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Extended r-central Bell polynopmials with umbral calculus viewpoint

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Abstract

Recently, extended r-central factorial numbers of the second kind and extended r-central Bell polynomials were introduced and various results of them were investigated. The purpose of this paper is to further derive properties, recurrence relations and identities related to these numbers and polynomials using umbral calculus techniques. Especially, we will represent the extended r-central Bell polynomials in terms of quite a few families of well-known special polynomials.

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1 Introduction and preliminaries

In [5], the extended r-central factorial numbers of the second kind and the extended r-central Bell polynomials were introduced and various properties and identities related to these numbers and polynomials were investigated by means of generating functions. The extended r-central factorial numbers of the second kind are an extended version of the central factorial numbers of the second kind and also a central analogue of the r-Stirling numbers of the second kind (see [2, 7, 10, 11, 14, 15, 17, 22]). The extended r-central Bell polynomials are an extended version of the central Bell polynomials and also a central analogue of r-Bell polynomials (see [3, 8, 11, 12, 15, 18]).

Here we study the extended r-central factorial numbers of the second kind and the extended r-central Bell polynomials by making use of umbral calculus techniques. In particular, we represent the extended r-central Bell polynomials in terms of many well-known special polynomials. Here the special polynomials are Bernoulli polynomials, Euler polynomials, falling factorial polynomials, Abel polynomials, ordered Bell polynomials, Laguerre polynomials, Daehee polynomials, Hermite polynomials, polynomials closely related to the reverse Bessel polynomials and studied by Carlitz, and Bernoulli polynomials of the second kind. The necessary facts about umbral calculus will be briefly reviewed in the next section.

The central factorials $x^{[n]}$ $(n \geq 0)$ are given by (see [1, 4–6, 10, 12, 20])

$$x^{[0]} = 1, \qquad x^{[n]} = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right)\cdots\left(x - \frac{n}{2} + 1\right) \quad (n \geq 1). \tag{1.1}$$

For nonnegative integers n, k, with $n \geq k$, the central factorial numbers of the second kind $T(n,k)$ are given by the coefficients in the expansion (see [1, 4–6, 10, 12, 20])

$$x^n = \sum_{k=0}^{n} T(n,k)x^{[k]}. \tag{1.2}$$

It is well known that $T(2n, 2n-2k)$ enumerates the number of ways to pace k rooks on a 3 D-triangle board of size $(n-1)$ (see [1, 16]). The central factorial numbers of the second kind $T(n,k)$ are given by the generating function

$$\frac{1}{k!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}. \tag{1.3}$$

From (1.3), we can easily deduce that they are explicitly given by

$$\begin{aligned} T(n,k) &= \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\left(\frac{k}{2}-i\right)^n \\ &= \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(-\frac{k}{2}+i\right)^n \\ &= \frac{(-1)^{n-k}}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(-\frac{k}{2}+i\right)^n. \end{aligned} \tag{1.4}$$

Thus (1.4) yields

$$T(n,k) = 0, \quad \text{if } n \not\equiv k \pmod 2, \tag{1.5}$$

and that

$$T(n,k) = \frac{1}{k!}\triangle^k\left(-\frac{k}{2}\right)^n = \frac{1}{k!}\delta^k 0^n, \tag{1.6}$$

where $\triangle^k(-\frac{k}{2})^n = \triangle^k x^n|_{x=-\frac{k}{2}}$, $\triangle f(x) = f(x+1) - f(x)$ is the forward difference operator, $\delta^k 0^n = \delta^k x^n|_{x=0}$, and $\delta f(x) = f(x+\frac{1}{2}) - f(x-\frac{1}{2})$ is the central difference operator.

For these results, one may refer to [19, 20].

Let r be any nonnegative integer. The central factorial numbers of the second kind $T(n,k)$ were generalized to the extended r-central factorial numbers of the second kind $T_r(n+r, k+r)$ (see [5]). For nonnegative integers n, k, with $n \geq k$, $T_r(n+r, k+r)$ are given by the coefficients in the expansion

$$(x+r)^n = \sum_{k=0}^{n} T_r(n+r, k+r)x^{[k]}. \tag{1.7}$$

The extended r-central factorial numbers of the second kind, $T_r(n+r, k+r)$, are also given by the generating function

$$\frac{1}{k!}e^{rt}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k = \sum_{n=k}^{\infty} T_r(n+r, k+r)\frac{t^n}{n!}. \tag{1.8}$$

An explicit expression for $T_r(n+r,k+r)$ can be deduced from (1.8) as follows:

$$\begin{aligned} T_r(n+r,k+r) &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(r+l-\frac{k}{2}\right)^n \\ &= \frac{1}{k!}\delta^k r^n, \end{aligned} \tag{1.9}$$

where $\delta^k r^n = \delta^k x^n|_{x=r}$, and by induction we can show

$$\delta^k f(x) = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}f\left(x+l-\frac{k}{2}\right). \tag{1.10}$$

For some details on these, one may refer to [4, 5, 20]. The central Bell polynomials $B_n^{(c)}(x)$ are defined by (see [5, 10, 12, 13])

$$e^{x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} B_n^{(c)}(x)\frac{t^n}{n!}. \tag{1.11}$$

Then it is immediate from (1.11) that

$$B_n^{(c)}(x) = \sum_{k=0}^{n} T(n,k)x^k, \tag{1.12}$$

and that (see [12])

$$B_n^{(c)}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left(-\frac{k}{2}\right)^{n-k}x^k, \tag{1.13}$$

where $S_2(l,k)$ are the Stirling numbers of the second kind, given by

$$\frac{1}{k!}\left(e^t-1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{1.14}$$

Further, the central Bell polynomials are given by the following Dobinski-like formula (see [12]):

$$B_n^{(c)}(x) = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\binom{l+j}{j}(-1)^j\frac{1}{(l+j)!}\left(\frac{l}{2}-\frac{j}{2}\right)^n x^{l+j}. \tag{1.15}$$

On the other hand, the extended r-central Bell polynomials $B_n^{(c,r)}(x)$ are defined by (see [5, 6, 21])

$$e^{rt}e^{x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} B_n^{(c,r)}(x)\frac{t^n}{n!}. \tag{1.16}$$

Then it is easy from (1.16) that

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n} T_r(n+r,k+r)x^k, \tag{1.17}$$

and that (see [5])

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left(r-\frac{k}{2}\right)^{n-k}x^k. \tag{1.18}$$

In addition, the extended r-central Bell polynomials are given by the following Dobinski-like formula:

$$B_n^{(c,r)}(x) = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\binom{l+j}{j}(-1)^j\frac{1}{(l+j)!}\left(\frac{l}{2}-\frac{j}{2}+r\right)^n x^{l+j}, \tag{1.19}$$

which can be observed immediately from the proof of Theorem 2.3 in [12].

2 Quick review of umbral calculus

Here we will briefly recall some of the basic facts about umbral calculus. The reader is advised to refer to [20] for a complete treatment. Let $\mathbb{C}$ be the field of complex numbers, and let $\mathfrak{F}$ be the algebra of all formal power series in the variable t with the coefficients in $\mathbb{C}$:

$$\mathfrak{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k\frac{t^k}{k!} \middle| a_k \in \mathbb{C}\right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ denote the ring of polynomials in x with the coefficients in $\mathbb{C}$, and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. For $L \in \mathbb{P}^*$ and $p(x) \in \mathbb{P}$, the notation $\langle L|p(x)\rangle$ will be used for the action of the linear functional L on $p(x)$.

For $f(t) = \sum_{k=0}^{\infty} a_k\frac{t^k}{k!} \in \mathfrak{F}$, the linear functional $\langle f(t)|\cdot\rangle$ on $\mathbb{P}$ is defined by

$$\langle f(t)|x^n\rangle = a_n \quad (n \geq 0). \tag{2.1}$$

In particular, from (2.2) we see that

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{2.2}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $L \in \mathbb{P}^*$, let $f_L(t) = \sum_{k=0}^{\infty}\langle L|x^k\rangle\frac{t^k}{k!} \in \mathfrak{F}$. Then we note that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$, and the map $L \to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ to $\mathfrak{F}$. Henceforth, $\mathfrak{F}$ denotes both the algebra of all formal power series in t and the vector space of all linear functionals on $\mathbb{P}$. Thus an element $f(t)$ of $\mathfrak{F}$ will be thought of as both a formal power series and a linear functional on $\mathbb{P}$. $\mathfrak{F}$ is called the umbral algebra, the study of which the umbral calculus is.

The order $o(f(t))$ of $0 \neq f(t) \in \mathfrak{F}$ is the smallest integer k such that the coefficient of t^k does not vanish. In particular, for $0 \neq f(t) \in \mathfrak{F}$ it is called an invertible series if $o(f(t)) = 0$ and a delta series if $o(f(t)) = 1$. Let $f(t), g(t) \in \mathfrak{F}$, with $o(g(t)) = 0$, $o(f(t)) = 1$. Then there exists a unique sequence of polynomials $s_n(x)(\deg s_n(x) = n)$ such that

$$\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}, \quad \text{for } n,k \geq 0. \tag{2.3}$$

Such a sequence is called the Sheffer sequence for the Sheffer pair $(g(t),f(t))$, which we denote by $s_n(x) \sim (g(t),f(t))$. Then $s_n(x) \sim (g(t),f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{2.4}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

$$t^k p(x) = p^{(k)}(x), \qquad e^{yt} p(x) = p(x+y), \qquad \langle e^{yt} | p(x) \rangle = p(y). \tag{2.5}$$

Let $s_n(x) \sim (g(t),f(t))$. Then we have the following: The Sheffer identity is given by

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y), \tag{2.6}$$

where $p_n(x) = g(t)s_n(x) \sim (1,f(t))$. Then the conjugate representation says that

$$s_n(x) = \sum_{k=0}^{n} \frac{1}{k!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^k | x^n \rangle x^k. \tag{2.7}$$

We also have the recurrence formula

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x), \tag{2.8}$$

$$f(t)s_n(x) = n s_{n-1}(x) \quad (n \geq 0), \tag{2.9}$$

$$\begin{aligned} x s_n(x) = {} & \sum_{k=0}^{n} \binom{n}{k} \langle g(\bar{f}(t))^{-1} g'(\bar{f}(t)) | x^{n-k} \rangle s_k(x) \\ & + \sum_{k=1}^{n+1} \binom{n}{k-1} \langle f'(\bar{f}(t)) | x^{n-k+1} \rangle s_k(x). \end{aligned} \tag{2.10}$$

The derivative of $s_n(x)$ is given by

$$\frac{d}{dx} s_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} \langle \bar{f}(t) | x^{n-k} \rangle s_k(x) \quad (n \geq 1). \tag{2.11}$$

Assume that $s_n(x) \sim (g(t),f(t))$, $r_n(x) \sim (h(t), l(t))$. Then $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, where

$$c_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^k \middle| x^n \right\rangle. \tag{2.12}$$

Let $p_n(x) \sim (1,f(t))$, $q_n(x) \sim (1, l(t))$. Then the transfer formula says that

$$q_n(x) = x \left(\frac{f(t)}{l(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 1). \tag{2.13}$$

Finally, for $h(t) \in \mathfrak{F}$, $p(x) \in \mathbb{P}$,

$$\langle h(t) | x p(x) \rangle = \langle \partial_t h(t) | p(x) \rangle. \tag{2.14}$$

3 Main results

Here we will derive some properties, identities and recurrence relations for the extended r-central Bell polynomials by making use of umbral calculus techniques and the formulas in Sect. 2. In addition, we will express those polynomials as linear combinations of quite a few well-known special polynomials. Here the special polynomials are Bernoulli polynomials, Euler polynomials, falling factorial polynomials, Abel polynomials, ordered Bell polynomials, Laguerre polynomials, Daehee polynomials, Hermite polynomials, polynomials closely related to the reverse Bessel polynomials and studied by Carlitz, and Bernoulli polynomials of the second kind.

We first note from (1.16) and (2.4) that

$$\begin{aligned} B_n^{(c,r)}(x) &\sim \left(\left(\frac{t+\sqrt{t^2+4}}{2}\right)^{-2r}, \log\left(\frac{t+\sqrt{t^2+4}}{2}\right)^2\right) \\ &= \big(g(t), f(t)\big). \end{aligned} \tag{3.1}$$

Using (2.14) for $n \geq 1$, we have

$$\begin{aligned} B_n^{(c,r)}(y) &= \left\langle \sum_{m=0}^{\infty} B_m^{(c,r)}(y)\frac{t^m}{m!} \middle| x^n \right\rangle \\ &= \langle e^{rt}e^{y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} | x^n\rangle \\ &= \langle \partial_t \big(e^{rt}e^{y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\big) | x^{n-1}\rangle \\ &= \langle re^{rt}e^{y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} | x^{n-1}\rangle + \left\langle e^{rt}e^{y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\frac{1}{2}y\big(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\big) \middle| x^{n-1}\right\rangle \\ &= r\langle e^{rt}e^{y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} | x^{n-1}\rangle + \frac{1}{2}y\langle e^{\frac{t}{2}} | e^{rt}e^{y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}x^{n-1}\rangle \\ &\quad + \frac{1}{2}y\langle e^{-\frac{t}{2}} | e^{rt}e^{y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}x^{n-1}\rangle \\ &= rB_{n-1}^{(c,r)}(y) + \frac{1}{2}y\left\langle e^{\frac{t}{2}} \middle| \sum_{m=0}^{\infty} B_m^{(c,r)}(y)\frac{t^m}{m!}x^{n-1}\right\rangle \\ &\quad + \frac{1}{2}y\left\langle e^{-\frac{t}{2}} \middle| \sum_{m=0}^{\infty} B_m^{(c,r)}(y)\frac{t^m}{m!}x^{n-1}\right\rangle \\ &= rB_{n-1}^{(c,r)}(y) + \frac{1}{2}y\sum_{m=0}^{n-1}\frac{1}{m!}B_m^{(c,r)}(y)(n-1)_m\langle e^{\frac{t}{2}} | x^{n-1-m}\rangle \\ &\quad + \frac{1}{2}y\sum_{m=0}^{n-1}\frac{1}{m!}B_m^{(c,r)}(y)(n-1)_m\langle e^{-\frac{t}{2}} | x^{n-1-m}\rangle \\ &= rB_{n-1}^{(c,r)}(y) + \frac{1}{2}y\sum_{m=0}^{n-1}\binom{n-1}{m}B_m^{(c,r)}(y)\left(\frac{1}{2}\right)^{n-1-m}\big(1+(-1)^{n-1-m}\big) \\ &= rB_{n-1}^{(c,r)}(y) + y\sum_{0\leq m\leq n-1, n\not\equiv m (\mathrm{mod}\, 2)}\binom{n-1}{m}B_m^{(c,r)}(y)\left(\frac{1}{2}\right)^{n-1-m}. \end{aligned} \tag{3.2}$$

Thus by replacing n by $n+1$, we obtained the following theorem.

Theorem 3.1 *For any nonnegative integer n, we have*

$$B_{n+1}^{(c,r)}(x) = rB_n^{(c,r)}(x) + x \sum_{0\le m\le n, n\equiv m (\mathrm{mod}\, 2)} \binom{n}{m} B_m^{(c,r)}(x) \left(\frac{1}{2}\right)^{n-m}. \tag{3.3}$$

For $n \geq 1$ and from (2.11), we get

$$\begin{aligned}
\frac{d}{dx} B_n^{(c,r)}(x) &= \sum_{k=0}^{n-1} \binom{n}{k} \langle e^{\frac{t}{2}} - e^{-\frac{t}{2}} | x^{n-k} \rangle B_k^{(c,r)}(x) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{1}{2}\right)^{n-k} \bigl(1-(-1)^{n-k}\bigr) B_k^{(c,k)}(x) \\
&= \sum_{0\le k\le n-1, n\not\equiv k (\mathrm{mod}\, 2)} \binom{n}{k} \left(\frac{1}{2}\right)^{n-1-k} B_k^{(c,r)}(x).
\end{aligned}$$

Again, by replacing n by $n + 1$, we have shown the next result.

Theorem 3.2 *For any nonnegative integer n, we have the following expression*

$$\frac{d}{dx} B_{n+1}^{(c,r)}(x) = \sum_{0\le k\le n, n\equiv k (\mathrm{mod}\, 2)} \binom{n+1}{k} \left(\frac{1}{2}\right)^{n-k} B_k^{(c,r)}(x). \tag{3.4}$$

From (3.2), we first observe that

$$f'(t) = \frac{2}{\sqrt{t^2+4}}, \qquad \frac{g'(t)}{g(t)} = \frac{-2r}{\sqrt{t^2+4}}. \tag{3.5}$$

Then, by using (2.8) and (2.9), we obtain

$$\begin{aligned}
B_{n+1}^{(c,r)}(x) &= \left(x + \frac{2r}{\sqrt{t^2+4}}\right) \frac{\sqrt{t^2+4}}{2} B_n^{(c,r)}(x) \\
&= x\left(\frac{t+\sqrt{t^2+4}}{2} - \frac{t}{2}\right) B_n^{(c,r)}(x) + rB_n^{(c,r)}(x) \\
&= x\left(e^{\frac{1}{2}f(t)} - \frac{t}{2}\right) B_n^{(c,r)}(x) + rB_n^{(c,r)}(x) \\
&= xe^{\frac{1}{2}f(t)} B_n^{(c,r)}(x) - \frac{1}{2}x\frac{d}{dx} B_n^{(c,r)}(x) + rB_n^{(c,r)}(x) \\
&= x\sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{2^k} f(t)^k B_n^{(c,r)}(x) - \frac{1}{2}x\frac{d}{dx} B_n^{(c,r)}(x) + rB_n^{(c,r)}(x) \\
&= x\sum_{k=0}^{n} \binom{n}{k}\frac{1}{2^k} B_{n-k}^{(c,r)}(x) - \frac{1}{2}x\frac{d}{dx} B_n^{(c,r)}(x) + rB_n^{(c,r)}(x).
\end{aligned} \tag{3.6}$$

This does not give us a new result. In fact, combining (3.3) and (3.6) yields Eq. (3.4).

In order to apply Eq. (2.10), we first note from (3.5) and $\bar{f}(t) = e^{\frac{t}{2}} - e^{-\frac{t}{2}}$ that

$$g\bigl(\bar{f}(t)\bigr)^{-1} g'\bigl(\bar{f}(t)\bigr) = \frac{-2r}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}, \qquad f'\bigl(\bar{f}(t)\bigr) = \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}. \tag{3.7}$$

Then, by (2.10), we have

$$\begin{aligned} xB_n^{(c,r)}(x) &= -r\sum_{k=0}^{n}\binom{n}{k}\left\langle \frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\,\middle|\, x^{n-k}\right\rangle B_k^{(c,r)}(x) \\ &\quad + \sum_{k=1}^{n+1}\binom{n}{k-1}\left\langle \frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\,\middle|\, x^{n-k+1}\right\rangle B_k^{(c,r)}(x) \\ &= -r\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{*}B_k^{(c,r)}(x) + \sum_{k=1}^{n+1}\binom{n}{k-1}E_{n-k+1}^{*}B_k^{(c,r)}(x) \\ &= \sum_{k=0}^{n+1}\left\{-r\binom{n}{k}E_{n-k}^{*} + \binom{n}{k-1}E_{n-k+1}^{*}\right\}B_k^{(c,r)}(x). \end{aligned}$$

Here E_n^* are the type 2 Euler numbers, which are given by

$$\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}} = \sum_{n=0}^{\infty}E_n^{*}\frac{t^n}{n!}. \tag{3.8}$$

Thus we have shown the following theorem.

Theorem 3.3 *For $n \geq 0$, we have the following identity*:

$$xB_n^{(c,r)}(x) = \sum_{k=0}^{n+1}\left\{-r\binom{n}{k}E_{n-k}^{*} + \binom{n}{k-1}E_{n-k+1}^{*}\right\}B_k^{(c,r)}(x),$$

where E_n^* are the type 2 Euler numbers given in (3.8).

Noting that $B_n^{(c)}(x) = g(t)B_n^{(c,r)}(x)$, from the Sheffer identity in (2.6), we have

$$B_n^{(c,r)}(x+y) = \sum_{k=0}^{n}\binom{n}{k}B_k^{(c,r)}(x)B_{n-k}^{(c)}(y). \tag{3.9}$$

Noting that the Bernoulli polynomials $B_n(x)$ is Sheffer for the pair $(\frac{e^t-1}{t}, t)$, we write $B_n^{(c,r)}(x) = \sum_{k=0}^{n} C_{n,k}B_k(x)$. Then

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle e^{rt}\frac{e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k \,\middle|\, x^n\right\rangle \\ &= \left\langle \frac{e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\,\middle|\, \sum_{l=k}^{\infty}T_r(l+r,k+r)\frac{t^l}{l!}x^{n-l}\right\rangle \\ &= \sum_{l=k}^{n}\binom{n}{l}T_r(l+r,k+r)\left\langle \frac{e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1}{t}\,\middle|\, \frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}x^{n-l}\right\rangle. \end{aligned} \tag{3.10}$$

Before proceeding, let us recall that B_n^* are the type 2 Bernoulli numbers, which are defined by

$$\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \sum_{n=0}^{\infty} B_n^* \frac{t^n}{n!}. \tag{3.11}$$

From (3.10) and (3.12), we have

$$\begin{aligned} C_{n,k} &= \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) \left\langle \frac{e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1}{t} \middle| \sum_{m=0}^{\infty} B_m^* \frac{t^m}{m!} x^{n-l} \right\rangle \\ &= \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} T_r(l+r,k+r) B_m^* \left\langle \frac{e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1}{t} \middle| x^{n-l-m} \right\rangle \\ &= \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} T_r(l+r,k+r) B_m^* \frac{1}{n-l-m+1} \langle e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1 | x^{n-l-m+1} \rangle \\ &= \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} T_r(l+r,k+r) B_m^* \frac{1}{n-l-m+1} \left\langle \sum_{s=1}^{\infty} B_s^{(c)} \frac{t^s}{s!} \middle| x^{n-l-m+1} \right\rangle. \end{aligned} \tag{3.12}$$

Here $B_n^{(c)}$ are the central Bell numbers, given by

$$e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \sum_{n=0}^{\infty} B_n^{(c)} \frac{t^n}{n!}. \tag{3.13}$$

Finally, from (3.12) we obtain

$$\begin{aligned} C_{n,k} &= \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} \frac{1}{n-l-m+1} T_r(l+r,k+r) B_m^* B_{n-l-m+1}^{(c)} \\ &= \frac{1}{n+1} \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n+1}{l} \binom{n-l+1}{m} T_r(l+r,k+r) B_m^* B_{n-l-m+1}^{(c)}. \end{aligned} \tag{3.14}$$

This completes the proof for the next theorem.

Theorem 3.4 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $B_k(x)$:*

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n} \left(\frac{1}{n+1} \sum_{l=k}^{n} \sum_{m=0}^{n-l} \binom{n+1}{l} \binom{n-l+1}{m} T_r(l+r,k+r) B_m^* B_{n-l-m+1}^{(c)} \right) B_k(x),$$

where B_n^* are the type 2 Bernoulli numbers in (3.11) and $B_n^{(c)}$ are the central Bell numbers in (3.13).

Let us write $B_n^{(c,r)}(x) = \sum_{k=0}^{n} C_{n,k} E_k(x)$. Here $E_n(x)$ are the Euler polynomials with $E_n(x) \sim (\frac{e^t+1}{2}, t)$. Then

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle e^{rt}\frac{e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}+1}{2}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k \Big| x^n\right\rangle \\ &= \frac{1}{2}\sum_{l=k}^{n}\binom{n}{l} T_r(l+r,k+r)\left\langle e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}+1 | x^{n-l}\right\rangle \\ &= \frac{1}{2}\sum_{l=k}^{n}\binom{n}{l} T_r(l+r,k+r)\left(B_{n-l}^{(c)}+\delta_{n,k}\right). \end{aligned} \tag{3.15}$$

This shows the next result.

Theorem 3.5 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $E_k(x)$:*

$$B_n^{(c,r)}(x) = \frac{1}{2}\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l} T_r(l+r,k+r)\left(B_{n-l}^{(c)}+\delta_{n,l}\right)E_k(x).$$

We let $B_n^{(c,r)}(x) = \sum_{k=0}^{n} C_{n,k}(x)_k$. Here $(x)_n$ is the falling factorial sequence with $(x)_n \sim (1, e^t - 1)$. Then

$$\begin{aligned} C_{n,k} &= \left\langle \frac{1}{k!}e^{rt}\left(e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1\right)^k \Big| x^n\right\rangle \\ &= \left\langle \sum_{l=k}^{\infty} S_{2,r}(l+r,k+r)\frac{1}{l!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^l \Big| x^n\right\rangle \\ &= \sum_{l=k}^{n} S_{2,r}(l+r,k+r)\left\langle \frac{1}{l!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^l \Big| x^n\right\rangle \\ &= \sum_{l=k}^{n} S_{2,r}(l+r,k+r)\left\langle \sum_{m=l}^{\infty} T(m,l)\frac{t^m}{m!} \Big| x^n\right\rangle \\ &= \sum_{l=k}^{n}\sum_{m=l}^{n} S_{2,r}(l+r,k+r)T(m,l)\frac{1}{m!}n!\delta_{n,m} \\ &= \sum_{l=k}^{n} S_{2,r}(l+r,k+r)T(n,l). \end{aligned} \tag{3.16}$$

Here $S_{2,r}(l+r,k+r)$ are the r-Stirling numbers of the second kind, given by

$$\frac{1}{k!}e^{rt}\left(e^t-1\right)^k = \sum_{n=k}^{\infty} S_{2,r}(n+r,k+r)\frac{t^n}{n!}. \tag{3.17}$$

This gives the next result.

Theorem 3.6 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $(x)_k$:*

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n} S_{2,r}(l+r,k+r)T(n,l)(x)_k.$$

To obtain another expression, we compute $C_{n,k}$ in (3.16) in a different way as follows:

$$\begin{aligned}
&\frac{1}{k!}\left\langle e^{rt}\left(e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1\right)^{k}\Big|x^{n}\right\rangle \\
&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left\langle e^{rt}e^{l(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\Big|x^{n}\right\rangle \\
&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left\langle \sum_{m=0}^{\infty}B_{m}^{(c,r)}(l)\frac{t^{m}}{m!}\Big|x^{n}\right\rangle \\
&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}B_{n}^{(c,r)}(l).
\end{aligned}\tag{3.18}$$

This finishes the proof of the next theorem.

Theorem 3.7 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $(x)_k$:*

$$B_n^{(c,r)}(x)=\sum_{k=0}^{n}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}B_n^{(c,r)}(l)(x)_k.$$

Let $B_n^{(c,r)}(x)=\sum_{k=0}^{n}C_{n,k}A_k(x;a)$. Here $A_n(x;a)$ are the Abel polynomials with $A_n(x;a)\sim(1,te^{at})$, $(a\neq 0)$. Then

$$\begin{aligned}
C_{n,k}&=\frac{1}{k!}\left\langle e^{rt}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^{k}e^{ak(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\Big|x^{n}\right\rangle \\
&=\sum_{l=k}^{n}\binom{n}{l}T_r(l+r,k+r)\left\langle e^{ak(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\Big|x^{n-l}\right\rangle \\
&=\sum_{l=k}^{n}\binom{n}{l}T_r(l+r,k+r)\left\langle \sum_{m=0}^{\infty}B_{m}^{(c)}(ak)\frac{t^{m}}{m!}\Big|x^{n-l}\right\rangle \\
&=\sum_{l=k}^{n}\binom{n}{l}T_r(l+r,k+r)B_{n-l}^{(c)}(ak).
\end{aligned}\tag{3.19}$$

Thus the following has been verified.

Theorem 3.8 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $A_k(x;a)$:*

$$B_n^{(c,r)}(x)=\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}T_r(l+r,k+r)B_{n-l}^{(c,r)}(ak)A_k(x;a).$$

Let us write $B_n^{(c,r)}(x)=\sum_{k=0}^{n}C_{n,k}Ob_k(x)$. Here $Oh_n(x)$ are the ordered Bell polynomials with $Ob_n(x)\sim(2-e^t,t)$. Then

$$\begin{aligned}
C_{n,k}&=\frac{1}{k!}\left\langle e^{rt}\left(2-e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^{k}\Big|x^{n}\right\rangle \\
&=\sum_{l=k}^{n}\binom{n}{l}T_r(l+r,k+r)\left\langle 2-e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\Big|x^{n-l}\right\rangle
\end{aligned}$$

$$= \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r)\big(2\delta_{n,l} - B_{n-l}^{(c)}\big). \tag{3.20}$$

This shows the following result.

Theorem 3.9 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $Ob_k(x)$:*

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n} \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r)\big(2\delta_{n,l} - B_{n-l}^{(c)}\big) Ob_k(x).$$

We write $B_n^{(c,r)}(x) = \sum_{k=0}^{n} C_{n,k} L_k^{(\alpha)}(x)$, where $L_k^{(\alpha)}(x)$ are the Laguerre polynomials with $L_k^{(\alpha)}(x) \sim ((1-t)^{-\alpha-1}, \frac{t}{t-1})$. Then

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle e^{rt}\big(1 - \big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)\big)^{-\alpha-1} \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}} - 1}\right)^{k} \middle| x^n \right\rangle \\ &= (-1)^k \left\langle \big(1 - \big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)\big)^{-(\alpha+k+1)} \middle| \frac{1}{k!} e^{rt}\big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)^k x^n \right\rangle \\ &= (-1)^k \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r)\big\langle \big(1 - \big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)\big)^{-(\alpha+k+1)} \big| x^{n-l}\big\rangle. \end{aligned} \tag{3.21}$$

Before proceeding, we recall several definitions. The central Fubini polynomials $F_n^{(c)}(x)$ are defined by (see [9])

$$\frac{1}{1 - x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} F_n^{(c)}(x) \frac{t^n}{n!}. \tag{3.22}$$

For $x = 1$, $F_n^{(c)} = F_n^{(c)}(1)$ are called the central Fubini numbers.

More generally, for any real number α, the central Fubini polynomials $F_n^{(c,x)}(x)$ of order α are given by

$$\left(\frac{1}{1 - x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})}\right)^{\alpha} = \sum_{n=0}^{\infty} F_n^{(c,\alpha)}(x) \frac{t^n}{n!}. \tag{3.23}$$

For $x = 1$, $F_n^{(c,\alpha)} = F_n^{(c,\alpha)}(1)$ are called the central Fubini numbers of order α. Now, from (3.21) and (3.23) we have

$$\begin{aligned} C_{n,k} &= (-1)^k \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) \left\langle \sum_{m=0}^{\infty} F_m^{(c,\alpha+k+1)} \frac{t^m}{m!} \middle| x^{n-l} \right\rangle \\ &= (-1)^k \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) F_{n-l}^{(c,\alpha+k+1)}. \end{aligned}$$

This verifies the following theorem.

Theorem 3.10 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $L_k^{(\alpha)}(x)$.*

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n}(-1)^k \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) F_{n-l}^{(c,\alpha+k+1)} L_k^{(\alpha)}(x).$$

Let us write $B_n^{(c,r)}(x) = \sum_{k=0}^{n} C_{n,k} D_k(x)$. Here $D_n(x)$ are the Daehee polynomials with $D_n(x) \sim (\frac{e^t-1}{t}, e^t-1)$. Then

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle e^{rt}\frac{e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\left(e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1\right)^k \middle| x^n\right\rangle \\
&= \frac{1}{k!}\left\langle e^{rt}\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\frac{e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1}{t}\left(e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1\right)^k \middle| x^n\right\rangle \\
&= \frac{k+1}{n+1}\left\langle e^{rt}\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \middle| \frac{1}{(k+1)!}\left(e^{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}-1\right)^{k+1} x^{n+1}\right\rangle \\
&= \frac{k+1}{n+1}\left\langle \frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \middle| \sum_{l=k+1}^{\infty} S_2(l,k+1)\frac{1}{l!}e^{rt}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^l x^{n+1}\right\rangle \\
&= \frac{k+1}{n+1}\sum_{l=k+1}^{n+1} S_2(l,k+1)\left\langle \frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \middle| \sum_{m=l}^{\infty} T_r(m+r,l+r)\frac{t^m}{m!} x^{n+1}\right\rangle \\
&= \frac{k+1}{n+1}\sum_{l=k+1}^{n+1} S_2(l,k+1)\sum_{m=l}^{n+1}\binom{n+1}{m} T_r(m+r,l+r)\left\langle \frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \middle| x^{n+1-m}\right\rangle \\
&= \frac{k+1}{n+1}\sum_{l=k+1}^{n+1}\sum_{m=l}^{n+1} S_2(l,k+1) T_r(m+r,l+r)\binom{n+1}{m} B_{n+1-m}^{*}.
\end{aligned} \tag{3.24}$$

This completes the proof for the next result.

Theorem 3.11 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $D_k(x)$.*

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n}\frac{k+1}{n+1}\sum_{l=k+1}^{n+1}\sum_{m=l}^{n+1} S_2(l,k+1) T_r(m+r,l+r)\binom{n+1}{m} B_{n+1-m}^{*} D_k(x).$$

Let us put $B_n^{(c,r)}(x) = \sum_{k=0}^{n} C_{n,k} H_k^{(\nu)}(x)$. Here $H_k^{(\nu)}(x)$ are the Hermite polynomials with $H_k^{(\nu)}(x) \sim (e^{\frac{\nu t^2}{2}}, t)$. Then

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle e^{rt} e^{\nu(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^2/2}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^k \middle| x^n\right\rangle \\
&= \sum_{l=k}^{n}\binom{n}{l} T_r(l+r,k+r)\left\langle e^{\nu(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^2/2} \middle| x^{n-l}\right\rangle \\
&= \sum_{l=k}^{n}\binom{n}{l} T_r(l+r,k+r)\left\langle \sum_{m=0}^{\infty}\frac{\nu^m}{m!}\frac{(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^{2m}}{2^m} \middle| x^{n-l}\right\rangle
\end{aligned}$$

$$= \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) \sum_{m=0}^{[\frac{n-l}{2}]} \frac{(2m)!\nu^m}{m!2^m} \left\langle \frac{1}{(2m)!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^{2m} \middle| x^{n-l} \right\rangle$$

$$= \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) \sum_{m=0}^{[\frac{n-l}{2}]} \frac{(2m)!\nu^m}{m!2^m} T(n-l,2m). \tag{3.25}$$

This shows the following theorem.

Theorem 3.12 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $H_k^{(\nu)}(x)$.*

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n} \sum_{l=k}^{n} \sum_{m=0}^{[\frac{n-l}{2}]} \binom{n}{l} T_r(l+r,k+r) m! \binom{2m}{m} \left(\frac{\nu}{2}\right)^m T(n-l,2m) H_k^{(\nu)}(x).$$

The Bessel polynomials $y_n(x)$ are given by

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k,$$

which satisfies the differential equation

$$x^2 y'' + (2x+2)y' - n(n+1)y = 0.$$

The reverse Bessel polynomials are known to be

$$\theta_n(x) = x^n y_n\left(\frac{1}{x}\right) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \frac{x^{n-k}}{2^k},$$

which obey

$$xy'' - 2(x+n)y' + 2ny = 0.$$

On the other hand, Carlitz defined a related set of polynomials,

$$P_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right) = x\theta_{n-1}(x) \quad (n \geq 1), \qquad P_0(x) = 1.$$

Then we can show that $P_n(x) \sim (1, t - \frac{1}{2}t^2)$ (see [20]). Let us write $B_n^{(c,r)}(x) = \sum_{k=0}^{n} C_{n,k} P_k(x)$. Then

$$C_{n,k} = \frac{1}{k!} \left\langle e^{rt} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}} - \frac{1}{2}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^2 \right)^k \middle| x^n \right\rangle$$

$$= \left(-\frac{1}{2}\right)^k \left\langle \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}} - 2\right)^k \middle| \frac{1}{k!} e^{rt} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^k x^n \right\rangle$$

$$= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) \left\langle \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}} - 2\right)^k \middle| x^{n-l} \right\rangle$$

$$= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) \left\langle \sum_{m=0}^{k} \binom{k}{m} (-2)^{k-m} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^m \middle| x^{n-l} \right\rangle$$

$$= \left(-\frac{1}{2}\right)^k \sum_{l=k}^{n} \binom{n}{l} T_r(l+r,k+r) \sum_{m=0}^{k} \binom{k}{m} (-2)^{k-m} m! \left\langle \frac{1}{m!} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^m \middle| x^{n-l} \right\rangle$$

$$= \sum_{l=k}^{n} \sum_{m=0}^{k} \binom{n}{l} \binom{k}{m} \left(-\frac{1}{2}\right)^m m! T_r(l+r,k+r) T(n-l,m). \tag{3.26}$$

This completes the proof for the next theorem.

Theorem 3.13 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $P_k(x)$:*

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{m=0}^{k} \binom{n}{l} \binom{k}{m} \left(-\frac{1}{2}\right)^m m! T_r(l+r,k+r) T(n-l,m) \right) P_k(x).$$

We write $B_n^{(c,r)}(x) = \sum_{k=0}^{n} C_{n,k} b_k(x)$. Here $b_n(x)$ are the Bernoulli polynomials of the second kind with $b_n(x) \sim (\frac{t}{e^t-1}, e^t - 1)$. Then

$$C_{n,k} = \frac{1}{k!} \left\langle e^{rt} \frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1} \left(e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1\right)^k \middle| x^n \right\rangle$$

$$= \left\langle \frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1} \middle| \frac{1}{k!} e^{rt} \left(e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1\right)^k x^n \right\rangle$$

$$= \left\langle \frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1} \middle| \sum_{l=k}^{\infty} S_2(l,k) \frac{1}{l!} e^{rt} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^l x^n \right\rangle$$

$$= \sum_{l=k}^{n} S_2(l,k) \left\langle \frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} - 1} \middle| \sum_{m=l}^{\infty} T_r(m+r,l+r) \frac{t^m}{m!} x^n \right\rangle$$

$$= \sum_{l=k}^{n} S_2(l,k) \sum_{m=l}^{n} \binom{n}{m} T_r(m+r,l+r) \left\langle \sum_{s=0}^{\infty} B_s \frac{1}{s!} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^s \middle| x^{n-m} \right\rangle$$

$$= \sum_{l=k}^{n} S_2(l,k) \sum_{m=l}^{n} \binom{n}{m} T_r(m+r,l+r) \sum_{s=0}^{n-m} B_s \left\langle \frac{1}{s!} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^s \middle| x^{n-m} \right\rangle$$

$$= \sum_{l=k}^{n} \sum_{m=l}^{n} \sum_{s=0}^{n-m} \binom{n}{m} S_2(l,k) B_s T_r(m+r,l+r) T(n-l,s), \tag{3.27}$$

where $S_2(l,k)$ are the Stirling numbers of the second and B_s are the Bernoulli numbers, given by

$$\frac{t}{e^t - 1} = \sum_{s=0}^{\infty} B_s \frac{t^s}{s!}.$$

Thus we have shown the next theorem.

Theorem 3.14 *For $n \geq 0$, we have the following representation of $B_n^{(c,r)}(x)$ in terms of $b_k(x)$.*

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=l}^{n}\sum_{s=0}^{n-m}\binom{n}{m}S_2(l,k)B_s T_r(m+r,l+r)T(n-l,s)\right)b_k(x).$$

From (3.2), $B_n^{(c,r)}(x) = (\frac{t+\sqrt{t^2+4}}{2})^{2r}B_n^{(c)}(x)$, and hence

$$B_n^{(c,r+s)}(x) = \left(\frac{t+\sqrt{t^2+4}}{2}\right)^{2s}B_n^{(c,r)}(x). \tag{3.28}$$

In particular, for $s = 1$ we have

$$\begin{aligned} B_n^{(c,r+1)}(x) &= \left(\frac{t+\sqrt{t^2+4}}{2}\right)^{2}B_n^{(c,r)}(x) \\ &= e^{f(t)}B_n^{(c,r)}(x) \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}f(t)^k B_n^{(c,r)}(x) \\ &= \sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{(c,r)}(x). \end{aligned} \tag{3.29}$$

Thus we have shown the following theorem.

Theorem 3.15 *The following identity holds true.*

$$B_n^{(c,r+1)}(x) = \sum_{k=0}^{n}\binom{n}{k}B_k^{(c,r)}(x).$$

4 Conclusions

In this paper, we studied the extended r-central factorial numbers of the second kind and the extended r-central Bell polynomials by making use of umbral calculus techniques. We noted that the extended r-central factorial numbers of the second kind are an extended version of the central factorial numbers of the second kind and also a central analog of the r-Stirling numbers of the second kind. Also, the extended r-central Bell polynomials are an extended version of the central Bell polynomials and also a central analogue of r-Bell polynomials.

We derived some properties, identities and recurrence relations. In addition, we represented the extended r-central Bell polynomials in terms of many well-known special polynomials. Here the special polynomials are Bernoulli polynomials, Euler polynomials, falling factorial polynomials, Abel polynomials, ordered Bell polynomials, Laguerre polynomials, Daehee polynomials, Hermite polynomials, polynomials closely related to the reverse Bessel polynomials and studied by Carlitz, and Bernoulli polynomials of the second kind.

Finally, along the same line as this paper, we will continue to investigate some special numbers and polynomials from the umbral calculus viewpoint.

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Authors' contributions
Each of the authors, L-CJ, TK, DSK, and HYK, contributed to each part of this study equally and read and approved the final version of the manuscript.

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