Zhang and Zhang *Advances in Difference Equations* (2018) 2018:359
https://doi.org/10.1186/s13662-018-1823-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH | Open Access

Existence of nontrivial solution for a nonlocal problem with subcritical nonlinearity

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Abstract

In this paper, we consider the following new nonlocal Dirichlet boundary value problem:

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda u + g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where a and b are positive, λ is a positive parameter, $0 \leq \lambda < a\lambda_1$, λ_1 is the first eigenvalue of operator $-\Delta$. Under appropriate assumptions on the function g which is of subcritical growth, we obtain a nontrivial solution.

MSC: Primary 35B33; secondary 35B38; 35B09

Keywords: Nonlocal problem; Nontrivial solution; Subcritical nonlinearity

1 Introduction and main result

In this paper, we consider the following new nonlocal Dirichlet boundary value problem:

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda u + g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where a and b are positive, λ is a positive parameter.

The search for a nontrivial solution of problem (1.1) is a new subject and of great significance. We put forward a new nonlocal term $a - b\int_\Omega |\nabla u|^2\,dx$, which is different from the well known nonlocal term $a + b\int_\Omega |\nabla u|^2\,dx$ and presents a lot of interesting difficulties.

Recently, mathematical studies have focused on the existence of solutions of the Kirchhoff type problem

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2\,dx)\Delta u = g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $a > 0$, $b > 0$ and Ω is either a smooth bounded domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. The results about problem with subcritical nonlinearity can be seen in [1–5] and the critical cases in [6–13]. Here we do not present the results in detail, someone who is interested in them can consult the references therein.

However, there are only few results about problem (1.1). When $\lambda = 0$ and $g(x,u) = |u|^{p-2}u$ was of subcritical growth, Yin and Liu [14] considered

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2\,dx)\Delta u = |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

and obtained existence and multiplicity of nontrivial solutions. When $\lambda = 0$ and $g(x,u) = f_\lambda(x)|u|^{p-2}u$, Lei [15] considered

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2\,dx)\Delta u = f_\lambda(x)|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Under some special conditions and for $1 < p < 2$, the author obtained two solutions. Lei [16] also investigated

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2\,dx)\Delta u = \frac{\lambda}{u^\gamma}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

and, when $0 < \gamma < 1$ and $0 < \lambda < \lambda_*$, at least two positive solutions were obtained. Wang [17] studied a nonlocal problem involving critical exponent, namely

$$\begin{cases} -(a - b\int_\Omega |\nabla u|^2\,dx)\Delta u = |u|^2u + \mu f(x), & x \in \mathbb{R}^4, \\ u \in D^{1,2}(\mathbb{R}^4), \end{cases}$$

for which infinitely many positive solutions and at least two positive solutions were found for $\mu = 0$ and $\mu \in (0, \mu_*]$. For some other important results the interested reader is also referred to [18–21].

We are inspired by the above articles and consider a new problem which is different from the mentioned above. Assume that nonlinearity g satisfies the following assumptions:

(g_1) g is continuous, $1 \le i \le N$, $|g(x,u)| \le C(1 + |u|^{p-1})$ for some $C > 0$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ if $N \ge 3$, $2^* = \infty$ if $N = 1$ or 2;

(g_2) $g(x,u) = o(u)$ uniformly in x as $u \to 0$;

(g_3) $u \mapsto \frac{g(x,u)}{u}$ is positive for $u \ne 0$, nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, +\infty)$.

Now, we state our main result.

Theorem 1.1 *Suppose that conditions* (g_1)–(g_3) *and* $0 \le \lambda < a\lambda_1$ *hold, then problem* (1.1) *has a nontrivial solution.*

2 Preliminary results

In this section, we present the variational results which will be used in the proof of Theorem 1.1. Let $E := H_0^1(\Omega)$ be endowed with the usual norm

$$\|u\| = \langle u,u\rangle^{1/2} = \left(\int_\Omega |\nabla u|^2\right)^{1/2}.$$

The usual norm in the Lebesgue space $L^p(\Omega)$ is denoted by $|u|_p$.

A function $u \in E$ is called a weak solution of problem (1.1) if

$$a\int_\Omega \nabla u\nabla v\,dx - b\|u\|^2\int_\Omega \nabla u\nabla v\,dx = \lambda\int_\Omega uv\,dx - \int_\Omega g(x,u)v\,dx, \quad \forall v \in E.$$

Moreover, our assumptions imply that the solutions of (1.1) are the critical points of the functional defined in E by

$$I(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega |u|^2\,dx - \int_\Omega G(x,u)\,dx.$$

It is easy to see for $\forall u, v \in E$,

$$\langle I'(u), v\rangle = a\int_\Omega \nabla u\nabla v\,dx - b\|u\|^2\int_\Omega \nabla u\nabla v\,dx - \lambda\int_\Omega uv\,dx - \int_\Omega g(x,u)v\,dx.$$

Let λ_i $(i = 1,2,\dots)$ be the eigenvalues of operator $-\Delta$ with zero Dirichlet boundary condition. It is well known that each eigenvalue λ_i is positive, isolated and has finite multiplicity, the smallest eigenvalue λ_1 being simple and $\lambda_i \to \infty$ as $i \to \infty$. Here we only need the first eigenvalue of $-\Delta$, where $\lambda_1 = \inf_{u\neq 0}\frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$ and assume that $0 \le \lambda < a\lambda_1$.

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1, so from now on we always suppose that (g_1)–(g_3) hold. First, (g_1) and (g_2) imply that for each $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$|g(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1} \quad \text{for all } u \in \mathbb{R}. \tag{3.1}$$

And using (g_2) and (g_3), one can easily check that

$$G(x,u) \ge 0 \quad \text{and} \quad g(x,u)u \ge 2G(x,u) > 0 \quad \text{if } u \neq 0. \tag{3.2}$$

Lemma 3.1 *If* $0 \le \lambda < a\lambda_1$, *then there exists a sequence* $\{u_n\} \subset E$ *satisfying* $I(u_n) \to c$, $I'(u_n) \to 0$, *where* $0 < c < \frac{a^2}{4b}$.

Proof For $\lambda_1 = \inf_{u\neq 0}\frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$, then

$$\left(a - \frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^2 \le a\int_\Omega |\nabla u|^2 - \lambda\int_\Omega |u|^2 \le a\int_\Omega |\nabla u|^2.$$

Also by (3.1), we can choose a sufficiently small $\varepsilon = \frac{\lambda_1}{2}(a - \frac{\lambda}{\lambda_1})$, and then

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\int_\Omega |u|^2 - \int_\Omega G(x,u) \\ &\geq \frac{1}{2}\left(a - \frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^2 - \frac{b}{4}\left(\int_\Omega |\nabla u|^2\right)^2 - \frac{\varepsilon}{2}\int_\Omega |u|^2 - \frac{C_\varepsilon}{p}\int_\Omega |u|^p \\ &\geq \frac{1}{2}\left(a - \frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^2 - \frac{b}{4}\|u\|^4 - \frac{\varepsilon}{2\lambda_1}\int_\Omega |\nabla u|^2 - \frac{C_1 C_\varepsilon}{p}\|u\|^p \\ &\geq \frac{1}{4}\left(a - \frac{\lambda}{\lambda_1}\right)\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{C_1 C_\varepsilon}{p}\|u\|^p, \end{aligned}$$

Since $4 < p < 2^*$, for small enough $\rho > 0$, for all $u \in E$ and $\|u\| = \rho$, it holds that $I(u) = \gamma > 0$. On the other hand, for $u \neq 0$ and $t \in \mathbb{R}$,

$$I(tu) = \frac{at^2}{2}\|u\|^2 - \frac{bt^4}{4}\|u\|^4 - \frac{\lambda t^2}{2}\int_\Omega |u|^2 - \int_\Omega G(x,tu),$$

so that when $t \to \infty$, we have $I(tu) \to -\infty$. This means that there is a t_1 such that $u_1 = t_1 u \in E$, $\|u_1\| > \rho$ and $I(u_1) < 0$. As a consequence, by the mountain pass lemma without (PS) condition [22], there exists a sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$, $I'(u_n) \to 0$ for

$$c = \inf_{h\in\Gamma}\max_{u\in h([0,1])} I(u) \geq \gamma > 0,$$

where

$$\Gamma = \bigl\{h \in C\bigl([0,1],E\bigr) : h(0) = 0, h(1) = u_1\bigr\}.$$

Because

$$\begin{aligned} \max_{t\in[0,1]} I(tu_1) &= \max_{t\in[0,1]}\left\{\frac{at^2}{2}\|u_1\|^2 - \frac{bt^4}{4}\|u_1\|^4 - \frac{\lambda t^2}{2}\int_\Omega |u_1|^2 - \int_\Omega G(x,tu_1)\right\} \\ &< \max_{t\in[0,1]}\left\{\frac{at^2}{2}\|u_1\|^2 - \frac{bt^4}{4}\|u_1\|^4\right\} \\ &\leq \frac{a^2}{4b}, \end{aligned}$$

it is easy to obtain that $0 < c < \frac{a^2}{4b}$ according to the definition of c. □

Lemma 3.2 *Under the condition $c < \frac{a^2}{4b}$, I satisfies the $(PS)_c$ condition, i.e., any $(PS)_c$ sequence of I has a convergent subsequence.*

Proof We drew on the experience of [14]. Let $\{u_n\} \subset E$ be such that $I(u_n) \to c$, $I'(u_n) \to 0$. Since by (3.2)

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{2}\bigl\langle I'(u_n), u_n\bigr\rangle \\ &= \frac{a}{2}\|u_n\|^2 - \frac{b}{4}\|u_n\|^4 - \frac{\lambda}{2}\int_\Omega |u_n|^2 - \int_\Omega G(x,u_n) \end{aligned}$$

$$
\begin{aligned}
&- \left[\frac{a}{2}\|u_n\|^2 - \frac{b}{2}\|u_n\|^4 - \frac{\lambda}{2}\int_\Omega |u_n|^2 - \frac{1}{2} g(x,u_n) \right] \\
&\geq \frac{b}{4}\|u_n\|^4,
\end{aligned}
$$

we know that $\{u_n\}$ is bounded in E. By passing to a subsequence, still denoted $\{u_n\}$, we may assume that there is a $u \in E$ such that

$$
\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } E, \\
&u_n \to u \quad \text{in } L^s(\Omega) \text{ for } s \in [1, 2^*), \\
&u_n(x) \to u(x) \quad \text{for a.e. } x \in \Omega.
\end{aligned}
$$

On account of

$$
\begin{aligned}
o(1) &= \langle I'(u_n), u_n - u \rangle \\
&= \left(a - b\|u_n\|^2\right) \int_\Omega \nabla u_n \nabla (u_n - u) - \lambda \int_\Omega u_n(u_n - u) - \int_\Omega g(x,u_n)(u_n - u)
\end{aligned}
$$

and

$$
\left| \int_\Omega u_n(u_n - u) \right| \leq \left(\int_\Omega |u_n|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |u_n - u|^2 \right)^{\frac{1}{2}},
$$

also by (3.1)

$$
\begin{aligned}
&\left| \int_\Omega g(x,u_n)(u_n - u) \right| \\
&\quad \leq \varepsilon \left| \int_\Omega u_n(u_n - u) \right| + C_\varepsilon \left| \int_\Omega |u_n|^{p-2} u_n(u_n - u) \right| \\
&\quad \leq \varepsilon \left(\int_\Omega |u_n|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |u_n - u|^2 \right)^{\frac{1}{2}} + C_\varepsilon \left(\int_\Omega \left(|u_n|^{p-1}\right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_\Omega \left(|u_n - u|^p\right) \right)^{\frac{1}{p}},
\end{aligned}
$$

because $u_n \to u$ in $L^s(\Omega)$, $s \in [1, 2^*)$, the above two formulas show that when $n \to \infty$,

$$
\left(a - b\|u_n\|^2\right) \int_\Omega \nabla u_n \nabla (u_n - u) \to 0. \tag{3.3}
$$

If there exists a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, satisfying $\|u_n\|^2 \to \frac{a}{b}$, define a functional

$$
\varphi(u) = \frac{\lambda}{2}\int_\Omega |u|^2 + \int_\Omega G(x,u), \quad u \in E.
$$

Then

$$
\langle \varphi'(u), v \rangle = \lambda \int_\Omega uv + \int_\Omega g(x,u)v, \quad u, v \in E,
$$

and

$$
\langle \varphi'(u_n) - \varphi'(u), v \rangle = \lambda \int_\Omega (u_n - u)v + \int_\Omega \left[g(x,u_n) - g(x,u) \right] v.
$$

Claim. $\langle \varphi'(u_n) - \varphi'(u), v\rangle \to 0$, $\forall v \in E$.

Firstly,

$$\lambda \int_\Omega (u_n - u)v \le \lambda \left(\int_\Omega |u_n - u|^2\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2\right)^{\frac{1}{2}},$$

since $u_n \to u$ in $L^2(\Omega)$, thus $\lambda \int_\Omega (u_n - u)v \to 0$.

Secondly, to prove the claim, we only need to prove

$$\lim_{n\to\infty} \int_\Omega \big|g(x,u_n) - g(x,u)\big||v| = 0. \tag{3.4}$$

If (3.4) is not true, then there exist a constant $\varepsilon_0 > 0$ and a subsequence u_{k_i} such that

$$\int_\Omega \big|g(x,u_{k_i}) - g(x,u)\big||v| \ge \varepsilon_0, \quad \forall i \in \mathbb{N}, \tag{3.5}$$

Since $u_n \to u$ in $L^p(\Omega)$, passing to a subsequence if necessary, we can assume that $\sum_{i=1}^\infty |u_{k_i} - u|_p^p < +\infty$. Set

$$\omega(x) = \left[\sum_{i=1}^\infty \big|u_{k_i}(x) - u(x)\big|^p\right]^{\frac{1}{p}}, \quad \forall x \in \Omega.$$

Then $\omega \in L^p(\Omega)$. Note that for $\forall i \in \mathbb{N}$, $x \in \Omega$,

$$\begin{aligned}
&\big|g(x,u_{k_i}) - g(x,u)\big||v| \\
&\quad\le \big(\big|g(x,u_{k_i})\big| + \big|g(x,u)\big|\big)|v| \\
&\quad\le \big[\varepsilon\big(|u_{k_i}| + |u|\big) + C_\varepsilon\big(|u_{k_i}|^{p-1} + |u|^{p-1}\big)\big]|v| \\
&\quad\le \big[2^2\varepsilon\big(|u_{k_i} - u| + |u|\big) + 2^p C_\varepsilon\big(|u_{k_i} - u|^{p-1} + |u|^{p-1}\big)\big]|v| \\
&\quad\le \big[2^2\varepsilon\big(|\omega| + |u|\big) + 2^p C_\varepsilon\big(|\omega|^{p-1} + |u|^{p-1}\big)\big]|v| \\
&\quad:= f(x),
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\int_\Omega f(x)\,dx &= \int_\Omega \big[2^2\varepsilon\big(|\omega| + |u|\big) + 2^p C_\varepsilon\big(|\omega|^{p-1} + |u|^{p-1}\big)\big]|v| \\
&\le 2^2\varepsilon\big(|\omega|_2 + |u|_2\big)|v|_2 + 2^p C_\varepsilon\big(|\omega|_p^{p-1} + |u|_p^{p-1}\big)|v|_p < +\infty.
\end{aligned} \tag{3.7}$$

Since $u_{k_i} \to u$ a.e. in Ω, then by (3.6), (3.7) and Lebesgue Dominated Convergence Theorem, we have

$$\lim_{i\to\infty} \int_\Omega \big|g\big(x,u_{k_i}(x)\big) - g\big(x,u(x)\big)\big||v| = 0,$$

which contradicts (3.5). Hence (3.4) holds. Then the claim follows. By arbitrariness of v, then

$$\|\varphi'(u_n) - \varphi'(u)\|_{E'} \to 0,$$

and $\varphi'(u_n) \to \varphi'(u)$ in E'. While $\langle I'(u_n), v\rangle = (a - b\|u_n\|^2)\langle u_n, v\rangle - \langle \varphi'(u_n), v\rangle$, $\langle I'(u_n), v\rangle \to 0$, $a - b\|u_n\|^2 \to 0$, hence $\varphi'(u_n) \to 0$, i.e.,

$$\langle \varphi'(u), v\rangle = \lambda \int_\Omega uv + \int_\Omega g(x,u)v = 0, \quad \forall v \in E,$$

and then we have

$$\lambda u(x) + g(x, u(x)) = 0 \quad \text{for a.e. } x \in \Omega,$$

by the fundamental lemma of the variational method (see [23]). It follows that $u = 0$. So

$$\varphi(u_n) = \frac{\lambda}{2} \int_\Omega |u_n|^2 + \int_\Omega G(x,u_n) \to \frac{\lambda}{2} \int_\Omega |u|^2 + \int_\Omega G(x,u) = 0.$$

Hence we see that $I(u_n) = \frac{a}{2}\|u_n\|^2 - \frac{b}{4}\|u_n\|^4 - \frac{\lambda}{2}\int_\Omega |u_n|^2 - \int_\Omega G(x,u_n) \to \frac{a^2}{4b}$ from $\|u_n\|^2 \to \frac{a}{b}$. This is a contradiction to $I(u_n) \to c < \frac{a^2}{4b}$. Then $a - b\|u_n\|^2 \to 0$ is not true and any subsequence of $\{a - b\|u_n\|^2 \to 0\}$ does not converge to zero. Therefore there exists a $\delta > 0$ such that $|a - b\|u_n\|^2| > \delta > 0$ when n is large enough. It is clear that $\{a - b\|u_n\|^2 \to 0\}$ is bounded. It follows from (3.3) that $\int_\Omega \nabla u_n \nabla (u_n - u) \to 0$. So $\|u_n\| \to \|u\|$. Hence $u_n \to u$ in E due to the uniform convexity of E. □

Proof of Theorem 1.1 According to Lemma 3.1, there exists a sequence $\{u_n\} \in E$ satisfying $I(u_n) \to c > 0$, $I'(u_n) \to 0$. By Lemma 3.2, $\{u_n\}$, which is the sequence obtained by Lemma 3.1, possesses a convergent to u subsequence (still denoted by $\{u_n\}$). So it follows from the continuity that $I(u_n) \to c > 0$, $I'(u_n) \to 0$. But $I(0) = 0$, therefore $u \neq 0$, that is, u is a nontrivial solution of problem (1.1). □

Acknowledgements

We would like to thank the referee for his/her valuable comments and helpful suggestions which have led to an improvement of the presentation of this paper. The authors are supported by The Inner Mongolia Autonomous Region university scientific research project (NJZY18021) and Postdoctoral research project of Inner Mongolia University (21100-5175504) and Inner Mongolia Normal University introduces high-level scientific research projects (2016YJRC005) and Research project of Inner Mongolia Normal University (2016ZRYB001).

Funding

Not applicable.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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Received: 4 August 2018 Accepted: 28 September 2018 Published online: 10 October 2018

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