G-stability one-leg hybrid methods for solving DAEs

P. Agarwal[1,2,3*] , Iman H. Ibrahim[4] and Fatma M. Yousry[4]

*Correspondence:
goyal.praveen2011@gmail.com
[1]Department of Mathematics,
Anand International College of
Engineering, Jaipur, India
[2]International Center for Basic and
Applied Sciences, Jaipur, India
Full list of author information is
available at the end of the article

Abstract

This paper introduces the solution of differential algebraic equations using two hybrid classes and their twin one-leg with improved stability properties. Physical systems of interest in control theory are sometimes described by systems of differential algebraic equations (DAEs) and ordinary differential equations (ODEs) which are zero index DAEs. The study of the first hybrid class includes the order of convergence, A(α)-stability, stability regions, and G-stability for its one-leg twin in two cases: for step ($k = 1$) and steps ($k = 2$). For the second class, G-stability of its one-leg twin is studied in two cases: for steps ($k = 2$) and steps ($k = 3$). Test problems are introduced with different step size at different end points.

MSC: 65L05; 65L06; 65L20

Keywords: Hybrid methods; One-leg methods; DAEs; G-stability; A(α)-stability

1 Introduction

Consider the initial value problems of the form

$$f\big(x'(t); x(t); t\big) = 0; \qquad x(t_0) - a = 0, \quad t \in [t_0; T], \tag{1}$$

where $a \in R^m$ is a consistent initial value for (1) and the function $f \colon R^m \times R^m \times [t_0; T] \to R^m$ is assumed to be sufficiently smooth. If ($\partial f / \partial x$) is nonsingular, then it is possible to formally solve (1) for x in order to obtain an ordinary differential equation. However, if ($\partial f / \partial x$) is singular, it is no longer possible and the solution x has to satisfy certain algebraic constraints; therefore, equations (1) are referred to as differential algebraic equations.

Systems of differential algebraic equations arise from many applications such as physics, engineering, and circuit analysis. Some systems can be reduced to an ODE system, which are zero index DAEs, and can be solved by numerical ODE methods after reduction. Other systems, in which reductions to an explicit differential systems are in the form $x' = f(x; t)$, are either impossible or impractical, that is because the problem is more naturally posed in the form

$$f\big(t, x', x, y\big) = 0; \tag{2-a}$$

$$g(t, x, y) = 0; \tag{2-b}$$

and a reduction might reduce the sparseness of Jacobian matrices. These systems are then solved directly [16, 17].

A fundamentally important concept in the algorithms of the numerical solutions of DAEs is the index of a DAE. In a sense, this tells us how far away the DAE is from being an ODE. The index of a DAE is the minimum number of times all or part of the DAE system must be differentiated with respect to time in order to convert the DAE into an explicit ODE. The higher the index is, the further it is from an ODE and the more difficult it is in general to solve the DAE [12].

The first general method applied to the numerical solution of DAEs is backward differentiation formula (BDF). Ebadi andGokhale presented in [9–11] class 2 + 1 hybrid BDF-like methods, hybrid BDF methods (HBDF), and new hybrid methods for the numerical solution of IVPs. These methods have wide stability regions and good performance in solving CPU time compared to the extended BDF (EBDF) and modified extended BDF (MEBDF) methods [3].

In Sect. 2 the first hybrid class is derived, its orders of convergence are investigated, and its stability analysis is studied. In Sect. 3 some basic notions of one-leg schemes and G-stability are mentioned. The one-leg twin of the first class is derived and its G-stability is discussed in Sect. 4. The one-leg twin of the second class [14] is derived and its G-stability is discussed in Sect. 5. Numerical tests are investigated in Sect. 6. Finally, a conclusion is introduced.

2 The first hybrid class

The first hybrid class takes the form

$$y_{n+s} = h\mu f_n + \sum_{j=0}^{k} \gamma_{n-j} y_{n-j}, \tag{3}$$

$$y_n + \sum_{j=1}^{k} \alpha_{n-j} y_{n-j} = h(\beta_s f_{n+s} + \beta_1 f_n + \beta_0 f_{n-1}), \tag{4}$$

where $f_{n+s} = f(t_{n+s}; y_{n+s})$; $t_{n+s} = t_n + sh$; $-1 < s$ and β_s, β_1, α_{n-j}, $j = 1, 2, \ldots, k$, are parameters to be determined as functions of s and β_0. The methods for step k and order $p = k + 1$ (from $k = 1$ up to 6) will be derived and y_{n+s} has order $k - 1$. To evaluate the value of y_{n+s} at an off-step point, i.e. t_{n+s}, we will consider the nodes t_n (double node), $t_{n-1}, \ldots, t_{n-k}$ (simple nodes).

Applying Newton's interpolation formula for this data gives the following scheme:

$$
\begin{aligned}
y(t) = {}& y_n + (t - t_n)y'_n + (t - t_n)^2 \frac{hy'_n - \nabla y_n}{h^2} \\
& + (t - t_n)^2(t - t_{n-1}) \frac{hy'_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n}{2!h^3} \\
& + (t - t_n)^2(t - t_{n-1})(t - t_{n-2}) \frac{hy'_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{3}\nabla^3 y_n}{3!h^4} + \cdots.
\end{aligned}
\tag{5}
$$

Differentiate (5) with respect to t:

$$y'(t) = y'_n + 2(t - t_n)\frac{hy'_n - \nabla y_n}{h^2} + \left(2(t - t_n)(t - t_{n-1}) + (t - t_n)^2\right)\frac{hy'_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n}{2!h^3}$$

$$+ \left(2(t - t_n)(t - t_{n-1})(t - t_{n-2}) + (t - t_n)^2(t - t_{n-2}) + (t - t_n)^2(t - t_{n-1})\right)$$

$$\times \frac{hy'_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{3}\nabla^3 y_n}{3!h^4} + \cdots. \tag{6}$$

Using (5) and (6) to evaluate y_{n+s} and f_{n+s} gives

$$y(t_n + sh) = y_n + shf_n + s^2(hf_n - \nabla y_n) + \frac{s^2(s + 1)}{2!}\left(hf_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n\right)$$

$$+ \frac{s^2(s + 1)(s + 2)}{3!}\left(hf_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{3}\nabla^3 y_n\right) + \cdots, \tag{7}$$

$$f(t_{n+s}) = f_n + 2s\frac{hf_n - \nabla y_n}{h} + s(2 + 3s)\frac{hf_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n}{2!h}$$

$$+ s\left(4 + 9s + 4s^2\right)\frac{hf_n - \nabla y_n - \frac{1}{2}\nabla^2 y_n - \frac{1}{3}\nabla^3 y_n}{3!h} + \cdots, \tag{8}$$

where f (or $f(t, y)$) is considered as a derivative of the solution $y(t)$, $\nabla y_n = y_n - y_{n-1}$.

Method (4) is of order p if and only if

$$1 + \sum_{j=1}^{k} \alpha_{n-j} = 0, \qquad \sum_{j=1}^{k} -j\alpha_{n-j} = (\beta_s + \beta_1 + \beta_0),$$

$$\sum_{j=1}^{k} \alpha_{n-j}(-j)^q = q\left(\beta_s s^{q-1} + (-1)^{q-1}\beta_0\right), \quad \text{where } q = 2, \ldots, p.$$

The coefficients of the methods for steps $k = 1$ up to 6 are tabulated in Tables 1, 2, 3, and 4.

Since formula (3) is of order k and formula (4) is of order $k + 1$, then it is easy to see that method (3)–(4) has order $k + 1$.

Table 1 The coefficients of method (2.2) for orders 2, 3, and 4

k	1	2	3
α_{n-1}	-1	$\frac{2(-8 - 6s + 3\beta_0 + 3s\beta_0)}{14 + 9s}$	$\frac{-18(12 + s(15 + 4s)) + (1 + s)(119 + 46s)\beta_0}{2(85 + 90s + 22s)}$
α_{n-2}	0	$\frac{2 + 3s - 6(1 + s)\beta_0}{14 + 9s}$	$\frac{9(3 + 2s(3 + s)) - 4(1 + s)(17 + 7s)\beta_0}{85 + 90s + 22s}$
α_{n-3}	0	0	$\frac{-2(4 + s(9 + 4s)) + (1 + s)(17 + 10s)\beta_0}{2(85 + 90s + 22s)}$
β_1	$\frac{1 + 2s - 2(1 + s)\beta_0}{2s}$	$\frac{4 + 6s(2 + s) - (1 + s)(5 + 3s)}{s(14 + 9s)}$	$\frac{-6(3 + 2s)(1 + s(3 + s)) + (1 + s)(17 + s(17 + 4s))\beta_0}{s(85 + 90s + 22s)}$
β_s	$-\frac{1 - 2\beta_0}{2s}$	$\frac{-4 + 5\beta_0}{s(14 + 9s)}$	$\frac{-18 + 17\beta_0}{s(85 + 90s + 22s)}$

Table 2 The coefficients of method (2.2) for order 5

k	4
α_{n-1}	$\frac{-288(48+s(78+s(36+5s)))+(1+s)(8996+6055s+985s)\beta_0}{6(1660+s(2265+s(952+125s)))}$
α_{n-2}	$\frac{72(24+s(57+s(32+5s)))-3(1+s)(1274+s(913+155s))\beta_0}{(3320+2s(2265+s(952+125s)))}$
α_{n-3}	$\frac{-32(8+5s)(2+s(4+s))+3(1+s)(316+s(281+55s))\beta_0}{(3320+2s(2265+s(952+125s)))}$
α_{n-4}	$\frac{18(12+s(33+s(24+5s)))-(1+s)(374+s(367+85s))\beta_0}{6(1660+s(2265+s(952+125s)))}$
β_1	$\frac{12(24+5s(4+s)(5+s(4+s))-3(1+s)(74+s(96+s(39+5s)))\beta_0}{s(1660+s(2265+s(952+125s)))}$
β_s	$\frac{6(-48+37\beta_0)}{s(1660+s(2265+s(952+125s)))}$

Table 3 The coefficients of method (2.2) for order 6

k	5
α_{n-1}	$\frac{(-7200(120+s(231+s(142+s(35+3s))))+(1+s)(615,436+s(553,907+3s(53,483+5002s))))\beta_0}{(12(48,076+s(77,175+s(42,980+9975s+822s))))}$
α_{n-2}	$\frac{(300(120+s(321+s(236+s(65+6s))))-4(1+s)(17,929+s(17,174+s(5194+501s))))\beta_0}{(48,076+s(77,175+s(42,980+9975s+822s)))}$
α_{n-3}	$\frac{(-400(40+s(117+s(98+3s(10+s))))+9(1+s)(2948+s(3325+s(1127+118s)))\beta_0}{(48,076+s(77,175+s(42,980+9975s+822s)))}$
α_{n-4}	$\frac{(225(60+s(183+s(164+s(55+6s))))-4(1+s)(5221+s(6362+3s(794+91s)))\beta_0}{(3(48,076+s(77,175+s(42,980+9975s+822s))))}$
α_{n-5}	$\frac{(-96(24+s(75+s(70+s(25+3s))))+(1+s)(3436+s(4379+s(1753+222s))))\beta_0}{(4(48,076+s(77,175+s(42,980+9975s+822s))))}$
β_1	$\frac{(60(5+2s)(24+s(5+s)(20+3s(5+s)))-24(1+s)(197+s(302+s(163+s(37+3s)))))\beta_0}{(s(48,076+s(77,175+s(42,980+9975s+822s))))}$
β_s	$\frac{-7200+4728\beta_0}{s(48,076+s(77,175+s(42,980+9975s+822s)))}$

Table 4 The coefficients of method (2.2) for order 7

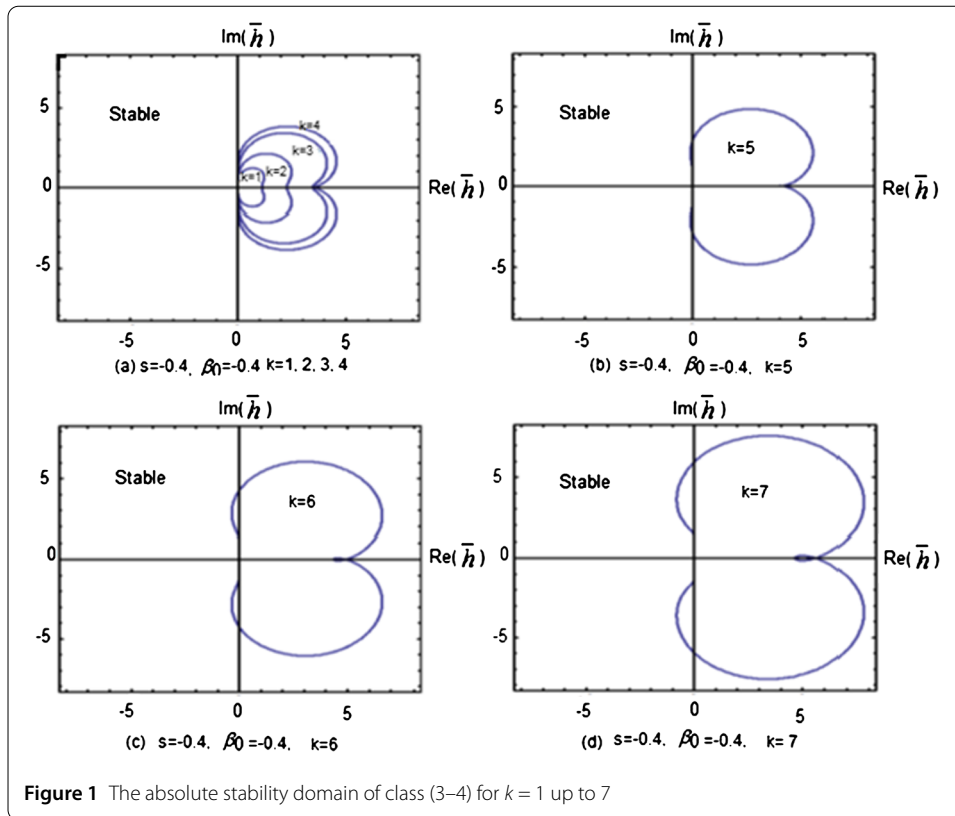| k	6
α_{n-1}	$\frac{-7200(1440+s(3132+s(2320+s(775+s(120+7s)))))+(1+s)(7,796,104+s(8,479,052+s(3,336,768+7s(80,701+4973s))))\beta_0}{(60(107,912+s(194,628+s(130,060+s(40,775+s(6054+343s))))))}$
α_{n-2}	$\frac{(300(720+s(2106+s(1844+s(685+s(114+7s)))))-5(1+s)(78,796+s(91,175+s(37,473+s(6547+413s))))\beta_0}{(215,824+2s(194,628+s(130,060+s(40,775+s(6054+343s)))))}$
α_{n-3}	$\frac{(-1200(480+s(1524+s(1488+s(605+s(108+7s)))))+5(1+s)(174,152+s(230,788+s(104,616+7s(2807+187s))))\beta_0}{(9(107,912+s(194,628+s(130,060+s(40,775+s(6054+343s))))))}$
α_{n-4}	$\frac{225(360+s(1188+s(1228+s(535+s(102+7s)))))-20(1+s)(5701+s(8066+s(3942+s(793+56s))))\beta_0}{3(107,912+s(194,628+s(130,060+s(40,775+s(6054+343s)))))}$
α_{n-5}	$\frac{-96(288+s(972+s(1040+s(475+s(96+7s)))))+5(1+s)(7496+s(11,020+s(5664+7s(173+13s))))\beta_0}{4(107,912+s(194,628+s(130,060+s(40,775+s(6054+343s)))))}$
α_{n-6}	$\frac{300(240+s(822+s(900+s(425+s(90+7s)))))-(1+s)(95,356+s(143,753+s(76,527+s(17,173+1379s))))\beta_0}{90(107,912+s(194,628+s(130,060+s(40,775+s(6054+343s)))))}$
β_1	$\frac{(60(720+7s(6+s)(84+s(6+s))(17+s(6+s)))-10(1+s)(2484+s(4292+s(2785+s(855+s(125+7s)))))\beta_0}{(3s(107,912+s(194,628+s(130,060+s(40,775+s(6054+343s))))))}$
β_s	$\frac{360(-40+23\beta_0)}{s(107,912+s(194,628+s(130,060+s(40,775+s(6054+343s)))))}$

2.1 Stability analysis

Consider the scalar test problem $y' = \lambda y$, $y(0) = y_0$. From equations (3) and (4) the corresponding characteristic equation is as follows:

$$y_n + \sum_{j=1}^{k} \alpha_{n-j} y_{n-j} = \overline{h}\left(\beta_s \left(h\mu y_n + \sum_{j=0}^{k} \gamma_{n-j} y_{n-j} \right) + \beta_1 y_n + \beta_0 y_{n-1} \right), \tag{9}$$

where $\overline{h} = \lambda y$, i.e.

$$A\overline{h}^2 + B\overline{h} + C = 0, \tag{10}$$

Figure 1 The absolute stability domain of class (3–4) for $k = 1$ up to 7

where

$$A = \beta_s \mu r, \qquad B = \beta_s \sum_{j=0}^{k} \gamma_{n-j} r^{1-j} + \beta_1 r + \beta_0, \qquad C = r + \sum_{j=1}^{k} a_{n-j} r^{1-j}. \qquad (11)$$

The absolute stability regions for this class for $k = 1$ up to 7 are given in Fig. 1 for the optimal s and β_0 using the boundary locus method.

The angle α of A(α)-stability for different methods, BDF, EBDF, A-EBDF, MEBDF, Enright methods, HEBDF, and The class (3–4) for various orders are tabulated in Table 5.

We recall some basic notions of one-leg schemes and G-stability.

3 One-leg schemes

Suppose that a linear k-step method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f(t_{n+i}, y_{n+i}) \qquad (12)$$

is given. One-leg methods can be formulated in a compact form by introducing the polynomials

$$\rho(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i, \qquad \sigma(\xi) = \sum_{i=0}^{k} \beta_i \xi^i, \qquad (13)$$

Table 5 A(α)-stability for BDF, EBDF, A-EBDF, MEBDF, Enright methods, HEBDF, and The class (3–4) for various orders

k	1	2	3	4	5	6	7
BDF							
Order	1	2	3	4	5	6	-
α	90°	90°	88°	73°	51°	18°	-
EBDF							
Order	2	3	4	5	6	7	8
α	90°	90°	90°	87.61°	80.2°	67.7°	48.8°
A-EBDF							
Order	2	3	4	5	6	7	8
α	90°	90°	90°	88.85°	84.2°	75°	61°
MEBDF							
Order	2	3	4	5	6	7	8
α	90°	90°	90°	88.4°	82.5°	74.5°	62°
Enright methods							
Order	3	4	5	6	7	8	9
α	90°	90°	87.88°	82.03°	73.10°	59.95°	37.61°
HEBDF							
Order	2	3	4	5	6	7	8
α	90°	90°	90°	89.013°	85.2°	77.195°	60.686°
The class (3–4)							
Order	2	3	4	5	6	7	8
α	90°	90°	90°	90°	86.1º	81.6º	75.2º

with real coefficients $\alpha_i, \beta_i \in R$ and no common divisor. There is also the assumption throughout the normalization that

$$\sigma(1) = 1. \tag{14}$$

The associated one-leg methods are defined by

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = hf\left(\sum_{i=0}^{k} \beta_i t_{n+i}, \sum_{i=0}^{k} \beta_i y_{n+i}\right). \tag{15}$$

In the one-leg methods, the derivative f is evaluated at one point only, which makes it easier to analyze. The one-leg method (15) may have stronger nonlinear stability properties such as G-stability [12, 15]. On the other hand, it is known that to obtain a one-leg method of high order, the parameters α_i, β_i have to satisfy more constraints than those for linear multistep methods, see [7, 8, 14]. The conditions $\rho(1) = 0$, $\rho'(1) = \sigma(1) = 1$ imply the consistency of the scheme (ρ, σ).

3.1 G-stability analysis

The G-stability analysis, announced at the 1975 Dundee conference and published in [6], uses the test problem $dy/dx = f(x, y)$, where $\langle y - z, f(x,y) - f(x,z) \rangle \leq 0$. In the same publication, one-leg methods were introduced and related to corresponding linear multistep methods. Stable behavior for this problem was defined as G-stability. A more detailed account of this work will be given.

If the differential equation satisfies the one-sided Lipschitz condition

$$\langle f(x,y) - f(x,z) \rangle \leq v \|y - z\|^2, \tag{16}$$

with $\nu = 0$, then the exact solutions are contractive. Consider the multistep method as a mapping $R^{n,k} \to R^{n,k}$. Let $Y_m = (y_{m+k-1}, \dots, y_m)^T$ and consider inner product norms on $R^{n,k}$

$$\|Y_m\|_G^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} g_{ji} \langle y_{m+i-1}, y_{m+j-1} \rangle, \tag{17}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on R^n used in (16) and k-dimensional matrix $G = (g_{ij})$ $i, j = 1, \dots, k$ is assumed to be real, symmetric, and positive definite. The inner product $\langle \cdot, \cdot \rangle$, on which $\langle \cdot, \cdot \rangle_G$ is built, is supposed to have a corresponding norm defined by $\|u\|_2 = \langle u, u \rangle$. Similarly we will write $\| \cdot, \cdot \|_G$ as the norm corresponding to $\langle \cdot, \cdot \rangle_G$.

Definition 1 ([6]) The one-leg method (15) is called G-stable if there exists a real, symmetric, and positive definite matrix G such that, for two numerical solutions $\{Y_m\}$ and $\{\hat{Y}_m\}$, we have

$$\|Y_{m+1} - \hat{Y}_{m+1}\|_G \le \|Y_m - \hat{Y}_m\|_G \tag{18}$$

for all step sizes $h > 0$ and for all differential equations satisfying (16) with $\nu = 0$.

Theorem 1 ([2]) *G-stability implies A-stability.*

Theorem 2 ([12]) *Consider a method (ρ, σ). If there exists a real, symmetric, and positive definite matrix G, and real numbers $a_0, \dots, a_k$ such that*

$$\frac{1}{2} \big(\rho(\xi)\sigma(\omega) + \rho(\omega)\sigma(\xi) \big) = (\xi\omega - 1) \sum_{i,j=1}^{k} g_{ij} \xi^{i-1} \omega^{j-1} + \left(\sum_{i=0}^{k} a_i \xi^i \right) \left(\sum_{i=0}^{k} a_j \omega^j \right), \tag{19}$$

then the corresponding one-leg method is G-sable.

Theorem 3 ([5]) *If ρ and σ have no common divisor, then the method (ρ, σ) is A-stable if and only if the corresponding one-leg method is G-stable.*

4 One-leg method for the first hybrid class
Here, the one-leg twin of the first class is studied when $k = 1$ and $k = 2$.

In the case of $k = 1$, method (4) takes the form

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} = h(\beta_s f_{n+s} + \beta_1 f_n + \beta_0 f_{n-1}), \tag{20}$$

$$y_{n+s} = y_n + shf_n, \tag{21}$$

where

$$\alpha_n = 1, \qquad \alpha_{n-1} = -1, \qquad \beta_s = \frac{-1 + 2\beta_0}{2s}, \quad \text{and} \quad \beta_1 = \frac{1 + 2s - 2(1 + s)\beta_0}{2s}.$$

Method (20) has order 2, its truncation error takes the form

$$T_2 = \frac{2 + 3s - 6(1 + s)\beta_0}{12} h^3 y^{(3)}(\eta).$$

The one-leg twin of (20) takes the form

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} = hf(\beta_s t_{n+s} + \beta_1 t_n + \beta_0 t_{n-1}) \tag{22}$$

and has order 2 and its truncation error takes the form

$$T_2 = (1/24)h^3 y^{(3)}(\eta).$$

To discuss G-stability of (22),

$$f_{n+s} = f_n.$$

Substitute f_{n+s} in equation (20), it becomes

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} = h(\beta_s f_n + \beta_1 f_n + \beta_0 f_{n-1}).$$

The corresponding characteristic equations are

$$\rho(\xi) = \alpha_n \xi + \alpha_{n-1}, \qquad \sigma(\xi) = (\beta_1 + \beta_s)\xi + \beta_0.$$

Applying Theorem 2, the variables a_i, $i = 0, 1$ and g_{ij}, $i, j = 1$ satisfy the relations

$$a_0 = \sqrt{\frac{1}{2} - \beta_0}, \qquad a_1 = -\sqrt{\frac{1}{2} - \beta_0} \quad \text{and} \quad g_{11} = a_0^2 + \beta_0.$$

Choose $\beta_0 < 1/2$, $g_{11} = 1/2 > 0$. So, method (22) is G-stable.

In the case of $k = 2$, method (4) takes the form

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2} = h(\beta_s f_{n+s} + \beta_1 f_n + \beta_0 f_{n-1}), \tag{23}$$

$$y_{n+s} = y_n + shf_n + s^2(hf_n - y_n + y_{n-1}). \tag{24}$$

After normalization

$$\alpha_n = \frac{14 + 9s}{6(2 + s + (1 + s)\beta_0)}, \qquad \alpha_{n-1} = \frac{-8 - 6s + 3(1 + s)\beta_0}{3(2 + s + (1 + s)\beta_0)},$$

$$\alpha_{n-2} = \frac{2 + 3s - 6(1 + s)\beta_0}{6(2 + s + (1 + s)\beta_0)}$$

$$\beta_s = \frac{-4 + 5\beta_0}{6s(2 + s + (1 + s)\beta_0)}, \quad \text{and} \quad \beta_1 = \frac{4 + 6s(2 + s) - (1 + s)(5 + 3s)\beta_0}{6s(2 + s + (1 + s)\beta_0)}.$$

Method (23) has order 3, its truncation error takes the form

$$T_3 = \frac{8 + 2s(9 + 4s) + \beta_0(-21 - s(33 + 10s) + 12(1 + s)\beta_0}{72(2 + s + (1 + s)\beta_0)} h^4 y^{(4)}(\eta).$$

The one-leg twin of (23) takes the form

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2} = hf(\beta_s t_{n+s} + \beta_1 t_n + \beta_0 t_{n-1}) \tag{25}$$

and has order 2 if $\beta_0 = (2 + 3s)/(6(1 + s))$ and its truncation error takes the form

$$T_3 = (1/24)h^3 y'''(\eta).$$

To discuss G-stability of (25),

$$f_{n+s} = f_n + 2s(hf_n - y_n + y_{n-1})/h.$$

Substitute f_{n+s} in equation (23), it becomes

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2} = h\big(\beta_s\big(f_n + 2s(hf_n - y_n + y_{n-1})/h\big) + \beta_1 f_n + \beta_0 f_{n-1}\big).$$

The corresponding characteristic equations are

$$\rho(\xi) = (\alpha_n + 2s\beta_s)\xi^2 + (\alpha_{n-1} - 2s\beta_s)\xi + \alpha_{n-2},$$

$$\sigma(\xi) = \big(\beta_1 + \beta_s(1 + 2s)\big)\xi^2 + \beta_0\xi.$$

Applying Theorem 2, the variables a_i, $i = 0, 1, 2$ and g_{ij}, $i, j = 1, 2$ satisfy the relations

$$g_{11} = a_0^2,$$

$$g_{12} = \big(12a_0 a_1\big(2 + s + (1 + s)\beta_0\big) + \beta_0\big(-2 - 3s + 6(1 + s)\beta_0\big)\big)/\big(12\big(2 + s + (1 + s)\beta_0\big)\big),$$

$$g_{22} = -\big(36a_2^2\big(2 + s + (1 + s)\beta_0\big)^2 + (6 + 9s + 10\beta_0)\big(-4 - 6s + (-2 + 3s)\beta_0\big)\big)$$

$$/\big(36\big(2 + s + (1 + s)\beta_0\big)^2\big).$$

Choosing $\beta^* = 0.4$ and $s = 0.9$ makes that $a_0 = 0.079714$, $a_1 = -0.0561286$, $a_2 = -0.0235855$, $g_{11} > 0$, and $\mathrm{Det}\big(\big(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\big)\big) > 0$. Therefore, the matrix G is positive definite and method (25) is G-stable.

5 The second hybrid class

The second hybrid class takes the form

$$y_{n+s} = h\mu f_n + \sum_{j=0}^{k-2} \gamma_{n-j} y_{n-j}, \tag{26}$$

$$y_n + \sum_{j=1}^{k} \alpha_{n-j} y_{n-j} = h\beta_s\big(f_{n+s} - \beta^* f_{n-1}\big), \tag{27}$$

where $f_{n+s} = f(t_{n+s}; y_{n+s})$; $t_{n+s} = t_n + sh$; $-1 < s < 1$ and β_s, α_{n-j}, $j = 1, 2, \ldots, k$, are parameters to be determined as functions of s and β^*. The method with step k has order $p = k$ and y_{n+s} has order $k - 1$. To evaluate the value of y_{n+s} at off-step point, i.e. t_{n+s}, consider the nodes t_n (double node), $t_{n-1}, \ldots, t_{n-k}$ (simple nodes) [14].

Here, the one-leg twin of the second class is studied when $k = 2$ and $k = 3$.

In the case of $k = 2$, the method takes the form

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2} = h\beta_s\big(f_{n+s} - \beta^* f_{n-1}\big), \tag{28}$$

$$y_{n+s} = y_n + shf_n, \tag{29}$$

where

$$\alpha_n = \frac{3 + 2s - \beta^*}{2(1 - \beta^*)}, \qquad \alpha_{n-1} = \frac{-2(1 + s)}{(1 - \beta^*)},$$

$$\alpha_{n-2} = \frac{-(-1 - 2s - \beta^*)}{2(1 - \beta^*)} \quad \text{and} \quad \beta_s = \frac{1}{(1 - \beta^*)}.$$

Method (28) has order 2, its truncation error takes the form

$$T_3 = \frac{2 + 3s(2 + s) + \beta^*}{6(-1 + \beta^*)} h^3 y'''(\eta).$$

The one-leg twin of (28) takes the form

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2} = hf\left(\beta_s t_{n+s} - \beta_s \beta^* t_{n-1}\right) \tag{30}$$

and has order 2 and its truncation error takes the form

$$\bar{T}_3 = \left(\frac{1}{6} - \frac{(1 + s)^2}{2(-1 + \beta^*)^2}\right) h^3 y'''(\eta).$$

If

$$s = \frac{1}{3}\left(-3 + \sqrt{3}\sqrt{1 - 2\beta^* + \beta^{*2}}\right),$$

then method (30) has order 3 and its truncation error becomes

$$\bar{T}_4 = \frac{1}{36\sqrt{3}} h^4 y^{(4)}(\eta).$$

To discuss G-stability of (30), using (8), we have

$$f_{n+s} = f_n.$$

Substitute f_{n+s} in equation (28), it becomes

$$\sum_{i=0}^{2} \alpha_{n-i} y_{n-i} = h\beta_s\left(f_n - \beta^* f_{n-1}\right).$$

The corresponding characteristic equations are

$$\rho(\xi) = \alpha_n \xi^2 + \alpha_{n-1}\xi + \alpha_{n-2} \quad \text{and} \quad \sigma(\xi) = \beta_s \xi\left(\xi - \beta^*\right).$$

Applying Theorem 2, the variables a_i, $i = 0, 1, 2$ and g_{ij}; $i, j = 1, 2$ satisfy the relations

$$g_{11} = a_0^2,$$

$$g_{21} = \frac{-4(1 + s) - 4a_1 a_2(-1 + \beta^*)^2 + \beta^*(-3 - 2s + \beta^*)}{4(-1 + \beta^*)^2},$$

$$g_{22} = -a_2^2 + \frac{3 + 2s - \beta^*}{2(-1 + \beta^*)^2}.$$

Choosing $\beta^* = 0.3$ and $s = -0.1$ makes $a_0 = -0.583636$,

$$a_1 = 1.54524, \qquad a_2 = -0.9616, \qquad g_{11} > 0 \quad \text{and}$$

$$\text{Det}\left(\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\right) > 0.$$

Therefore, the matrix G is positive definite and method (30) is G-stable.

In the case of $k = 3$, the method takes the form

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2} + \alpha_{n-3} y_{n-3} = h\beta_s \left(f_{n+s} - \beta^* f_{n-1}\right), \tag{31}$$

$$y_{n+s} = y_n + shf_n + s^2(hf_n - y_n + y_{n-1}), \tag{32}$$

where

$$\alpha_n = \frac{11 + 12s + 3s^2 - 2\beta^*}{6(1 - \beta^*)}, \qquad \alpha_{n-1} = \frac{-(6 + 10s + 3s^2 + \beta^*)}{2(1 - \beta^*)},$$

$$\alpha_{n-2} = \frac{3 + 8s + 3s^2 + 2\beta^*}{2(1 - \beta^*)}, \qquad \alpha_{n-3} = \frac{-(2 + 6s + 3s^2 + \beta^*)}{6(1 - \beta^*)} \quad \text{and}$$

$$\beta_s = \frac{1}{(1 - \beta^*)}.$$

Method (31) has order 3 and its truncation error takes the form

$$T_4 = \frac{(3 + 2s)(1 + s(3 + s)) + \beta^*}{12(-1 + \beta^*)} h^4 y^{(4)}(\eta).$$

The one-leg twin of (31) takes the form

$$\alpha_n y_n + \alpha_{n-1} y_{n-1} + \alpha_{n-2} y_{n-2} + \alpha_{n-3} y_{n-3}$$
$$= hf\left(\beta_s t_{n+s} - \beta_s \beta^* t_{n-1}\right) \tag{33}$$

and has order 2, its truncation error takes the form

$$\bar{T}_3 = -\frac{(1 + s)^2 \beta^*}{2(-1 + \beta^*)^2} h^3 y'''(\eta).$$

To discuss G-stability of (33), using (8), we have

$$hf_{n+s} = hf_n + 2s(hf_n - y_n + y_{n-1}).$$

Substitute f_{n+s} in equation (31), it becomes

$$\sum_{i=0}^{3} \alpha_{n-i} y_{n-i} = \beta_s\left((hf_n + 2s(hf_n - y_n + y_{n-1})) - h\beta^* f_{n-1}\right).$$

The corresponding characteristic equations are

$$\rho(\xi) = (\alpha_n + 2s\beta_s)\xi^3 + (\alpha_{n-1} - 2s\beta_s)\xi^2 + \alpha_{n-2}\xi + \alpha_{n-3} \quad \text{and}$$

$$\sigma(\xi) = (1 + 2s)\beta_s\xi^3 - \beta^*\beta_s\xi^2.$$

Applying Theorem 2, the variables a_i, $i = 0, 1, 2, 3$ and g_{ij}; $i, j = 1, 2, 3$ satisfy the relations

$$g_{11} = a_0^2,$$

$$g_{12} = g_{21} = a_0 a_1,$$

$$g_{13} = g_{31} = \frac{12a_0 a_2(-1 + \beta^*)^2 - \beta^*(2 + 3s(2 + s) + \beta^*)}{12(-1 + \beta^*)^2},$$

$$g_{23} = g_{32}$$

$$= \frac{-3(1 + 2s)(6 + s(14 + 3s)) - 12a_2 a_3(-1 + \beta^*)^2 + \beta^*(-14 - 3s(10 + s) + 2\beta^*)}{12(-1 + \beta^*)^2},$$

$$g_{22} = a_0^2 + a_1^2,$$

$$g_{33} = a_0^2 + a_1^2 + \frac{2a_2^2(-1 + \beta^*)^2 - \beta^*(6 + s(14 + 3s) + \beta^*)}{2(-1 + \beta^*)^2}.$$

Choosing $s = -0.3$ and $\beta^* = 0.2$ makes $a_0 = -0.231455$;

$$a_1 = 0.613677, \qquad a_2 = -0.53299,$$

$$a_3 = 0.150767, \qquad g_{11} > 0,$$

$$\mathrm{Det}\left(\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\right) > 0 \quad \text{and} \quad \mathrm{Det}\left(\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}\right) > 0.$$

Therefore, the matrix G is positive definite and method (33) is G-stable.

6 Numerical tests

Here, some numerical results are presented to evaluate the performance of the proposed technique [1, 4, 13]. The numerical tests are solved after reduction.

Test 1 Consider the differential algebraic equations:

$$y_1'(t) - ty_2'(t) + t^2 y_3'(t) + y_1(t) - (t + 1)y_2(t) + (t^2 + 2t)y_3(t) = 0;$$

$$y_2'(t) - ty_3'(t) - y_2(t) + (t - 1)y_3(t) = 0;$$

$$y_3(t) = \sin(t);$$

with the initial condition $y_1(0) = 1$; $y_2(0) = 1$; $y_3(0) = 0$; and the exact solution is

$$y_1(t) = \exp(-t) + t\exp(t); \qquad y_2(t) = \exp(t) + t\sin(t); \qquad y_3(t) = \sin(t).$$

Test 2 Consider the differential algebraic equations:

$$x'(t) = 2(1 - y)\sin(y) + x/\sqrt{(1 - y)}; \qquad 0 = x^2 + (y - 1)\cos^2(y);$$

with the initial condition $x(1) = 1$; $y(1) = 0$; and the exact solution is $x(t) = t\cos(1 - t^2)$, $y(t) = 1 - t^2$.

Test 3 Consider the differential algebraic equations:

$$x'(t) = f(x;t) - B(x;t)y; \qquad 0 = g(x;t);$$

where $x'(t) = (x_1, x_2)^{\mathrm{T}}$; $f(x;t) = (1 + (t - 1/2)\exp(t), 2t + (t^2 - 1/4)\exp(t))^{\mathrm{T}}$; $B(x;t) = (x'_1, x'_2)^{\mathrm{T}}$; $g(x;t) = 1/2(x_1^2 + x_2^2 - (t - 1/2)^2 - (t^2 - 1/4)^2)$; with the initial condition $x_1(0) = -1/2$; $x_2(0) = -1/4$; and the exact solution is $x_1(t) = (t - 1/2)$; $x_2(t) = t^2 - 1/4$; $y(t) = \exp(t)$.

The above tests are solved by the two hybrid classes and their one-leg twins of the two classes with $k = 2$ at different values of t. In the first method, $\beta_0 = 0.8$, $s = 2$, and in the second method, $\beta^* = -0.4$, $s = -0.3$. The errors of numerical solutions of tests 1, 2, and 3 are tabulated in Tables 6, 7, and 8, with different step-sizes h, respectively.

7 Conclusion

In this paper, the first hybrid class is studied for $k = 1$ up to 6. It has large stability regions. Its one-leg twin for $k = 1$ ($p = 2$) and $k = 2$ ($p = 3$) is G-stable. In the second class, for $k = p = 2$, the one-leg twin has order 2 except when $s = \frac{1}{3}(-3 + \sqrt{3}\sqrt{(1 - 2\beta^* + \beta^{*2})})$ it has order 3. For $k = p = 3$, the one-leg twin has order 2 and if $\beta^* = 0$, it leads to one leg hybrid BDF, and they are G-stable. The numerical tests show that the hybrid method (4) gives good result with small steps.

Table 6 The error of the first test

	t	h	$Er(y_1(t))$	$Er(y_2(t))$
Hybrid Method (4)	3	0.001	4.78965E-8	1.53733E-8
	6	0.001	2.07019E-6	3.41682E-7
	3	0.0001	1.67133E-10	4.69775E-11
	6	0.0001	1.26784E-9	1.45462E-10
One-Leg of Hybrid Method (4)	3	0.001	8.87175E-6	4.70979E-6
	6	0.001	4.45514E-4	8.88051E-5
	3	0.0001	8.92249E-8	4.72626E-8
	6	0.0001	4.48537E-6	8.9293E-7
Hybrid Method (27)	3	0.001	1.15024E-7	3.66907E-8
	6	0.001	5.31317E-6	8.72459E-7
	3	0.0001	3.5984E-10	1.107355E-10
	6	0.0001	1.12391E-8	1.93154E-9
One-Leg of Hybrid Method (27)	3	0.001	1.28838E-6	9.03448E-8
	6	0.001	2.06502E-4	2.96481E-5
	3	0.0001	1.22774E-8	1.09555E-9
	6	0.0001	2.02582E-6	2.90166E-7
HBDF	3	0.001	4.67957E-9	1.52812E-8
	6	0.001	7.17848E-6	1.01547E-6
	3	0.0001	2.87855E-10	9.16494E-11
	6	0.0001	4.06857E-8	6.55177E-9

Table 7 The error of the second test

	t	h	$Er(y_1(t))$	$Er(y_2(t))$
Hybrid Method	1.3	0.001	6.79234E-13	4.38633E-10
	2.3	0.001	3.66441E-8	1.53328E-8
	1.5	0.0001	1.23906E-12	7.22089E-13
	3	0.0001	7.66911E-10	1.98713E-9
One-Leg Hybrid Method	1.3	0.001	1.20205E-6	1.07885E-6
	2.3	0.001	1.08490E-5	3.18015E-6
	1.5	0.0001	1.54044E-8	1.13384E-8
	3	0.0001	8.71812E-8	2.76648E-8
Class1	1.3	0.001	8.48159E-11	4.33369E-10
	2.3	0.001	1.37322E-7	3.42503E-8
	1.5	0.0001	1.41808E-12	2.87503E-12
	3	0.0001	1.36406E-8	3.2887E-8
One-Leg Class1	1.3	0.001	2.17487E-7	1.48626E-7
	2.3	0.001	2.25127E-6	1.99805E-6
	1.5	0.0001	3.73525E-9	6.91022E-10
	3	0.0001	2.8067E-7	1.95566E-6
HBDF	1.3	0.001	3.24911E-9	2.8293E-9
	2.3	0.001	8.05334E-6	1.9571E-6
	1.5	0.0001	4.8609E-11	3.10769E-11
	3	0.0001	7.08909E-3	2.92056E-1

Table 8 The error of the third test

	t	h	$Er(x_1(t))$	$Er(x_2(t))$	$Er(y(t))$
Hybrid Method	1	0.001	1.22665E-10	1.44872E-10	2.37014E-9
	3	0.001	8.34595E-8	2.85294E-7	8.03989E-7
	1	0.0001	1.63203E-13	2.12274E-13	2.45715E-12
	3	0.0001	1.02396E-10	3.50464E-10	9.74921E-10
One-Leg Hybrid Method	1	0.001	3.66007E-8	9.76405E-8	6.03067E-7
	3	0.001	1.38646E-7	2.52238E-7	3.53319E-6
	1	0.0001	3.62941E-10	9.81447E-10	5.9809E-9
	3	0.0001	3.0906E-10	1.16064E-9	2.49718E-8
Class1	1	0.001	5.5670E-6	6.86347E-6	3.45989E-5
	3	0.001	3.41734E-5	1.18007E-4	2.28748E-4
	1	0.0001	8.08261E-8	9.76428E-8	4.405E-7
	3	0.0001	3.46347E-7	1.19601E-6	2.32459E-6
One-Leg Class1	1	0.001	6.7857E-9	1.47176E-8	1.12592E-7
	3	0.001	1.69542E-7	5.4232E-7	1.85398E-6
	1	0.0001	2.48221E-7	1.85705E-7	7.07125E-8
	3	0.0001	3.66093E-9	1.22613E-8	3.38426E-8
HBDF	1	0.001	5.68471E-10	1.28144E-9	6.63956E-9
	3	0.001	1.07333E-8	4.18462E-8	8.94225E-8
	1	0.0001	7.12319E-13	1.52255E-12	6.35358E-12
	3	0.0001	1.43312E-11	5.25517E-11	1.13875E-10

Abbreviations
DAEs, Differential algebraic equations; ODEs, Ordinary differential equations; BDF, Backward differentiation formula; HBDF, Hybrid BDF methods; EBDF, Extended BDF; MEBDF, Modified extended BDF.

Author details
[1]Department of Mathematics, Anand International College of Engineering, Jaipur, India. [2]International Center for Basic and Applied Sciences, Jaipur, India. [3]Department of Mathematics, Harish-Chandra Research Institute (HRI), Allahbad, India. [4]Department of Mathematics, Faculty of Women for Arts, Science and Education, Ain Shams University, Cairo, Egypt.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Ascher, U., Lin, P.: Sequential regularization methods for nonlinear higher index DAEs. SIAM J. Sci. Comput. **18**, 160–181 (1995)
2. Baiocchi, C., Crouzeix, M.: On the equivalence of A-stability and G-stability. Appl. Numer. Math. **5**, 19–22 (1989)
3. Cash, J.R.: Modified extended backward differentiation formulae for the numerical solution of stiff initial value problems in ODEs and DAEs. J. Comput. Appl. Math. **125**, 117–130 (2000)
4. Celik, E.: On the numerical solution of differential-algebraic equation with index-2. Appl. Math. Comput. **156**, 541–550 (2004)
5. Dahlquist, G.: G-stability is equivalent to A-stability. BIT Numer. Math. **18**, 384–401 (1978)
6. Dahlqusit, G.: Error analysis for a class of methods for stiff nonlinear initial value problems. In: Numerical Analysis Dundee. Lecture Notes in Mathematics, vol. 506, pp. 60–74. Springer, Berlin (1975)
7. Dahlqusit, G.: Some properties of linear multistep and one-leg methods for ODEs. In: Skeel, R.D. (ed.) Proc. 1979 SIGNUM Meeting on Numerical ODEs (1979) Report R 79/963
8. Dahlqusit, G., Liniger, W., Nevanlinna, O.: Stability of two step methods for variable integration steps. SIAM J. Numer. Anal. **20**, 1071–1085 (1983)
9. Ebadi, M., Gokhale, M.Y.: Hybrid BDF methods for the numerical solutions of ordinary differential equations. Numer. Algorithms **55**, 1–17 (2010)
10. Ebadi, M., Gokhale, M.Y.: Class 2 + 1 hybrid BDF-like methods for the numerical solutions of ordinary differential equations. Calcolo **48**, 273–291 (2011)
11. Ebadi, M., Gokhale, M.Y.: A class of multistep methods based on a super-future points technique for solving IVPs. Comput. Math. Appl. **61**, 3288–3297 (2011)
12. Hairer, E., Lubich, C., Roche, M.: The Numerical Solution of Differential-Algebraic Systems by Runge–Kutta Methods. Lecture Notes in Mathematics, vol. 1409. Springer, Berlin (1989)
13. Husein, M.M.J.: Numerical solution of linear differential-algebraic equations using Chebyshev polynomials. Int. Math. Forum **3**(39), 1933–1943 (2008)
14. Ibrahim, I.H., Yousry, F.M.: Hybrid special class for solving differential algebraic equations. Numer. Algorithms **69**(2), 301–320 (2015)
15. Liniger, W.: Multistep and one-leg methods for implicit mixed differential algebraic system. IEEE Trans. Circuits Syst. **26**(9), 755–762 (1979)
16. Michael, H., Roswitha, M., Caren, T., Ewa, W., Stefan, W.: Least-squares collocation for linear higher-index differential–algebraic equations. J. Comput. Appl. Math. **317**, 403–431 (2017)
17. Paolo, M., Pierangelo, M.: Analysis systems of differential algebraic equations (DAEs). Ph.D. Thesis in Aerospace Engineering, Mechanical systems Engineering and Rotary Wing Aircraft (2011)