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Causal difference equations with upper and lower solutions in the reverse order

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Abstract

This paper is devoted to studying the existence conditions for difference equations involving causal operators in the presence of upper and lower solutions in the reverse order. To this end, we prove some new comparison theorems and develop the upper and lower solutions method. Our results improve and extend some relevant results in difference equations. Two examples are given to illustrate the obtained results.

MSC: 34B15; 39A10

Keywords: Upper and lower solutions method; Causal operators; Extremal solutions; Coupled extremal quasi-solutions

1 Introduction

In this paper, we are concerned with the existence of solutions for the following difference equations with causal operators:

$$\begin{cases} \Delta x(k) = (Qx)(k), & k \in \mathbb{Z}[0, T-1] = \{0, 1, \dots, T-1\}, \\ g(x(0), x(T)) = 0, \end{cases} \tag{1}$$

where $\Delta x(k) = x(k+1) - x(k)$, $E_0 = C(\mathbb{Z}[0, T-1], \mathbb{R})$, $Q \in C(E_0, E_0)$ is a causal operator, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, and the following type of equations:

$$\begin{cases} \Delta x(k-1) = (Qx)(k), & k \in \mathbb{Z}[1, T] = \{1, 2, \dots, T\}, \\ g(x(0), x(T)) = 0, \end{cases} \tag{2}$$

where $\Delta x(k-1) = x(k) - x(k-1)$, $E_1 = C(\mathbb{Z}[1, T], \mathbb{R})$, $Q \in C(E_1, E_1)$ is a causal operator, and $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

With the development of boundary value problems (BVPs) for differential equations and for difference equations [18, 19, 25, 26], and the theory of causal differential equations [6–9, 14, 21, 23], many authors have focused their attention on BVPs for causal difference equations [11, 12, 24]. In particular, in 2011, Jankowski [11] investigated first-order BVPs of difference equations with causal operators and developed the monotone iterative technique. In 2006, Atici, Cabada, and Ferreiro [2] considered the difference equations with functional boundary value conditions. Inspired by this paper, in 2015, Wang and Tian [24] established some existence criteria for the following difference equations involving causal operators with nonlinear boundary conditions:

$$\begin{cases} \Delta y(k-1) = (Qy)(k), & k \in \mathbb{Z}[1,T], \\ B(y(0),y) = 0, \end{cases}$$

and

$$\begin{cases} \Delta y(k) = (Qy)(k), & k \in \mathbb{Z}[0,T-1], \\ B(y(0),y) = 0. \end{cases}$$

To obtain existence results of causal difference equations for problem (1) and (2), we use the method of lower and upper solutions coupled with the monotone iterative technique. This method is well known not only for the continuous case but also for the discrete case, see [1, 10, 13, 15, 17, 20, 22]. However, in the above papers, the definition of lower and upper solutions is not perfect, for example, in [2], and most results only discuss the case when lower solution is less than upper solution. In fact, in many cases, the lower and upper solutions often occur in the reverse order, which is a fundamentally different situation. So far only a few papers have investigated the existence results for the non-ordered case [3–5, 16, 27]. In this paper, we shall consider the causal difference equations with nonlinear periodic boundary conditions under the assumption of the existing upper and lower solutions for the reverse case.

We shall divide the results of this paper into six sections. First, some comparison principles are established. Next, by using the notion of lower and upper solutions $v(k)$, $w(k)$ and the monotone iterative technique, we testify the existence of the extremal solutions for (1) and (2) with $v(k) \geq w(k)$. Then, by using the definition of coupled lower and upper solutions $v(k)$, $w(k)$, we obtain the existence of the coupled quasi-solutions of (1) and (2) with lower and upper solutions in the reverse order. Finally, two examples are given to illustrate the results.

2 Lemmas

Let $\mathbb{R}$ be a real numbers set, $\mathbb{Z}$ denote the set of nonnegative integer numbers, $\mathbb{Z}[m,n] = \{m, m+1, \dots, n\}$, $E = C(\mathbb{Z}[m,n],\mathbb{R})$, where $m,n \in \mathbb{Z}$ and $m < n$. We define $\|x\| = \max_{k\in\mathbb{Z}[m,n]} |x(k)|$. Moreover, in the paper, we only consider the discrete topology for the set $\mathbb{Z}[0,T]$.

A function $x \in C(\mathbb{Z}[0,T],\mathbb{R})$ is said to be a solution of problem (1) if it satisfies (1). Similarly the solution of problem (2) is defined analogously above.

Definition 2.1 Assume that $Q \in C(E,E)$, then Q is said to be a causal operator if the following property holds: if $u, v \in E$ are such that $u(s) = v(s)$ for $m \leq s \leq k < n$, $k \in \mathbb{Z}[m,n]$ arbitrary, then $(Qu)(s) = (Qv)(s)$ for $m \leq s \leq k$.

Lemma 2.2 *Suppose that* $M \geq 0$, $p \in C(\mathbb{Z}[0,T],\mathbb{R})$ *and*

$$\begin{cases} \Delta p(k) \geq Mp(k) + (\mathcal{L}p)(k), & k \in \mathbb{Z}[0,T-1], \\ \lambda p(0) \geq p(T), \end{cases} \tag{3}$$

where $\mathcal{L} \in C(E_0, E_0)$ is a positive linear operator, that is, $\mathcal{L}m \geq 0$ whenever $m \geq 0$, and

$$\sum_{j=0}^{T-1}\bigl(M + (\mathcal{L}1)(j)\bigr) \leq \frac{\lambda}{\lambda+1}, \quad 0 < \lambda \leq 1, 1(k) = 1 \text{ for all } k \in \mathbb{Z}[0, T]. \tag{4}$$

Then $p(k) \leq 0$ for $k \in \mathbb{Z}[0, T]$.

Proof Suppose that the conclusion is not true, then $p(k) \geq 0$ for some $k \in \mathbb{Z}[0, T]$. We have two cases as follows.

Case I: There is $\bar{k} \in \mathbb{Z}[0, T]$ satisfying $p(\bar{k}) > 0$ and $p(k) \geq 0$ for all $k \in \mathbb{Z}[0, T]$.

By (3), we know that $\Delta p(k) \geq 0$ on $\mathbb{Z}[0, T-1]$ and $p(k)$ is nondecreasing on $\mathbb{Z}[0, T]$. So, we have

$$\begin{aligned} p(k) &= p(0) + \sum_{j=0}^{k-1} \Delta p(j) \geq p(0) + \sum_{j=0}^{k-1}\bigl(Mp(j) + (\mathcal{L}p)(j)\bigr) \\ &\geq p(0) + p(0)\sum_{j=0}^{k-1}\bigl(M + (\mathcal{L}1)(j)\bigr) \\ &= p(0)\left(1 + \sum_{j=0}^{k-1}\bigl(M + (\mathcal{L}1)(j)\bigr)\right). \end{aligned}$$

Thus,

$$\lambda p(0) \geq p(T) \geq p(0)\left(1 + \sum_{j=0}^{T-1}\bigl(M + (\mathcal{L}1)(j)\bigr)\right) > p(0),$$

so $\lambda > 1$, this is in contradiction with (4).

Case II: There exist k_* and k^* such that $p(k_*) < 0$ and $p(k^*) > 0$.

Set $\min_{k\in\mathbb{Z}(0,T)} p(k) = -r$, $r > 0$. In general, let $p(k_*) = -r$.

From (3), we have

$$\begin{aligned} p(k) &= p(0) + \sum_{j=0}^{k-1} \Delta p(j) \geq p(0) + \sum_{j=0}^{k-1}\bigl(Mp(j) + (\mathcal{L}p)(j)\bigr) \\ &\geq p(0) - r\sum_{j=0}^{T-1}\bigl(M + (\mathcal{L}1)(j)\bigr). \end{aligned}$$

Set $k = k_*$, we obtain

$$-r \geq p(0) - r\sum_{j=0}^{T-1}\bigl(M + (\mathcal{L}1)(j)\bigr).$$

Thus, we get

$$p(0) \leq -r + r\sum_{j=0}^{T-1}\bigl(M + (\mathcal{L}1)(j)\bigr).$$

In addition,

$$p(k) = p(T) - \sum_{j=k}^{T-1} \Delta p(j).$$

Take $k = k^*$, we have

$$0 < p\big(k^*\big) = p(T) - \sum_{j=k^*}^{T-1} \Delta p(j).$$

Then

$$p(T) > \sum_{j=k^*}^{T-1} \Delta p(j) \geq -r \sum_{j=0}^{T-1} \big(M + (\mathcal{L}1)(j)\big).$$

Using the fact $p(0) \geq \lambda^{-1} p(T)$, we obtain

$$-r + r \sum_{j=0}^{T-1} \big(M + (\mathcal{L}1)(j)\big) \geq p(0) \geq \lambda^{-1} p(T) > -\lambda^{-1} r \sum_{j=0}^{T-1} \big(M + (\mathcal{L}1)(j)\big),$$

which is a contradiction with (4). Then we get $p(k) \leq 0$ on $\mathbb{Z}[0, T]$, this completes the proof. □

Lemma 2.3 *Let* $M \geq 0$, $p \in C(\mathbb{Z}[0, T], \mathbb{R})$, *and*

$$\begin{cases} \Delta p(k-1) \geq Mp(k) + (\mathcal{L}p)(k), & k \in \mathbb{Z}[1, T], \\ \lambda p(0) \geq p(T), \end{cases}$$

where $\mathcal{L} \in C(E_1, E_1)$ *is a positive linear operator and*

$$\sum_{j=1}^{T} \big(M + (\mathcal{L}1)(j)\big) \leq \frac{\lambda}{\lambda + 1}, \quad 0 < \lambda \leq 1, 1(k) = 1 \text{ for all } k \in \mathbb{Z}[0, T]. \tag{5}$$

Then $p(k) \leq 0$ *for* $k \in \mathbb{Z}[0, T]$.

The proof is analogous to Lemma 2.2, so it is omitted.

3 Existence results to (1.1)

In this section, to prove the existence of extremal solutions for (1), we first give the following linear equations:

$$\begin{cases} \Delta x(k) = Mx(k) + (\mathcal{L}x)(k) + \sigma_{\bar{\eta}}(k), & k \in \mathbb{Z}[0, T-1], \\ g(\eta(0), \eta(T)) + M_1(x(0) - \eta(0)) - M_2(x(T) - \eta(T)) = 0, \end{cases} \tag{6}$$

where $\sigma_\eta(k) = (Q\eta)(k) - M\eta(k) - (\mathcal{L}\eta)(k)$.

Lemma 3.1 *A function $x \in E_0$ is a solution of* (6) *if and only if x is a solution of the summation equation below*:

$$x(k) = \frac{C_\eta (1+M)^k}{M_1 - M_2(1+M)^T} + \sum_{i=0}^{T-1} G(k,i)\bigl[\sigma_\eta(i) + (\mathcal{L}x)(i)\bigr],$$

where $C_\eta = -g(\eta(0),\eta(T)) + M_1\eta(0) - M_2\eta(T)$, M, M_1, M_2 are constants satisfying $M \geq 0$, $M_1 \neq M_2(1+M)^T$, and

$$G(k,i) = \frac{1}{M_1 - M_2(1+M)^T} \begin{cases} \frac{M_1(1+M)^k}{(1+M)^{i+1}}, & 0 \leq i \leq k-1 \leq T, \\ \frac{M_2(1+M)^{T+k}}{(1+M)^{i+1}}, & 0 \leq k \leq i \leq T-1. \end{cases}$$

Proof Assume that $x \in E_0$ is a solution of (6). Set $x(k) = y(k)(1+M)^k$, $k \in \mathbb{Z}[0,T]$. From (6), we see that $y(k)$ satisfies

$$\begin{cases} \Delta y(k) = \frac{\sigma_\eta(k) + (\mathcal{L}(1+M)y)(k)}{(1+M)^{k+1}}, & k \in \mathbb{Z}[0,T-1], \\ y(0) = \frac{C_\eta}{M_1} + \frac{M_2(1+M)^T}{M_1} y(T). \end{cases} \tag{7}$$

By applying (7), one arrives at

$$y(k) = y(0) + \sum_{i=0}^{k-1} \frac{\sigma_\eta(k) + (\mathcal{L}(1+M)y)(k)}{(1+M)^{k+1}}. \tag{8}$$

Let $k = T$ in (8). Then one has

$$y(T) = y(0) + \sum_{i=0}^{T-1} \frac{\sigma_\eta(k) + (\mathcal{L}(1+M)y)(k)}{(1+M)^{k+1}}.$$

From the boundary condition $y(T) = \frac{M_1 y(0) - C_\eta}{M_2(1+M)^T}$, we get

$$y(0) = \frac{C_\eta}{M_1 - M_2(1+M)^T} + \frac{M_2(1+M)^T}{M_1 - M_2(1+M)^T} \sum_{i=0}^{T-1} \frac{\sigma_\eta(i) + (\mathcal{L}(1+M)y)(i)}{(1+M)^{i+1}}. \tag{9}$$

Substituting (9) into (8) and using $y(k) = \frac{x(k)}{(1+M)^k}$, $k \in \mathbb{Z}[0,T]$, we have

$$\begin{aligned} \frac{x(k)}{(1+M)^k} = {}& \frac{C_\eta}{M_1 - M_2(1+M)^T} + \frac{M_1}{M_1 - M_2(1+M)^T} \sum_{i=0}^{k-1} \frac{\sigma_\eta(i) + (\mathcal{L}x)(i)}{(1+M)^{i+1}} \\ & + \frac{M_2(1+M)^T}{M_1 - M_2(1+M)^T} \sum_{i=k}^{T-1} \frac{\sigma_\eta(i) + (\mathcal{L}x)(i)}{(1+M)^{i+1}}. \end{aligned}$$

Let

$$G(k,i) = \frac{1}{M_1 - M_2(1+M)^T} \begin{cases} \frac{M_1(1+M)^k}{(1+M)^{i+1}}, & 0 \leq i \leq k-1 \leq T, \\ \frac{M_2(1+M)^{T+k}}{(1+M)^{i+1}}, & 0 \leq k \leq i \leq T-1. \end{cases}$$

We see that x is a solution of (6) and the proof is complete. □

Apparently, $\|G(k,i)\| = \max\{|\frac{M_1(1+M)^T}{M_1-M_2(1+M)^T}|, |\frac{M_2(1+M)^T}{M_1-M_2(1+M)^T}|\}$. In the remainder of the paper, we denote $\tau = \|G(k,i)\| = \max\{|\frac{M_1(1+M)^T}{M_1-M_2(1+M)^T}|, |\frac{M_2(1+M)^T}{M_1-M_2(1+M)^T}|\}$.

Lemma 3.2 *Suppose that* $M \geq 0$, $M_1 \neq M_2(1+M)^T$, *and*

$$\tau \|\mathcal{L}\| T < 1. \tag{10}$$

Then problem (6) *has a unique solution.*

Proof Define an operator $F : E_0 \to E_0$ by

$$(Fx)(k) = \frac{C_\eta(1+M)^k}{M_1 - M_2(1+M)^T} + \sum_{i=0}^{T-1} G(k,i)\bigl[\sigma(i) + (\mathcal{L}x)(i)\bigr], \quad k \in \mathbb{Z}[0, T-1].$$

For any $x_1, x_2 \in E_0$, we have

$$|Fx_1 - Fx_2| \leq \left|\sum_{i=0}^{T-1} G(k,i)\bigl[\bigl(\mathcal{L}(x_2 - x_1)\bigr)(i)\bigr]\right| \leq \tau T \|\mathcal{L}\| \|x_2 - x_1\|.$$

Hence, by the Banach contraction principle, F has a unique fixed point and (6) has only one solution. We complete the proof. □

Next, we give the following definitions which help us to testify our main results.

Definition 3.3 A function w is called an upper solution of (1) if

$$\begin{cases} \Delta w(k) \geq (Qw)(k), & k \in \mathbb{Z}[0, T-1], \\ g(w(0), w(T)) \geq 0, \end{cases}$$

and a lower solution of (1) is defined similarly by reversing the inequalities above.

Theorem 3.4 *Suppose that* (4) *and* (10) *hold, and* $Q \in C[E_0, E_0]$

(H_1) *the functions* w, v *are upper and lower solutions of problem* (1) *with* $w(k) \leq v(k)$, $k \in \mathbb{Z}[0, T]$;

(H_2) Q *satisfies*

$$(Qy)(k) - (Qz)(k) \leq M\bigl(y(k) - z(k)\bigr) + \bigl(\mathcal{L}(y - z)\bigr)(k), \quad k \in \mathbb{Z}[0, T-1],$$

for $w(k) \leq z(k) \leq y(k) \leq v(k)$, *where* $M \geq 0$, $\mathcal{L} \in C[E_0, E_0]$ *is a positive linear operator;*

(H_3) *there exist constants* M_1, M_2 *such that* $M_2 \geq M_1 > 0$ *and*

$$g(\bar{y}, \bar{z}) - g(y, z) \geq M_1(\bar{y} - y) - M_2(\bar{z} - z))$$

for $w(0) \leq y \leq \bar{y} \leq v(0)$, $w(T) \leq z \leq \bar{z} \leq v(T)$, *and* $0 < \lambda \leq 1$ *with* $\lambda = \frac{M_1}{M_2}$.

Then problem (1) *has extremal solutions in the sector* $[w, v] = \{x : w(k) \leq x(k) \leq v(k), k \in \mathbb{Z}[0, T]\}$.

Proof First, we define the sequences $\{v_n(k)\}$, $\{w_n(k)\}$ as follows:

$$\begin{cases} \Delta v_n(k) = Mv_n(k) + (\mathcal{L}v_n)(k) + (Qv_{n-1})(k) - Mv_{n-1}(k) - (\mathcal{L}v_{n-1})(k), \\ g(v_{n-1}(0), v_{n-1}(T)) + M_1(v_n(0) - v_{n-1}(0)) - M_2(v_n(T) - v_{n-1}(T)) = 0 \end{cases} \tag{11}$$

and

$$\begin{cases} \Delta w_n(k) = Mw_n(k) + (\mathcal{L}w_n)(k) + (Qw_{n-1})(k) - Mw_{n-1}(k) - (\mathcal{L}w_{n-1})(k), \\ g(w_{n-1}(0), w_{n-1}(T)) + M_1(w_n(0) - w_{n-1}(0)) - M_2(w_n(T) - w_{n-1}(T)) = 0 \end{cases} \tag{12}$$

for $n = 1, 2, \ldots$, where $v_0 = v$, $w_0 = w$.

It follows from Lemma 3.2 that both (11) and (12) have unique solutions, respectively.

We have four steps to complete the proof.

Step 1. We demonstrate that $w_{n-1} \leq w_n$ and $v_n \leq v_{n-1}$, $n = 1, 2, \ldots$.

Set $p = v_1 - v$. Employing (H_1), we have

$$\begin{aligned} \Delta p(k) &= \Delta v_1(k) - \Delta v(k) \\ &\geq Mv_1(k) + (\mathcal{L}v_1)(k) + (Qv)(k) - Mv(k) - (\mathcal{L}v)(k) - (Qv)(k) \\ &= Mp(k) + (\mathcal{L}p)(k), \quad k \in \mathbb{Z}[0, T-1] \end{aligned}$$

and

$$p(0) = v_1(0) - v(0) = -\frac{1}{M_1} g\bigl(v(0), v(T)\bigr) + \frac{M_2}{M_1}\bigl(v_1(T) - v(T)\bigr) \geq \frac{M_2}{M_1} p(T).$$

From Lemma 2.2 and $M_2 \geq M_1 > 0$, we get $p \leq 0$, so $v_1 \leq v$.

Employing mathematical induction, it is readily seen that v_n is a nonincreasing sequence. Analogously, we can show w_n is a nondecreasing sequence.

Step 2. We prove that $w_1 \leq v_1$ if $w \leq v$.

Let $p = w_1 - v_1$. Using (H_2) and (H_3), we get

$$\begin{aligned} \Delta p(k) &= \Delta w_1(k) - \Delta v_1(k) \\ &= Mw_1(k) + (\mathcal{L}w_1)(k) + (Qw)(k) - Mw(k) - (\mathcal{L}w)(k) \\ &\quad - Mv_1(k) - (\mathcal{L}v_1)(k) - (Qv)(k) + Mv(k) + (\mathcal{L}v)(k) \\ &\geq Mp(k) + (\mathcal{L}p)(k), \quad k \in \mathbb{Z}[0, T-1] \end{aligned}$$

and

$$\begin{aligned} p(0) &= w_1(0) - v_1(0) \\ &= -\frac{1}{M_1} g\bigl(w(0), w(T)\bigr) + \frac{M_2}{M_1}\bigl(w_1(T) - w(T)\bigr) + w(0) \\ &\quad - \left[-\frac{1}{M_1} g\bigl(v(0), v(T)\bigr) + \frac{M_2}{M_1}\bigl(v_1(T) - v(T)\bigr) + v(0)\right] \\ &\geq \frac{M_2}{M_1} p(T). \end{aligned}$$

From Lemma 2.2, we obtain $p \leq 0$ and $w_1 \leq v_1$. By mathematical induction, we obtain $w_n \leq v_n$, $n = 1, 2, \ldots$.

Step 3. By the first two steps, we get

$$w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0,$$

and each v_n, w_n satisfies (10) and (11). It is easy to see that sequences $\{v_n(k)\}$, $\{w_n(k)\}$ are monotonously and bounded, passing to the limit when $n \to \infty$, we have $\lim_{n\to\infty} v_n(k) = \rho(k)$ and $\lim_{n\to\infty} w_n(k) = r(k)$ uniformly on $\mathbb{Z}[0, T]$. Clearly, $\rho(k)$, $r(k)$ satisfy problem (1).

Step 4. We show that ρ and r are extremal solutions of (1) in $[w, v]$.

Let $x(k)$ be any solution of (1) such that $w(k) \leq x(k) \leq v(k)$. Assume that there exists a positive integer n such that $w_n(k) \leq x(k) \leq v_n(k)$. Then, setting $p = w_{n+1} - x$, we have

$$\begin{aligned} \Delta p(k) &= \Delta w_{n+1}(k) - \Delta x(k) \\ &= Mw_{n+1}(k) + (\mathcal{L}w_{n+1})(k) + (Qw_n)(k) - Mw_n(k) - (\mathcal{L}w_n)(k) - (Qx)(k) \\ &\geq Mp(k) + (\mathcal{L}p)(k), \quad k \in \mathbb{Z}[0, T-1] \end{aligned}$$

and

$$\begin{aligned} p(0) &= w_{n+1}(0) - x(0) \\ &= -\frac{1}{M_1} g\bigl(w_n(0), w_n(T)\bigr) + \frac{M_2}{M_1}\bigl(w_{n+1}(T) - w_n(T)\bigr) \\ &\quad + w_n(0) - x(0) + \frac{1}{M_1} g\bigl(x(0), x(T)\bigr) \\ &\geq \frac{M_2}{M_1} p(T). \end{aligned}$$

By Lemma 2.2, $p \leq 0$, i.e., $w_{n+1} \leq x$. Similarly, we may get that $x \leq v_{n+1}$ on $\mathbb{Z}[0, T]$. Since $w_0(k) \leq x(k) \leq v_0(k)$, by induction we obtain $w_n(k) \leq x(k) \leq v_n(k)$ for every $n \in \mathbb{N}$, which implies $r(k) \leq x(k) \leq \rho(k)$, and the proof is complete. □

4 Existence results to (1.2)

In this section, to avoid repetition, we merely state the next lemmas and theorems without proofs since they are similar to those in Sect. 3.

Definition 4.1 Function w is called an upper solution of (2) if

$$\begin{cases} \Delta w(k-1) \geq (Qw)(k), \quad k \in \mathbb{Z}[1, T], \\ g(w(0), w(T)) \geq 0, \end{cases}$$

and a lower solution of (2) is defined similarly by reversing the inequalities above.

Consider the following linear problems:

$$\begin{cases} \Delta x(k-1) = Mx(k) + (\mathcal{L}x)(k) + h_u(k), \quad k \in \mathbb{Z}[1, T], \\ g(u(0), u(T)) + M_1(x(0) - u(0)) - M_2(x(T) - u(T)) = 0, \end{cases} \tag{13}$$

where $h_u(k) = (Qu)(k) - Mu(k) - (\mathcal{L}u)(k)$.

Lemma 4.2 *Let $C_u = -g(u(0), u(T)) + M_1 u(0) - M_2 u(T)$. A function $x \in E_1$ is a solution of* (13) *iff x is a solution of the following summation equation*:

$$x(k) = \frac{C_u(1-M)^{T-k}}{M_1(1-M)^T - M_2} + \sum_{i=1}^{T} H(k,i)\big[h_u(i) - (\mathcal{L}x)(i)\big],$$

where M, M_1, M_2 are constants satisfying $0 \le M < 1$, $M_2 \neq M_1(1-M)^T$, and

$$H(k,i) = \frac{1}{M_1(1-M)^T - M_2}\begin{cases} \frac{M_1(1-M)^{T+i-1}}{(1-M)^k}, & 1 \le i \le k \le T, \\ \frac{M_2(1-M)^{i-1}}{(1-M)^k}, & 0 \le k+1 \le i \le T. \end{cases}$$

In the remainder of the paper, we denote $\xi = \|H(k,i)\| = \max\{|\frac{M_1}{M_1(1-M)^T - M_2}|, |\frac{M_2}{M_1(1-M)^T - M_2}|\}$.

Lemma 4.3 *Assume that constants $0 \le M < 1$, $M_2 \neq M_1(1-M)^T$, and*

$$\xi \|\mathcal{L}\| T < 1. \tag{14}$$

Then problem (13) *has a unique solution.*

Theorem 4.4 *Suppose that* (14) *is satisfied, further*

(A_1) *w, v are upper and lower solutions of problem* (2) *and $w(k) \le v(k)$, $k \in \mathbb{Z}[1,T]$;*

(A_2) *there exist $0 \le M < 1$ and $\mathcal{L}, Q \in C[E_1, E_1]$ satisfying*

$$(Qy)(k) - (Qz)(k) \le M\big(y(k) - z(k)\big) + \big(\mathcal{L}(y-z)\big)(k), \quad k \in \mathbb{Z}[1,T],$$

for $w(k) \le z(k) \le y(k) \le v(k)$;

(A_3) *there exist constants M_1, M_2 such that $M_2 \ge M_1 > 0$ and*

$$g(\bar{y}, \bar{z}) - g(y,z) \ge M_1(\bar{y} - y) - M_2(\bar{z} - z))$$

for $w(0) \le y \le \bar{y} \le v(0)$, $w(T) \le z \le \bar{z} \le v(T)$, and $0 < \lambda \le 1$ with $\lambda = \frac{M_1}{M_2}$.

Then problem (2) *has extremal solutions in the sector $[w, v] = \{x : w(k) \le x(k) \le v(k), k \in \mathbb{Z}[0,T]\}$.*

5 Coupled lower and upper solutions

In this section, we shall prove the existence of the coupled quasi-solutions for problems (1) and (2).

Definition 5.1 Functions v, w are called coupled lower and upper solutions of (1) if

$$\begin{cases} \Delta v(k) \le (Qv)(k), & k \in \mathbb{Z}[0, T-1], \\ g(v(0), w(T)) \le 0 \end{cases}$$

and

$$\begin{cases} \Delta w(k) \ge (Qw)(k), & k \in \mathbb{Z}[0, T-1], \\ g(w(0), v(T)) \ge 0. \end{cases}$$

Definition 5.2 A pair (U, V) is said to be a coupled quasi-solution of problem (1) if

$$\begin{cases} \Delta U(k) = (QU)(k), \quad k \in \mathbb{Z}[0, T-1], \\ g(U(0), V(T)) = 0 \end{cases}$$

and

$$\begin{cases} \Delta V(k) = (QV)(k), \quad k \in \mathbb{Z}[0, T-1], \\ g(V(0), U(T)) = 0. \end{cases}$$

The definitions of coupled lower and upper solutions and coupled quasi-solution for (2) are similar to above.

Theorem 5.3 *Suppose that* (H_2), (4), *and* (10) *hold, let* $Q \in E_0$. *In addition, we assume that*

- (H_4) v, w *are coupled lower and upper solutions of* (1) *such that* $w \le v$;
- (H_5) *there exist* M_1, M_2 *such that* $M_2 \ge M_1 > 0$, $g(\cdot, z) \in C(\mathbb{R}^2, \mathbb{R})$ *is a nonincreasing function for each* $z \in [w(T), v(T)]$, *and*

$$g(\bar{x}, z) - g(x, z) \ge M_1(\bar{x} - x)), \quad \textit{if } w(0) \le x \le \bar{x} \le v(0).$$

Then there exist two monotone sequences $\{w_n(k)\}$ *and* $\{v_n(k)\}$ *such that* $w = w_0 \le w_1 \le w_2 \le \cdots \le w_n \le \cdots \le v_n \le \cdots \le v_2 \le v_1 \le v_0 = v$ *for every* $n \in N$, *which converge uniformly to the coupled extremal quasi-solutions.*

Proof Let

$$\begin{cases} \Delta v_n(k) = Mv_n(k) + (\mathcal{L}v_n)(k) + (Qv_{n-1})(k) - Mv_{n-1}(k) - (\mathcal{L}v_{n-1})(k), \\ g(v_{n-1}(0), w_{n-1}(T)) + M_1(v_n(0) - v_{n-1}(0)) - M_2(v_n(T) - v_{n-1}(T)) = 0 \end{cases}$$

and

$$\begin{cases} \Delta w_n(k) = Mw_n(k) + (\mathcal{L}w_n)(k) + (Qw_{n-1})(k) - Mw_{n-1}(k) - (\mathcal{L}w_{n-1})(k), \\ g(w_{n-1}(0), v_{n-1}(T)) + M_1(w_n(0) - w_{n-1}(0)) - M_2(w_n(T) - w_{n-1}(T)) = 0 \end{cases}$$

for $n = 1, 2, \ldots$, where $v_0 = v$, $w_0 = w$.

In regard to Lemma 3.1 and Lemma 3.2, it is easy to obtain that v, w are well defined. First we prove that $v_0 \le v_1 \le w_1 \le w_0$.

Let $p = v_1 - v_0$, applying (H_4) we have

$$\begin{aligned} \Delta p(k) &= \Delta v_1(k) - \Delta v_0(k) \\ &\ge Mv_1(k) + (\mathcal{L}v_1)(k) + (Qv_0)(k) - \big[(Qv_0)(k) - Mv_0(k) - (\mathcal{L}v_0)(k) - (Qv_0)(k)\big] \\ &= Mp(k) + (\mathcal{L}p)(k), \quad k \in \mathbb{Z}[0, T-1] \end{aligned}$$

and

$$\begin{aligned}
p(0) &= v_1(0) - v_0(0) \\
&= -\frac{1}{M_1} g\big(v_0(0), w_0(T)\big) + \frac{M_2}{M_1} p(T) \\
&\geq \frac{M_2}{M_1} p(T).
\end{aligned}$$

By Lemma 2.2, we obtain $p(k) \leq 0$ with $k \in \mathbb{Z}[0, T]$, that is, $v_1 \leq v_0$. Similar arguments prove that $w_0 \leq w_1$.

Now, set $p = w_1 - v_1$, using (H_2), we get

$$\begin{aligned}
\Delta p(k) &= \Delta w_1(k) - \Delta v_1(k) \\
&= \big[M w_1(k) + (\mathcal{L} w_1)(k) + (Q w_0)(k) - M w_0(k) - (\mathcal{L} w_0)(k)\big] \\
&\quad - \big[M v_1(k) + (\mathcal{L} v_1)(k) + (Q v_0)(k) - M v_0(k) - (\mathcal{L} v_0)(k)\big] \\
&\geq M p(k) + (\mathcal{L} p)(k), \quad k \in \mathbb{Z}[0, T-1].
\end{aligned}$$

Noticing $w_0 \leq v_0$ and (H_5), we obtain

$$\begin{aligned}
p(0) &= w_1(0) - v_1(0) \\
&= -\frac{1}{M_1} g\big(w_0(0), v_0(T)\big) + \frac{M_2}{M_1}\big(w_1(T) - w_0(T)\big) + w_0(0) \\
&\quad - \left[-\frac{1}{M_1} g\big(v_0(0), w_0(T)\big) + \frac{M_2}{M_1}\big(v_1(T) - v_0(T)\big) + v_0(0)\right] \\
&\geq \frac{M_2}{M_1} p(T).
\end{aligned}$$

From Lemma 2.2, we have $p(k) \leq 0$, $k \in \mathbb{Z}[0, T]$, i.e., $w_1 \leq v_1$.

In the following, we shall show that v_1, w_1 are the coupled lower and upper solutions of (1). Using H_4, H_5 and $v_1 \leq v_0$, $w_0 \leq w_1$, we obtain

$$\begin{aligned}
\Delta v_1(k) &= (Q v_1)(k) + (Q v_0)(k) - (Q v_1)(k) \\
&\qquad + M\big(v_1(k) - v_0(k)\big) + \big(\mathcal{L}(v_1 - v_0)\big)(k) \\
&\leq (Q v_1)(k), \\
\Delta w_1(k) &= (Q w_1)(k) + (Q w_0)(k) - (Q w_1)(k) \\
&\qquad + M\big(w_1(k) - w_0(k)\big) + \big(\mathcal{L}(w_1 - w_0)\big)(k) \\
&\geq (Q w_1)(k), \\
g\big(v_1(0), w_1(T)\big) &\leq g\big(v_1(0), w_0(T)\big) \\
&\leq g\big(v_0(0), w_0(T)\big) + M_1\big(v_1(0) - v_0(0)\big) \\
&\leq 0,
\end{aligned}$$

$$
\begin{aligned}
g\big(w_1(0), v_1(T)\big) &\geq g\big(w_1(0), v_0(T)\big) \\
&\geq g\big(w_0(0), v_0(T)\big) + M_1\big(w_1(0) - w_0(0)\big) \\
&\geq 0.
\end{aligned}
$$

We see that v_1, w_1 are coupled lower and upper solutions of (1).

Continuing this progress, by mathematical induction, we can get the sequences $\{v_n(k)\}$ and $\{w_n(k)\}$ such that

$$
w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0.
$$

Then we show that there exist ρ, r such that $\lim_{n\to\infty} v(k) = \rho(k)$, $\lim_{n\to\infty} w(k) = r(k)$ uniformly on $\mathbb{Z}[0,T]$, and ρ, r satisfy the equations

$$
\begin{cases}
\Delta\rho(k) = (Q\rho)(k), \\
g(\rho(0), r(T)) = 0,
\end{cases}
$$

and

$$
\begin{cases}
\Delta r(k) = (Qr)(k), \\
g(r(0), \rho(T)) = 0.
\end{cases}
$$

This proves that the pair (r,ρ) is a coupled quasi-solution of problem (1).

Finally, we prove that (r,ρ) is coupled minimal and maximal quasi-solutions of (1). Let $u_1, u_2 \in [w_0, v_0]$ be any coupled quasi-solutions of problem (1). Assume that there exists a positive integer n such that $w_n \leq u_1, u_2 \leq v_n$ on $\mathbb{Z}[0,T]$. Then, putting $p = w_{n+1} - u_1$ and employing the fact $g(u_1(0), u_2(T)) = 0$, $w_n \leq u_1$, and H_2, we have

$$
\begin{aligned}
\Delta p(k) &= \Delta w_{n+1}(k) - \Delta u_1(k) \\
&= Mw_{n+1}(k) + (\mathcal{L}w_{n+1})(k) + (Qw_n)(k) - Mw_n(k) - (\mathcal{L}w_n)(k) - (Qu_1)(k) \\
&\geq Mp(k) + (\mathcal{L}p)(k), \quad k \in \mathbb{Z}[0, T-1], \\
p(0) &= w_{n+1}(0) - u_1(0) \\
&= -\frac{1}{M_1} g\big(w_n(0), v_n(T)\big) + \frac{1}{M_1} g\big(u_1(0), u_2(T)\big) \\
&\quad + \frac{M_2}{M_1}\big(w_{n+1}(T) - w_n(T)\big) + w_n(0) - u_1(0) \\
&\geq \frac{M_2}{M_1}\big(w_{n+1}(T) - w_n(T)\big) \\
&\geq \frac{M_2}{M_1} p(T).
\end{aligned}
$$

By Lemma 2.2, $p(k) \leq 0$, which proves $w_{n+1}(k) \leq u_1(k)$ on $\mathbb{Z}[0,T]$. Using similar arguments, we can conclude $w_{n+1}(k) \leq u_1(k), u_2(k) \leq v_{n+1}(k)$ on $\mathbb{Z}[0,T]$. Since $w_0(k) \leq u_1(k), u_2(k) \leq v_0(k)$, by the principle of induction, $w_n(k) \leq u_1(k), u_2(k) \leq v_n(k)$, ($n = 0, 1, 2, \ldots$) hold for all $k \in \mathbb{Z}[0,T]$, which implies $r(k) \leq u_1(k), u_2(k) \leq \rho(k)$ on $\mathbb{Z}[0,T]$. It is clear that r, ρ are coupled minimal and maximal quasi-solutions of (1). We complete the proof. □

We can also obtain the existence of coupled extremal quasi-solutions for problem (2) by a way similar to the one we used in the proof of Theorem (5.3).

Theorem 5.4 *Assume condition* (A_2), (4), *and* (14) *hold, let* $Q \in E_1$. *In addition, we suppose that*

(A_4) *v, w are coupled lower and upper solutions of* (2) *such that* $w \leq v$;

(A_5) *there exist* M_1, M_2 *such that* $M_2 \geq M_1 > 0$, *and the function* $g(x,z) \in C(\mathbb{R}^2,\mathbb{R})$ *is nondecreasing in the second variable satisfying*

$$g(\bar{x},z) - g(x,z) \leq -M_1(\bar{x} - x)) \quad \text{if } w(0) \leq x \leq \bar{x} \leq v(0).$$

Then problem (2) *has coupled minimal and maximal quasi-solutions in the sector* $[w,v] = \{x : w(k) \leq x(k) \leq v(k), k \in \mathbb{Z}[0,T]\}$.

6 Two examples

In this section, we give two simple but illustrative examples, thereby validating the proposed theorems.

Example 6.1 Consider the problem of

$$\begin{cases} \Delta x(k) = 0.005x(k) + \frac{1}{0.01k}\sum_{i=1}^{k} ix(i) \equiv (Qx)(k), & k \in \mathbb{Z}[0,T], \\ g(x(0),x(T)) = \frac{1}{2}x^3(0) + 3x(0) - 4x(T) = 0. \end{cases} \tag{15}$$

Set $v(k) = 0$, $w(k) = -1$. We can easily prove that $v(k)$ is a lower solution, $w(k)$ is an upper solution with $w(k) \leq v(k)$. It is easy to see that (4), (10), H_1, H_2, and H_3 hold with $M = 0.005$, $M_1 = 3$, $M_2 = 4$, $\lambda = \frac{3}{4}$, $T = 37$. From Theorem (3.4), problem (15) has extremal solutions in the sector $[w,v]$.

Example 6.2 Consider the problem of

$$\begin{cases} \Delta x(k) = \frac{1}{800}x^2(k) + \frac{1}{400}x(k) + \frac{1}{100k^3}\sum_{i=1}^{k} i^2x(i) \equiv (Qx)(k), & k \in \mathbb{Z}[0,30], \\ g(x(0),x(T)) = \ln(2 - x(0)) + (x(T) - 1)^3 + \frac{3}{2}(x(T) - 1)^2 - \frac{1}{2}. \end{cases} \tag{16}$$

Taking $v(k) = 1$, $w(k) = 0$. We can easily prove that $v(k)$ is a coupled lower solution, $w(k)$ is a coupled upper solution with $w(k) \leq v(k)$. Let $(Qx)(k) = \frac{1}{800}x^2(k) + \frac{1}{400}x(k) + \frac{1}{100k^3}\sum_{i=1}^{k} i^2x(i)$, $(\mathcal{L}x)(k) = \frac{1}{100k^3}\sum_{i=1}^{k} i^2x(i)$. By computing, we get

$$(Qx)(k) - (Qz)(k) \leq \frac{1}{200}\bigl(x(k) - z(k)\bigr) + \bigl(\mathcal{L}(x - z)\bigr)(k),$$

where $v(k) \leq z(k) \leq x(k) \leq w(k)$ on $k \in \mathbb{Z}[0,30]$, $M = \frac{1}{200}$.

Set $g(x,z) = \ln(2-x) + (z-1)^3 + \frac{3}{2}(z-1)^2 - \frac{1}{2}$, we get that the function $g(x,z)$ is nonincreasing in the second variable and

$$g(\bar{x},z) - g(x,z) \geq (\bar{x} - x),$$

where $w(T) \leq x \leq \bar{x} \leq v(T)$, $M_1 = 1$, $M_2 = 2$, $\lambda = \frac{M_1}{M_2} = \frac{1}{2}$.

It is easy to prove that $\tau = \max_{k\in J}\{|\frac{(1+\frac{1}{200})^{30}}{1-2(1+\frac{1}{200})^{30}}|, |\frac{2(1+\frac{1}{200})^{30}}{1-2(1+\frac{1}{200})^{30}}|\} < 2$,

$$\sum_{j=0}^{30}\bigl(M + (\mathcal{L}1)(j)\bigr) = \sum_{j=0}^{30}\left(\frac{1}{200} + \frac{1}{600}\left(2 + \frac{1}{j}\right)\left(1 + \frac{1}{j}\right)\right) < \frac{\lambda}{1+\lambda} = \frac{1}{3},$$

and

$$\tau\|\mathcal{L}\|T = 30\tau\|\mathcal{L}\| = 30\tau\frac{1}{100} < 1.$$

Then all the conditions of Theorem 5.3 are satisfied. Hence problem (16) has coupled minimal and maximal quasi-solutions in the segment $[w,v]$.

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Authors' contributions
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