t) - \frac{p(t)}{r_1(t)}R_2(t,t_0)w(t) - km(t)q(t) \\
&\quad= \left[\frac{T_{\alpha}^{t_0}m(t)}{m(t)} - \frac{p(t)}{r_1(t)}R_2(t,t_0)\right]w(t) - km(t)q(t) \\
&\quad= A(t)w(t) - km(t)q(t).
\end{aligned}$$

By the definition of $L_1y(t)$ and (5.5) we obtain

$$\begin{aligned}
(t-t_0)^{1-\alpha}\bigl(y(g(t))\bigr)' = T_{\alpha}^{t_0}y\bigl(g(t)\bigr) &= \left(\frac{1}{r_1(g(t))}L_1y\bigl(g(t)\bigr)\right)^{1/\beta} \\
&\ge \left(\frac{R_2(g(t),t_0)}{r_1(g(t))}\right)^{1/\beta}\bigl(L_2y\bigl(g(t)\bigr)\bigr)^{1/\beta} \\
&\ge \left(\frac{R_2(g(t),t_0)}{r_1(g(t))}\right)^{1/\beta}\bigl(L_2y(t)\bigr)^{1/\beta}.
\end{aligned}$$

Then

$$\begin{aligned}\frac{y'(g(t))}{y(g(t))} &\geq (t-t_0)^{\alpha-1}\left(\frac{R_2(g(t),t_0)}{m(t)r_1(g(t))}\right)^{1/\beta}\frac{m^{1/\beta}(t)(L_2y)^{1/\beta}(t)}{y^{\gamma/\beta}(g(t))}y^{\gamma/\beta-1}\bigl(g(t)\bigr)\\ &\overset{(5.14)}{=} (t-t_0)^{\alpha-1}\left(\frac{R_2(g(t),t_0)}{m(t)r_1(g(t))}\right)^{1/\beta}w^{1/\beta}(t)y^{\gamma/\beta-1}\bigl(g(t)\bigr),\end{aligned}$$

and we obtain

$$\begin{aligned}T_\alpha^{t_0}w(t) &\leq A(t)w(t)-km(t)q(t)\\ &\quad -\gamma T_\alpha^{t_0}g(t)(t-t_0)^{\alpha-1}\left(\frac{R_2(g(t),t_0)}{m(t)r_1(g(t))}\right)^{1/\beta}w^{1/\beta}(t)y^{\gamma/\beta-1}\bigl(g(t)\bigr)w(t)\\ &\leq A(t)w(t)-km(t)q(t)\\ &\quad -\gamma T_\alpha^{t_0}g(t)(t-t_0)^{\alpha-1}w^{1/\beta+1}(t)y^{\gamma/\beta-1}\bigl(g(t)\bigr)\left(\frac{R_2(g(t),t_0)}{m(t)r_1(g(t))}\right)^{1/\beta}.\end{aligned}\tag{5.15}$$

Since $L_3y(t)<0$, we have $0<L_2y(t)\leq L_2y(t_1)$, $L_2y(t_1)=c_1$ for $t\geq t_1$. Then

$$r_2(t)T_\alpha^{t_0}(L_1y)(t)=L_2y(t)\leq c_1,\quad t\geq t_1,$$

and thus we get from Lemma 2.2 that

$$\begin{aligned}r_1(t)\bigl(T_\alpha^{t_0}y\bigr)^\beta(t)=L_1y(t) &= L_1y(t_1)+I_\alpha^{t_0}\left[\frac{r_2(t)T_\alpha^{t_0}(L_1y(t))}{r_2(t)}\right]\leq L_1y(t_1)+c_1I_\alpha^{t_0}\frac{1}{r_2(t)}\\ &= L_1y(t_1)+c_1R_2(t,t_0)=\left[\frac{L_1y(t_1)}{R_2(t,t_0)}+c_1\right]R_2(t,t_0)\\ &\leq \left[\frac{L_1y(t_1)}{R_2(t_2,t_0)}+c_1\right]R_2(t,t_0)=\tilde{c}_1R_2(t,t_0)\end{aligned}$$

(note that $L_1y(t_1)>0$), where

$$\tilde{c}_1=c_1+\frac{L_1y(t_1)}{R_2(t_2,t_0)}.$$

Therefore, we get for all $t\geq t_2$ that

$$\begin{aligned}y(t) &= y(t_2)+I_\alpha^{t_0}\bigl[T_\alpha^{t_0}y(t)\bigr]\leq y(t_2)+I_\alpha^{t_0}\left(\frac{\tilde{c}_1R_2(t,t_0)}{r_1(t)}\right)^{1/\beta}\\ &= y(t_2)+\tilde{c}_1^{\,1/\beta}R^*(t,t_0)=\left[\frac{y(t_2)}{R^*(t,t_0)}+\tilde{c}_1^{\,1/\beta}\right]R^*(t,t_0)\\ &\leq \left[\frac{y(t_2)}{R^*(t_2,t_0)}+\tilde{c}_1^{\,1/\beta}\right]R^*(t,t_0)\\ &= c_2R^*(t,t_0)\end{aligned}$$

(note that $y(t_2)>0$), where

$$c_2=\frac{y(t_2)}{R^*(t_2,t_0)}+\tilde{c}_1^{\,1/\beta}.$$

Then we get

$$y^{\gamma/\beta-1}\big(g(t)\big) \ge c_2^{\gamma/\beta-1}\big(R^*\big(g(t),t_0\big)\big)^{\gamma/\beta-1}, \quad t \ge t_2. \tag{5.16}$$

By (5.14) and (5.6) we have

$$\begin{aligned} w(t) &= m(t)\frac{L_2y(t)}{y^\gamma(g(t))} \le m(t)\frac{L_2y(g(t))}{y^\gamma(g(t))} \\ &\le m(t)\big(R^*\big(g(t),t_0\big)\big)^{-\beta}y^{\beta-\gamma}\big(g(t)\big), \quad t \ge t_2. \end{aligned} \tag{5.17}$$

Using (5.16) in (5.17), we get

$$w(t) \le c_2^{\beta-\gamma}m(t)\big(R^*\big(g(t),t_0\big)\big)^{-\gamma}, \quad t \ge t_2.$$

Then

$$w^{1/\beta-1}(t) \ge c_2^{(1/\beta-1)(\beta-\gamma)}m^{1/\beta-1}(t)\big(R^*\big(g(t),t_0\big)\big)^{-\gamma(1/\beta-1)}, \quad t \ge t_2. \tag{5.18}$$

Using (5.16) and (5.18) in (5.15), we get

$$\begin{aligned} &T_\alpha^{t_0}w(t) \\ &\quad \le A(t)w(t) - km(t)q(t) \\ &\qquad - \gamma c_2^{-\beta+\gamma}m^{-1}T_\alpha^{t_0}g(t)\big(R^*\big(g(t),t_0\big)\big)^{\gamma-1}\left(\frac{R_2(g(t),t_0)}{r_1(g(t))}\right)^{1/\beta}(t-t_0)^{\alpha-1}w^2(t) \\ &\quad = A(t)w(t) - km(t)q(t) - B(t)w^2(t) \\ &\quad = -km(t)q(t) - \left(\sqrt{B(t)}w(t) - \frac{A(t)}{2\sqrt{B(t)}}\right)^2 + \frac{A^2(t)}{4B(t)} \\ &\quad \le -km(t)q(t) + \frac{A^2(t)}{4B(t)}, \quad t \ge t_2, \end{aligned} \tag{5.19}$$

where $c^* = \gamma c_2^{\gamma-\beta}$, and A and B are as in (5.13). Applying I_α to (5.19) from t_0 to t, we get

$$I_\alpha^{t_0}\left[km(t)q(t) - \frac{A^2(t)}{4B(t)}\right] \le w(t_0) - w(t) \le w(t_0),$$

which contradicts (5.12).

Step 2. Let $L_1y(t) < 0$ on $[t_1,\infty)$. We consider the function $L_2y(t)$. The case $L_2y(t) \le 0$ cannot hold for all large t, say $t \ge t_2 \ge t_1$, since by double integration of

$$T_\alpha^{t_0}y(t) = \left(\frac{L_1y(t)}{r_1(t)}\right)^{1/\beta} \le \left(\frac{L_1y(t_2)}{r_1(t)}\right)^{1/\beta}, \quad t \ge t_2,$$

we get from (5.2) that $y(t) \le 0$ for all large t, which is a contradiction. Thus we assume that $y(t) > 0$, $L_1y(t) < 0$, and $L_2y(t) \ge 0$ for all large t, say $t \ge t_3 \ge t_2$. Now, for $v \ge u \ge t_3$, we have

$$\begin{aligned} y(u) > y(u) - y(v) &= -I_\alpha^{t_0}\left[\frac{r_1^{1/\beta}(v) T_\alpha^{t_0} y(v)}{r_1^{1/\beta}(v)}\right] \\ &\geq -I_\alpha^{t_0}\left[\frac{1}{r_1^{1/\beta}(v)}\right] r_1^{1/\beta}(v) T_\alpha^{t_0} y(v) \\ &= R_1(v,t_0)\bigl(-L_1 y(v)\bigr)^{1/\beta}. \end{aligned}$$

Letting $u = g(t)$ and $v = h(t)$, we obtain

$$\begin{aligned} y\bigl(g(t)\bigr) &\geq R_1\bigl(h(t),t_0\bigr)\bigl(-L_1 y\bigl(h(t)\bigr)\bigr)^{1/\beta} \\ &= R_1\bigl(h(t),t_0\bigr)x\bigl(h(t)\bigr), \quad \text{for } h(t) \geq g(t) \geq t_3, \end{aligned}$$

where $x(t) = (-L_1 y(t))^{1/\beta} > 0$ for $t \geq t_3$. By (5.1), since that $x(t)$ is decreasing and $g(t) \leq h(t) \leq t$, we get

$$T_\alpha^{t_0}\bigl(r_2 T_\alpha^{t_0} z\bigr)(t) + \frac{p(t)}{r_1(t)} z\bigl(h(t)\bigr) \geq kq(t)\bigl(R_1\bigl(h(t),t_0\bigr)\bigr)^\gamma z\bigl(h(t)\bigr) z^{\gamma/\beta-1}\bigl(h(t)\bigr),$$

where $z(t) = x^\beta(t)$. Because $z(t)$ is decreasing and $\beta \geq \gamma$, there exists a constant $c_4 > 0$ such that $z^{\gamma/\beta-1}(t) \geq c_4$ for $t \geq t_2$. Then we have

$$\begin{aligned} T_\alpha^{t_0}\bigl(r_2 T_\alpha^{t_0} z\bigr)(t) &\geq kq(t)\bigl(R_1\bigl(h(t),g(t)\bigr)\bigr)^\gamma z\bigl(h(t)\bigr) z^{\gamma/\beta-1}\bigl(h(t)\bigr) - \frac{p(t)}{r_1(t)} z\bigl(h(t)\bigr) \\ &\geq \left[c_4 kq(t)\bigl(R_1\bigl(h(t),g(t)\bigr)\bigr)^\gamma - \frac{p(t)}{r_1(t)}\right] z\bigl(h(t)\bigr). \end{aligned}$$

Proceeding exactly as in the proofs of Lemmas 5.3 and 5.4, we arrive at the desired conclusion, thus completing the proof. □

Example 5.1

$$\begin{aligned} &T_{\frac{1}{2}}\bigl(T_{\frac{1}{2}}\bigl(t^{-\frac{3}{2}} T_{\frac{1}{2}} y(t)\bigr)\bigr) + t^{-\frac{5}{2}} T_{\frac{1}{2}} y(t) \\ &\quad + \left[\frac{1}{2}(t-2)^{-2} t^{-\frac{1}{2}} + 2(t-2)^{-2} t^{-1} + 1\right] f\bigl(y(t-2)\bigr) = 0, \quad t > 0, \end{aligned} \tag{5.20}$$

where $r_1(t) = t^{-\frac{3}{2}}$, $r_2(t) = 1$, $q(t) = \frac{1}{2}(t-2)^{-2} t^{-\frac{1}{2}} + 2(t-2)^{-2} t^{-1} + 1$, $p(t) = t^{-\frac{5}{2}}$, $g(t) = t-2$, $h(t) = t-2$, $\alpha = \frac{1}{2}$, $\beta = 1$, $\gamma = 1$, $c^* = 1$. By taking $m(t) = 1$ we get

$$R_1(t,t_0) = I_\alpha^{t_0} \frac{1}{r_1^{1/\beta}(t)} = I_\alpha t^{\frac{3}{2}} = \frac{1}{2} t^2 \to \infty \quad \text{as } t \to \infty,$$

$$R_2(t,t_0) = I_\alpha^{t_0} \frac{1}{r_2(t)} = I_\alpha 1 = 2t^{\frac{1}{2}} \to \infty \quad \text{as } t \to \infty,$$

$$\begin{cases} A(t) = \frac{T_\alpha^{t_0} m(t)}{m(t)} - \frac{p(t)}{r_1(t)} R_2(t,t_0) = -t^{-1} R_2(t,t_0) = -t^{-1} 2t^{\frac{1}{2}} = 2t^{-\frac{1}{2}}, \\ B(t) = c^* m^{-1}(t) T_\alpha^{t_0} g(t) (R^*(g(t),t_0))^{\gamma-1} (\frac{R_2(g(t),t_0)}{r_1(g(t))})^{1/\beta} (t-t_0)^{\alpha-1} = 2(t-2)^2 t^{-\frac{1}{2}}, \end{cases}$$

$$I_\alpha^{t_0}\left[km(t)q(t)-\frac{A^2(t)}{4B(t)}\right]=I_\alpha\left(\frac{1}{2}(t-2)^{-2}t^{-\frac{1}{2}}+2(t-2)^{-2}t^{-1}+1-\frac{4t^{-1}}{8(t-2)^2t^{-\frac{1}{2}}}\right)$$
$$=I_\alpha\left(2(t-2)^{-2}t^{-1}+1\right),$$

so

$$\limsup_{t\to\infty} I_\alpha^{t_0}\left[km(t)q(t)-\frac{A^2(t)}{4B(t)}\right]=\infty.$$
$$Q(t)=ckq(t)\left(R_1\left(h(t),t_0\right)\right)^\gamma-\frac{p(t)}{r_1(t)}$$
$$=\left(\frac{1}{2}(t-2)^{-2}t^{-\frac{1}{2}}+2(t-2)^{-2}t^{-1}+1\right)\left(\frac{1}{2}(t-2)^2\right)-t^{-1}$$
$$=\frac{1}{4}t^{-\frac{1}{2}}+\frac{1}{2}(t-2)^2\geq 0,$$

and we obtain that

$$I_\alpha^{t_0}Q(t)=I_\alpha\frac{1}{4}t^{-\frac{1}{2}}+\frac{1}{2}(t-2)^2=\int_0^t\left(\frac{1}{4}s^{-1}+\frac{1}{2}(s-2)^2s^{-\frac{1}{2}}\right)ds$$
$$=\int_0^t\left(\frac{1}{4}s^{-1}+\frac{1}{2}s^{\frac{3}{2}}-2s^{\frac{1}{2}}+2s^{-\frac{1}{2}}\right)ds$$
$$=\frac{1}{4}\ln t+\frac{1}{5}t^{\frac{5}{2}}-\frac{4}{3}t^{\frac{3}{2}}+4t^{\frac{1}{2}}-\frac{1}{4}\ln 0.$$

Hence

$$I_\alpha^{t_0}Q(t)>I_\alpha^{t_0}Q(1)=0+\frac{1}{5}-\frac{4}{3}+4-\frac{1}{4}\ln 0>1,\quad t>1,$$
$$R_2\left(h(t),t_0\right)>R_2\left(h\left(\frac{9}{4}\right),t_0\right)=2\left(\frac{9}{4}-2\right)^{\frac{1}{2}}=1,\quad t>\frac{9}{4},$$
$$R_2(u,t_0)>R_2\left(\frac{1}{4},t_0\right)=2\left(\frac{1}{4}\right)^{\frac{1}{2}}=1,\quad u>\frac{1}{4}.$$

So

$$\limsup_{t\to\infty} R_2\left(h(t),t_0\right)I_\alpha^{t_0}Q(t)>1,$$
$$\limsup_{t\to\infty,u\to\infty} R_2(u,t_0)I_\alpha^{t_0}Q(t)>1,$$

Then we see that (5.8) and (5.10) are clearly satisfied, and it is easy to verify that the equation

$$T_{\frac{1}{2}}\left(T_{\frac{1}{2}}z(t)\right)+t^{-1}z(t)=0 \tag{5.21}$$

is nonoscillatory, and one nonoscillatory solution of (5.21) is $z(t)=18t^{\frac{1}{3}}$. Then we get that equation (5.20) is oscillatory.

Example 5.2

$$T_{\frac{1}{2}}\big(t^{-\frac{1}{2}}T_{\frac{1}{2}}\big(t^{-\frac{1}{2}}T_{\frac{1}{2}}y(t)\big)\big)+2t^{-\frac{1}{2}}T_{\frac{1}{2}}y(t)+3y(t)=0,\quad t\geq 0, \tag{5.22}$$

where $r_1(t)=r_2(t)=t^{-\frac{1}{2}}$, $p(t)=2t^{-\frac{1}{2}}$, $q(t)=3$, $k=1$, $g(t)=t$, $\alpha=\frac{1}{2}$, $\beta=\gamma=1$, $c^*=c=1$. Letting $m(t)=1$ and $h(t)=t$, we can obtain

$$R_2(t,t_0)=t,\qquad A(t)=-2t,\qquad B(t)=t,\qquad Q(t)=3t-2,$$

so all conditions except (5.12) are satisfied.

Equation (5.22) can be rewritten as

$$y'''(t)+2y'(t)+3y(t)=0.$$

It is obvious that the equation is nonoscillatory. It has a nonoscillatory solution $x=e^{\frac{1}{2}t}\cos\frac{\sqrt{2}}{2}t$. We can obtain that condition (5.12) indispensable.

6 Conclusion

In this paper, we study three kinds of different order conformable fractional equations and obtain oscillatory results of three equations. Those results unify the oscillation theory of the integral-order and fractional-order differential equations.

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