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Impulsive Hilfer fractional differential equations

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Abstract

Existence and controllability results for nonlinear Hilfer fractional differential equations are studied. Sufficient conditions for existence and approximate controllability for Sobolev-type impulsive fractional differential equations are established, where the time fractional derivative is the Hilfer derivative. An example for Sobolev-type Hilfer fractional delay partial differential equation with impulsive condition is considered.

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1 Introduction

Nonlinear fractional differential equations can be observed in many areas such as population dynamics, heat conduction in materials with memory, seepage flow in porous media, autonomous mobile robots, fluid dynamics, traffic models, electro magnetic, aeronautics, economics, and so on [1–10]. Controllability results for linear and nonlinear integer order differential systems were studied by several authors (see [11–21]). In Sect. 2, we shall present some basic definitions and lemmas concerning fractional calculus. In Sect. 3, we shall study the existence and controllability results for nonlinear Hilfer fractional differential equations. In Sect. 4, we shall investigate the sufficient conditions for existence and approximate controllability for Sobolev-type impulsive fractional differential equations. In Sect. 5, we consider an example for Sobolev-type Hilfer fractional delay partial differential equation with impulsive condition.

2 Preliminaries

In order to study the existence, controllability, and approximate controllability for delay Hilfer fractional differential equations with impulsive condition, we need the following basic definitions and lemmas.

Definition 2.1 (see [22]) The fractional integral operator of order $\mu > 0$ for a function f can be defined as

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)}{(t-s)^{1-\mu}} ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (see [23, 24]) The Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ is defined as

$$D_{0+}^{\nu,\mu} f(t) = I_{0+}^{\nu(1-\mu)} \frac{d}{dt} I_{0+}^{(1-\nu)(1-\mu)} f(t).$$

Through this paper, let E be a Banach space with $\| \cdot \|$, and let $PC(J, E)$ be the Banach space of all continuous maps from $J = (0, b]$ into E .

Define $Y = \{x : t^{(1-\nu)(1-\mu)} x(t) \in PC(J, E)\}$, with the norm $\| \cdot \|_Y$ defined by $\| \cdot \|_Y = \sup_{t \in J} \|t^{(1-\nu)(1-\mu)} x(t)\|$. Obviously, Y is a Banach space.

Introduce the set $B_r = \{x \in Y : \|x\|_Y \leq r\}$, where $r > 0$.

For $x \in E$, we define two families of operators $\{S_{\nu,\mu}(t) : t > 0\}$ and $\{P_\mu(t) : t > 0\}$ by

$$\begin{aligned} S_{\nu,\mu}(t) &= I_{0+}^{\nu(1-\mu)} P_\mu(t), & P_\mu(t) &= t^{\mu-1} T_\mu(t), \\ T_\mu(t) &= \int_0^\infty \mu \theta \Psi_\mu(\theta) S(t^\mu \theta) d\theta, \end{aligned} \tag{2.1}$$

where

$$\Psi_\mu(\theta) = \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)! \Gamma(1-n\mu)}, \quad 0 < \mu < 1, \theta \in (0, \infty) \tag{2.2}$$

is a function of Wright type which satisfies

$$\int_0^\infty \theta^\tau \Psi_\mu(\theta) d\theta = \frac{\Gamma(1+\tau)}{\Gamma(1+\mu\tau)}$$

for $\theta \geq 0$.

Lemma 2.1 (see [25]) *The operators $S_{\nu,\mu}$ and P_μ have the following properties.*

- (i) $\{P_\mu(t) : t > 0\}$ is continuous in the uniform operator topology.
- (ii) For any fixed $t > 0$, $S_{\nu,\mu}(t)$ and $P_\mu(t)$ are linear and bounded operators, and

$$\|P_\mu(t)x\| \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)} \|x\|, \quad \|S_{\nu,\mu}(t)x\| \leq \frac{Mt^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu) + \mu)} \|x\|. \tag{2.3}$$

- (iii) $\{P_\mu(t) : t > 0\}$ and $\{S_{\nu,\mu}(t) : t > 0\}$ are strongly continuous.

3 Existence and controllability results

We consider the following nonlinear delay Hilfer fractional differential equation with impulsive condition of the form

$$\begin{cases} D_{0+}^{\nu,\mu} x(t) = Ax(t) + f(t, x(\gamma_1(t)), \int_0^t h(t,s)g(s, x(\gamma_2(s))) ds), \\ t \in J = (0, b], t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m \\ I_{0+}^{(1-\nu)(1-\mu)} x(0) = x_0, \end{cases} \tag{3.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \leq \nu \leq 1, 0 < \mu < 1$, A is a closed, linear, and densely defined operator in E . The delay $\gamma_i(t) : J \rightarrow J, i = 1, 2$, are continuous functions, the

state $x(\cdot)$ takes values in the Banach space E , and $h : J \times J \rightarrow R$ is a continuous function, $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, and the nonlinear operators $f : J \times E \times E \rightarrow E$, $g : J \times E \rightarrow E$ are given. The operator A is the infinitesimal generator of a C_0 -semigroup $T(t)$ on E , and there exists a constant $M > 0$ such that $\|T(t)\| \leq M$.

To establish the results, we need the following hypotheses.

(H1) (i) $f : J \times E \times E \rightarrow E$ is continuous and there exist constants $N_1 > 0$ and $N_2 > 0$ such that, for all $t \in J$, $v_1, v_2, w_1, w_2 \in E$, we have

$$\|f(t, v_1, w_1) - f(t, v_2, w_2)\| \leq N_1[\|v_1 - v_2\| + \|w_1 - w_2\|], \quad N_2 = \|f(t, 0, 0)\|.$$

(ii) $g : J \times E \rightarrow E$ is continuous and there exist constants $L_1 > 0$ and $L_2 > 0$ such that, for all $t \in J$, $v_1, v_2 \in E$, we have

$$\|g(t, v_1) - g(t, v_2)\| \leq L_1\|v_1 - v_2\|, \quad L_2 = \|g(t, 0)\|.$$

(iii) The functions $I_k : E \rightarrow E$ are continuous and there exist constants $L_3 > 0, L_4 > 0$ such that, for all $t \in J$, $v_1, v_2 \in E$, we have

$$\|I_k(v_1) - I_k(v_2)\| \leq L_3\|v_1 - v_2\|, \quad L_4 = \|I_k(0)\|, t \in J.$$

(H2) There exists a constant L such that $|h(t, s)| \leq L$ for $(t, s) \in J \times J$.

(H3) There exists a constant q such that, for all $x_1, x_2 \in E$, $\|x_1(\gamma_i(t)) - x_2(\gamma_i(t))\| \leq q\|x_1(t) - x_2(t)\|$ for $i = 1, 2$.

(H4) There exists a constant $r > 0$ such that

$$\left\{ \frac{3M}{\Gamma(\nu(1-\mu) + \mu)} [\|x_0\| + m(L_3r + L_4)] + \frac{3Mb^{\mu+(1-\nu)(1-\mu)}}{\Gamma(\mu + 1)} [N_1(r + bL(L_1r + L_2)) + N_2] \right\} \leq r$$

and

$$\left\{ \frac{4M}{\Gamma(\nu(1-\mu) + \mu)} [\|x_0\| + m(L_3r + L_4)] + \frac{4Mb^{1-\nu(1-\mu)}}{\Gamma(\mu + 1)} [N_1(r + bL(L_1r + L_2)) + N_2] \right\} \times \left[1 + \frac{Mb^\mu \|B\| \|W^{-1}\|}{\Gamma(\mu + 1)} \right] + \frac{4Mb^{1-\nu(1-\mu)} \|x_1\| \|B\| \|W^{-1}\|}{\Gamma(\mu + 1)} \leq r.$$

Definition 3.1 (see [25]) We say that $x \in Y$ is a mild solution of equation (3.1) if it satisfies

$$x(t) = S_{\nu,\mu}(t)x_0 + \int_0^t P_\mu(t-s)f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right) ds + \sum_{0 < t_k < t} S_{\nu,\mu}(t - t_k)I_k(x(t_k^-)), \quad t \in J. \tag{3.2}$$

Theorem 3.1 *If hypotheses (H1)–(H4) are satisfied, then equation (3.1) has a unique mild solution on J provided that*

$$\chi_1 := \left\{ \frac{3qb^\mu MN_1}{\Gamma(\mu + 1)} [1 + bLL_1] + \frac{3mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu) + \mu)} \right\} < 1.$$

Proof Consider the operator Φ on Y defined as follows:

$$\begin{aligned} \Phi x(t) = & S_{v,\mu}(t)x_0 + \int_0^t P_\mu(t-s)f\left(s, x(\gamma_1(s)), \int_0^s k(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \\ & + \sum_{0 < t_k < t} S_{v,\mu}(t-t_k)I_k(x(t_k^-)). \end{aligned}$$

It will be shown that the operator Φ has a fixed point.

First we show that Φ maps B_r into itself. For $x \in B_r$,

$$\begin{aligned} \|\Phi x\|_Y = & \sup_{t \in J} t^{(1-v)(1-\mu)} \|(\Phi x)(t)\| \\ \leq & 3 \sup_{t \in J} t^{(1-v)(1-\mu)} \left\{ \|S_{v,\mu}(t)x_0\| \right. \\ & + \left. \int_0^t (\|P_\mu(t-s)\| \left\| f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right)\right\| ds \right. \\ & + \left. \sum_{0 < t_k < t} \|S_{v,\mu}(t-t_k)I_k(x(t_k^-))\| \right\} \\ \leq & \frac{3M}{\Gamma(v(1-\mu) + \mu)} [\|x_0\| + m(L_3r + L_4)] \\ & + \frac{3Mb^{\mu+(1-v)(1-\mu)}}{\Gamma(\mu + 1)} [N_1(r + bL(L_1r + L_2)) + N_2] \\ \leq & r. \end{aligned}$$

Thus Φ maps B_r into itself.

We show that $(\Phi x)(t)$ is continuous on J for any $x \in B_r$. Let $0 < t \leq b$ and $\epsilon > 0$ be sufficiently small, then

$$\begin{aligned} & \|(\Phi x)(\cdot + \epsilon) - (\Phi x)(\cdot)\|_Y \\ = & \sup_{t \in J} t^{(1-v)(1-\mu)} \|(\Phi x)(t + \epsilon) - (\Phi x)(t)\| \\ \leq & 3 \sup_{t \in J} t^{(1-v)(1-\mu)} \| (S_{v,\mu}(t + \epsilon) - S_{v,\mu}(t))x_0 \| \\ & + 3 \sup_{t \in J} t^{(1-v)(1-\mu)} \left\| \int_0^{t+\epsilon} P_\mu(t + \epsilon - s)f\left(s, x(\gamma_1(s)), \right. \right. \\ & \left. \left. \int_0^s h(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \right. \\ & \left. - \int_0^t P_\mu(t-s)f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \right\| \end{aligned}$$

$$\begin{aligned}
 &+ 3 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\| \sum_{0 < t_k < t + \epsilon} S_{\nu, \mu}(t + \epsilon - t_k) I_k(x(t_k^-)) \right. \\
 &\left. - \sum_{0 < t_k < t} S_{\nu, \mu}(t - t_k) I_k(x(t_k^-)) \right\|. \tag{3.3}
 \end{aligned}$$

Clearly, the right-hand side of (3.3) tends to zero as $\epsilon \rightarrow 0$. Hence, $(\Phi x)(t)$ is continuous on J .

We are going to show that $(\Phi x)(t)$ is a contraction on B_r .

Next, for $x_1, x_2 \in B_r$, we obtain

$$\begin{aligned}
 &\|(\Phi x_1)(t) - (\Phi x_2)(t)\| \\
 &\leq 3 \int_0^t \|P_\mu(t-s)\| \left\| f\left(s, x_1(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x_1(\gamma_2(\tau))) d\tau \right) \right. \\
 &\quad \left. - f\left(s, x_2(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x_2(\gamma_2(\tau))) d\tau \right) \right\| ds \\
 &\quad + 3 \sum_{0 < t_k < t} \|S_{\nu, \mu}(t - t_k)\| \|I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\| \\
 &\leq \left\{ \frac{3qb^\mu MN_1}{\Gamma(\mu + 1)} [1 + bLL_1] + \frac{3mMb^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu) + \mu)} \right\} \|x_1(t) - x_2(t)\| \\
 &\leq \chi_1 \|x_1(t) - x_2(t)\|.
 \end{aligned}$$

Therefore,

$$\sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Phi x_1)(t) - (\Phi x_2)(t)\| \leq \chi_1 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|x_1(t) - x_2(t)\|.$$

This implies that

$$\|\Phi x_1 - \Phi x_2\|_Y \leq \chi_1 \|x_1 - x_2\|_Y.$$

Then, Φ is a contraction mapping on B_r . From the Banach fixed point theorem, Φ has a unique fixed point $x(t)$ on J . Therefore system (3.1) has a unique mild solution on J , and the proof is completed. \square

Next, we will establish a set of sufficient conditions for controllability of impulsive delay Hilfer fractional differential equation in the following form:

$$\begin{cases}
 D_{0+}^{\nu, \mu} x(t) = Ax(t) + Bu(t) + f(t, x(\gamma_1(t)), \int_0^t h(t, s)g(s, x(\gamma_2(s))) ds), \\
 t \in J = (0, b], t \neq t_k, \\
 \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m \\
 I_{0+}^{(1-\nu)(1-\mu)} x(0) = x_0,
 \end{cases} \tag{3.4}$$

where the control function $u(\cdot)$ is given in $L^2(J, U)$, the Banach space of admissible control functions with U a Banach space. The symbol B stands for a bounded linear from U into E .

Definition 3.2 We say that $x \in Y$ is a mild solution of system (3.4) if it satisfies

$$\begin{aligned}
 x(t) &= S_{v,\mu}(t)x_0 + \int_0^t P_\mu(t-s)Bu(s) ds \\
 &+ \int_0^t P_\mu(t-s)f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \\
 &+ \sum_{0 < t_k < t} S_{v,\mu}(t-t_k)I_k(x(t_k^-)), \quad t \in J.
 \end{aligned} \tag{3.5}$$

Definition 3.3 System (3.4) is said to be controllable on J if, for every $x_0, x_1 \in E$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(t)$ of system (3.4) satisfies $x(b) = x_1$, where x_1 and b are the preassigned terminal state and time, respectively.

To establish the result, we need the following additional hypothesis:

(H5) The linear operator W from U into E defined by

$$Wu = \int_0^b P_\mu(b-s)Bu(s) ds$$

has an inverse operator W^{-1} which takes values in $L^2(J, U) \setminus \ker W$, where the kernel space of W is defined by $\ker W = \{x \in L^2(J, U) : Wx = 0\}$ and B is a bounded operator.

Theorem 3.2 If hypotheses (H1)–(H5) are satisfied and if

$$\chi_2 := \left\{ \frac{4qb^\mu MN_1}{\Gamma(\mu + 1)} [1 + bLL_1] + \frac{4mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu) + \mu)} \right\} \left\{ 1 + \frac{M\|B\|\|W^{-1}\|b^\mu}{\Gamma(\mu + 1)} \right\} < 1,$$

then system (3.4) is controllable on J .

Proof Using assumption (H5), define the control

$$\begin{aligned}
 u(t) &= W^{-1} \left\{ x_1 - S_{v,\mu}(b)x_0 \right. \\
 &- \int_0^b P_\mu(b-s)f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \\
 &\left. - \sum_{0 < t_k < b} S_{v,\mu}(b-t_k)I_k(x(t_k^-)) \right\} (t).
 \end{aligned}$$

It shall now be shown that when using this control, the operator Φ^* defined by

$$\begin{aligned}
 (\Phi^*x)(t) &= S_{v,\mu}(t)x_0 + \int_0^t P_\mu(t-s)f\left(s, x(\gamma_1(s)), \int_0^s k(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \\
 &+ \sum_{0 < t_k < t} S_{v,\mu}(t-t_k)I_k(x(t_k^-)) \\
 &+ \int_0^t P_\mu(t-\eta)BW^{-1} \left\{ x_1 - S_{v,\mu}(b)x_0 \right. \\
 &\left. - \int_0^b P_\mu(b-s)f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \right.
 \end{aligned}$$

$$- \sum_{0 < t_k < b} S_{v,\mu}(b - t_k) I_k(x(t_k^-)) \Big\}(\eta) d\eta \tag{3.6}$$

has a fixed point. This fixed point is then a solution of equation (3.4).

First we show that Φ^* maps B_r into itself. For $x \in B_r$,

$$\begin{aligned} \|\Phi^* x\|_Y &= \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Phi^* x)(t)\| \\ &\leq 4 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\{ \|S_{v,\mu}(t)x_0\| + \int_0^t \|P_\mu(t - \eta)\| \|B\| \|W^{-1}\| \|x_1 - S_{v,\mu}(b)x_0 \right. \\ &\quad \left. - \int_0^b P_\mu(b - s) f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \right. \\ &\quad \left. - \sum_{0 < t_k < b} S_{v,\mu}(b - t_k) I_k(x(t_k^-)) \right\}(\eta) d\eta \\ &\quad + \int_0^t \left(\|P_\mu(t - s)\| \left\| f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_2(\tau))) d\tau\right) \right\| ds \right. \\ &\quad \left. + \sum_{0 < t_k < t} \|S_{v,\mu}(t - t_k) I_k(x(t_k^-))\| \right\} \\ &\leq \frac{4M}{\Gamma(\nu(1 - \mu) + \mu)} [\|x_0\| + m(L_3 r + L_4)] + \frac{4Mb^\mu \|B\| \|W^{-1}\|}{\Gamma(\mu + 1)} \left\{ b^{(1-\nu)(1-\mu)} \|x_1\| \right. \\ &\quad \left. + \frac{M}{\Gamma(\nu(1 - \mu) + \mu)} [\|x_0\| + m(L_3 r + L_4)] \right. \\ &\quad \left. + \frac{Mb^{1-\nu(1-\mu)}}{\Gamma(\mu + 1)} [N_1(r + bL(L_1 r + L_2)) + N_2] \right\} \\ &\quad + \frac{4Mb^{1-\nu(1-\mu)}}{\Gamma(\mu + 1)} (N_1(r + bL(L_1 r + L_2)) + N_2) \\ &= \left\{ \frac{4M}{\Gamma(\nu(1 - \mu) + \mu)} [\|x_0\| + m(L_3 r + L_4)] \right. \\ &\quad \left. + \frac{4Mb^{1-\nu(1-\mu)}}{\Gamma(\mu + 1)} [N_1(r + bL(L_1 r + L_2)) + N_2] \right\} \\ &\quad \times \left[1 + \frac{Mb^\mu \|B\| \|W^{-1}\|}{\Gamma(\mu + 1)} \right] + \frac{4Mb^{1-\nu(1-\mu)} \|x_1\| \|B\| \|W^{-1}\|}{\Gamma(\mu + 1)} \leq r. \end{aligned}$$

Thus Φ^* maps B_r into itself.

We show that $(\Phi^* x)(t)$ is continuous on J for any $x \in B_r$. Let $0 < t \leq b$ and $\epsilon > 0$ be sufficiently small, then

$$\begin{aligned} &\|(\Phi^* x)(\cdot + \epsilon) - (\Phi^* x)(\cdot)\|_Y \\ &= \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Phi^* x)(t + \epsilon) - (\Phi^* x)(t)\| \\ &\leq 4 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\| (S_{v,\mu}(t + \epsilon) - S_{v,\mu}(t))x_0 \right\| \\ &\quad + 4 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\| \int_0^{t+\epsilon} P_\mu(t + \epsilon - s) Bu(s) ds - \int_0^t P_\mu(t - s) Bu(s) ds \right\| \end{aligned}$$

$$\begin{aligned}
 &+ 4 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\| \int_0^{t+\epsilon} P_\mu(t+\epsilon-s) f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \right. \\
 &- \left. \int_0^t P_\mu(t-s) f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_2(\tau))) d\tau\right) ds \right\| \\
 &+ 4 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\| \sum_{0 < t_k < t+\epsilon} S_{\nu, \mu}(t+\epsilon-t_k) I_k(x(t_k^-)) \right. \\
 &- \left. \sum_{0 < t_k < t} S_{\nu, \mu}(t-t_k) I_k(x(t_k^-)) \right\|. \tag{3.7}
 \end{aligned}$$

Clearly, the right-hand side of (3.7) tends to zero as $\epsilon \rightarrow 0$. Hence, $(\Phi^*x)(t)$ is continuous on J .

Next, for $x, y \in B_r$, we obtain

$$\begin{aligned}
 &\|(\Phi^*x)(t) - (\Phi^*y)(t)\| \\
 &\leq 4 \int_0^t \|P_\mu(t-\eta)\| \|B\| \|W^{-1}\| \left[\int_0^b \|P_\mu(b-s)\| \right. \\
 &\quad \times \left\| f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_2(\tau))) d\tau\right) \right. \\
 &\quad \left. - f\left(s, y(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, y(\gamma_2(\tau))) d\tau\right) \right\| ds \\
 &\quad + \sum_{0 < t_k < b} \|S_{\nu, \mu}(b-t_k)\| \|I_k(x(t_k^-)) - I_k(y(t_k^-))\|_Y \Big] d\eta + 4 \int_0^t \|P_\mu(t-s)\| \\
 &\quad \times \left\| f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_2(\tau))) d\tau\right) \right. \\
 &\quad \left. - f\left(s, y(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, y(\gamma_2(\tau))) d\tau\right) \right\| ds \\
 &\quad + 4 \sum_{0 < t_k < t} \|S_{\nu, \mu}(t-t_k)\| \|I_k(x(t_k^-)) - I_k(y(t_k^-))\| \\
 &\leq \frac{4M\|B\|\|W^{-1}\|b^\mu}{\Gamma(\mu+1)} \left\{ \frac{qb^\mu MN_1}{\Gamma(\mu+1)} [1 + bLL_1] + \frac{mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu)+\mu)} \right\} \|x(t) - y(t)\| \\
 &\quad + \frac{4qb^\mu MN_1}{\Gamma(\mu+1)} [1 + bLL_1] \|x(t) - y(t)\| + \frac{4mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu)+\mu)} \|x(t) - y(t)\| \\
 &\leq \left\{ \frac{4qb^\mu MN_1}{\Gamma(\mu+1)} [1 + bLL_1] + \frac{4mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu)+\mu)} \right\} \left\{ 1 + \frac{M\|B\|\|W^{-1}\|b^\mu}{\Gamma(\mu+1)} \right\} \|x(t) - y(t)\| \\
 &\leq \chi_2 \|x(t) - y(t)\|.
 \end{aligned}$$

Therefore,

$$\sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Phi^*x_1)(t) - (\Phi^*x_2)(t)\| \leq \chi_2 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|x_1(t) - x_2(t)\|.$$

This implies that

$$\|\Phi^*x_1 - \Phi^*x_2\|_Y \leq \chi_2 \|x_1 - x_2\|_Y.$$

Then Φ^* is a contraction mapping, and hence there exists a unique fixed point $x \in B_r$ such that $\Phi^*x(t) = x(t)$. Therefore system (3.4) has a mild solution satisfying $x(b) = x_1$. Thus, system (3.4) is controllable on J . □

4 Existence and approximate controllability

First, we study existence and uniqueness for Sobolev-type neutral Hilfer fractional differential equation with impulsive condition in the following form:

$$\begin{cases} D_{0+}^{\nu, \mu} [Zx(t) + G(t, x(\gamma_1(t)))] \\ \quad = Ax(t) + f(t, x(\gamma_2(t)), \int_0^t h(t, s)g(s, x(\gamma_3(s))) ds), \quad t \in J = (0, b], t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m \\ I_{0+}^{(1-\nu)(1-\mu)} x(0) = x_0, \end{cases} \tag{4.1}$$

where the delay $\gamma_i(t) : J \rightarrow J, i = 1, 2, 3$, are continuous functions, the state $x(\cdot)$ takes values in the Banach space E , the symbols A and Z are linear operators on E .

The operators $A : D(A) \subset E \rightarrow E$ and $Z : D(Z) \subset E \rightarrow E$ satisfy the following conditions:

(H6) A and Z are closed linear operators.

(H7) $D(Z) \subset D(A)$ and Z is bijective.

(H8) $Z^{-1} : E \rightarrow D(Z)$ is continuous.

Here, (H6) and (H7) together with the closed graph theorem imply the boundedness of the linear operator $AZ^{-1} : E \rightarrow E$.

(H9) For each $t \in J$ and for $\lambda \in \rho(AZ^{-1})$, the resolvent of AZ^{-1} , the resolvent $R(\lambda, AZ^{-1})$ is compact operator.

Lemma 4.1 (see [26]) *Let $T(t)$ be a uniformly continuous semigroup. If the resolvent set $R(\lambda, A)$ of A is compact for every $\lambda \in \rho(A)$, then $T(t)$ is a compact semigroup.*

From the above fact, AZ^{-1} generates a compact semigroup $\{S(t), t > 0\}$ in E , which means that there exists $M > 1$ such that $\sup_{t \in J} \|S(t)\| \leq M$.

We suppose that $0 \in \rho(AZ^{-1})$, the resolvent set of AZ^{-1} , and $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t > 0$. We define the fractional power $(AZ^{-1})^{-\gamma}$ by

$$(AZ^{-1})^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} S(t) dt, \quad \gamma > 0.$$

For $\gamma \in (0, 1]$, $(AZ^{-1})^\gamma$ is a closed linear operator on its domain $D((AZ^{-1})^\gamma)$. Furthermore, the subspace $D((AZ^{-1})^\gamma)$ is dense in E . We will introduce the following basic properties of $(AZ^{-1})^\gamma$.

Theorem 4.1 (see [27]) (1) *Let $0 < \gamma \leq 1$, then $E_\gamma := D((AZ^{-1})^\gamma)$ is a Banach space with the norm $\|x\|_\gamma = \|(AZ^{-1})^\gamma x\|, x \in E_\gamma$.*

(2) *If $0 < \beta < \gamma \leq 1$, then $D((AZ^{-1})^\gamma) \hookrightarrow D((AZ^{-1})^\beta)$ and the embedding is compact whenever the resolvent operator of (AZ^{-1}) is compact.*

(3) *For every $0 < \gamma \leq 1$, there exists a positive constant C_γ such that*

$$\|(AZ^{-1})^\gamma S(t)\| \leq \frac{C_\gamma}{t^\gamma}, \quad 0 < t \leq b.$$

Lemma 4.2 For any $x \in E$, $\beta \in (0, 1)$ and $\delta \in (0, 1]$, we have

$$(AZ^{-1})T_\mu(t)x = (AZ^{-1})^{1-\beta}T_\mu(t)(AZ^{-1})^\beta x, \quad 0 < t \leq b$$

and

$$\|(AZ^{-1})^\delta T_\mu(t)x\| \leq \frac{\mu C_\delta \Gamma(2 - \delta)}{t^{\delta\mu} \Gamma(1 + \mu(1 - \delta))} \|x\|, \quad 0 < t \leq b.$$

Definition 4.1 We say that $x \in Y$ is a mild solution of system (4.1) if the function $AZ^{-1}P_\mu(t - s)G(s, x(\gamma_1(s)), s \in (0, b)$ is integrable on $(0, b)$ and the following integral equation is verified:

$$\begin{aligned} x(t) &= Z^{-1}S_{v,\mu}(t)[Zx(0) + G(0, x(0))] - Z^{-1}G(t, x(\gamma_1(t))) \\ &\quad + \int_0^t Z^{-1}AZ^{-1}P_\mu(t - s)G(s, x(\gamma_1(s))) ds \\ &\quad + \int_0^t Z^{-1}P_\mu(t - s)f\left(s, x(\gamma_2(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_3(\tau))) d\tau\right) ds \\ &\quad + Z^{-1} \sum_{0 < t_k < t} S_{v,\mu}(t - t_k)I_k(x(t_k^-)), \quad t \in J. \end{aligned} \tag{4.2}$$

To establish the results, we need the following assumptions.

(H10) (i) $G : J \times E \rightarrow E$ is continuous and there exist constants $K_1 > 0$ and $K_2 > 0$ such that, for all $v_1, v_2 \in B_r$, we have

$$\begin{aligned} \|(AZ^{-1})^\beta G(t, v_1) - (AZ^{-1})^\beta G(t, v_2)\| &\leq K_1 \|v_1 - v_2\|_Y, \\ K_2 = \|(AZ^{-1})^\beta G(t, 0)\|, \quad t \in J. \end{aligned}$$

(ii) There exists a constant q such that, for all $x_1, x_2 \in E$,

$$\|x_1(\gamma_i(t)) - x_2(\gamma_i(t))\| \leq q \|x_1(t) - x_2(t)\| \quad \text{for } i = 1, 2, 3.$$

(H11) There exists a constant $r > 0$ such that

$$\begin{aligned} 5 \|Z^{-1}\| \left\{ \frac{M\varrho_1}{\Gamma(v(1 - \mu) + \mu)} \right. \\ \left. + \|b^{(1-v)(1-\mu)} \left[M_0 + \frac{C_{1-\beta} b^{\beta\mu} \Gamma(1 + \beta)}{\beta \Gamma(1 + \mu\beta)} \right] \varrho_2 + \frac{Mb^{1-v(1-\mu)} \varrho_3}{\mu \Gamma(\mu)} \right\} \leq r \end{aligned}$$

and

$$\begin{aligned} 6 \|Z^{-1}\| [1 + \varrho_4] \left\{ \frac{M\varrho_1}{\Gamma(v(1 - \mu) + \mu)} + \|b^{(1-v)(1-\mu)} \left[M_0 + \frac{C_{1-\beta} b^{\beta\mu} \Gamma(1 + \beta)}{\beta \Gamma(1 + \mu\beta)} \right] \varrho_2 \right. \\ \left. + \frac{Mb^{1-v(1-\mu)} \varrho_3}{\mu \Gamma(\mu)} \right\} + 6b^{(1-v)(1-\mu)} \|\bar{h}\| \varrho_4 \leq r, \end{aligned}$$

where

$$\begin{aligned} \varrho_1 &= (\|Z\| + M_0K_1)\|x_0\| + M_0K_2 + m(L_3r + L_4), & \varrho_2 &= (K_1r + K_2), \\ \varrho_3 &= N_1(r + bL(L_1r + L_2)) + N_2, \\ \varrho_4 &= \frac{M^2b^\mu \|Z^{-1}\| \|B\| \|B^*\|}{\lambda\mu\Gamma^2(\mu)}, & M_0 &= \|(AZ^{-1})^{-\beta}\|. \end{aligned}$$

Theorem 4.2 *If hypotheses (H1)–(H2) and (H6)–(H11) are satisfied, system (4.1) has a unique mild solution on J provided that*

$$\begin{aligned} \chi_3 := 5q \|Z^{-1}\| \left\{ M_0K_1 + \frac{C_{1-\beta}b^{\beta\mu}K_1\Gamma(1+\beta)}{\Gamma(1+\mu\beta)} + \frac{Mb^\mu N_1}{\mu\Gamma(\mu)}(1 + bLL_1) + \frac{mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu) + \mu)} \right\} < 1. \end{aligned}$$

Proof Consider the operator Ψ on Y defined as follows:

$$\begin{aligned} (\Psi x)(t) &= Z^{-1}S_{v,\mu}(t)[Zx(0) + G(0, x(0))] - Z^{-1}G(t, x(\gamma_1(t))) \\ &\quad + \int_0^t Z^{-1}AZ^{-1}P_\mu(t-s)G(s, x(\gamma_1(s))) ds \\ &\quad + \int_0^t Z^{-1}P_\mu(t-s)f\left(s, x(\gamma_2(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_3(\tau))) d\tau\right) ds \\ &\quad + Z^{-1} \sum_{0 < t_k < t} S_{v,\mu}(t-t_k)I_k(x(t_k^-)). \end{aligned}$$

It will be shown that the operator Ψ has a fixed point. This fixed point is then a mild solution of system (4.1). First we show that Ψ maps B_r into itself. For $x \in B_r$,

$$\begin{aligned} \|\Psi x\|_Y &= \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Psi x)(t)\| \\ &\leq 5 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\{ \|Z^{-1}S_{v,\mu}(t)[Zx(0) + G(0, x(0))]\| + \|Z^{-1}G(t, x(\gamma_1(t)))\| \right. \\ &\quad + \int_0^t \|Z^{-1}AZ^{-1}P_\mu(t-s)G(s, x(\gamma_1(s)))\| ds \\ &\quad + \int_0^t \|Z^{-1}P_\mu(t-s)\| \left\| f\left(s, x(\gamma_2(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_3(\tau))) d\tau\right) \right\| ds \\ &\quad \left. + \sum_{0 < t_k < t} \|Z^{-1}S_{v,\mu}(t-t_k)I_k(x(t_k^-))\| \right\} \\ &\leq \frac{5M\|Z^{-1}\|}{\Gamma(v(1-\mu) + \mu)} [\|Z\|\|x_0\| + M_0(K_1\|x_0\| + K_2) + m(L_3r + L_4)] \\ &\quad + 5M_0b^{(1-\nu)(1-\mu)} \|Z^{-1}\| [K_1r + K_2] \\ &\quad + \frac{5\|Z^{-1}\|C_{1-\beta}b^{\beta\mu+(1-\nu)(1-\mu)}\Gamma(1+\beta)}{\beta\Gamma(1+\mu\beta)} [K_1r + K_2] \\ &\quad + \frac{5\|Z^{-1}\|Mb^{1-\nu(1-\mu)}}{\mu\Gamma(\mu)} (N_1(r + bL(L_1r + L_2)) + N_2) \end{aligned}$$

$$\begin{aligned}
 &= 5 \|Z^{-1}\| \left\{ \frac{MQ_1}{\Gamma(\nu(1-\mu) + \mu)} + b^{(1-\nu)(1-\mu)} \left[M_0 + \frac{C_{1-\beta} b^{\beta\mu} \Gamma(1+\beta)}{\beta \Gamma(1+\mu\beta)} \right] Q_2 \right. \\
 &\quad \left. + \frac{Mb^{1-\nu(1-\mu)} Q_3}{\mu \Gamma(\mu)} \right\} \\
 &\leq r.
 \end{aligned}$$

Thus Ψ maps B_r into itself.

We show that $(\Psi x)(t)$ is continuous on J for any $x \in B_r$. Let $0 < t \leq b$ and $\epsilon > 0$ be sufficiently small, then

$$\begin{aligned}
 &\|(\Psi x)(\cdot + \epsilon) - (\Psi x)(\cdot)\|_Y \\
 &= \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Psi x)(t + \epsilon) - (\Psi x)(t)\| \\
 &\leq 5 \|Z^{-1}\| \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(S_{\nu,\mu}(t + \epsilon) - S_{\nu,\mu}(t))(Zx_0 + G(0, x(0)))\| \\
 &\quad + 5 \|Z^{-1}\| \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|G(t + \epsilon, x(\gamma_1(t + \epsilon))) - G(t, x(\gamma_1(t)))\| \\
 &\quad + 5 \|Z^{-1}\| \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\| \int_0^{t+\epsilon} AZ^{-1} P_\mu(t + \epsilon - s) G(s + \epsilon, x(\gamma_1(s + \epsilon))) ds \right. \\
 &\quad \left. - \int_0^t AZ^{-1} P_\mu(t - s) G(s, x(\gamma_1(s))) ds \right\| \\
 &\quad + 5 \|Z^{-1}\| \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\| \int_0^{t+\epsilon} P_\mu(t + \epsilon - s) f\left(s, x(\gamma_2(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_3(\tau))) d\tau\right) ds \right. \\
 &\quad \left. - \int_0^t P_\mu(t - s) f\left(s, x(\gamma_2(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_3(\tau))) d\tau\right) ds \right\| \\
 &\quad + 5 \|Z^{-1}\| \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\| \sum_{0 < t_k < t+\epsilon} S_{\nu,\mu}(t + \epsilon - t_k) I_k(x(t_k^-)) \right. \\
 &\quad \left. - \sum_{0 < t_k < t} S_{\nu,\mu}(t - t_k) I_k(x(t_k^-)) \right\|. \tag{4.3}
 \end{aligned}$$

Clearly, the right-hand side of (4.3) tends to zero as $\epsilon \rightarrow 0$. Hence, $(\Psi x)(t)$ is continuous on J .

Next, for $x_1, x_2 \in B_r$, we obtain

$$\begin{aligned}
 &\|(\Psi x_1)(t) - (\Psi x_2)(t)\| \\
 &\leq 5 \|Z^{-1}\| \left\{ \|(AZ^{-1})^{-\beta}\| \|(AZ^{-1})^\beta G(t, x_1(\gamma_1(t))) - (AZ^{-1})^\beta G(t, x_2(\gamma_1(t)))\| \right. \\
 &\quad + \int_0^t (t-s)^{\mu-1} \|(AZ^{-1})^{1-\beta} T_\mu(t-s)\| \|(AZ^{-1})^\beta G(s, x_1(\gamma_1(s))) \\
 &\quad \left. - (AZ^{-1})^\beta G(s, x_2(\gamma_1(s)))\| ds \right. \\
 &\quad \left. + \int_0^t \|P_\mu(t-s)\| \left\| f\left(s, x_1(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, x_1(\gamma_2(\tau))) d\tau\right) \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & \left\| -f\left(s, x_2(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x_2(\gamma_2(\tau))) d\tau\right) \right\| ds \\
 & + \sum_{0 < t_k < t} \|S_{v,\mu}(t - t_k)\| \|I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\| \Big\} \\
 & \leq 5q \|Z^{-1}\| \left\{ M_0 K_1 + \frac{C_{1-\beta} b^{\beta\mu} K_1 \Gamma(1 + \beta)}{\Gamma(1 + \mu\beta)} \right. \\
 & \quad \left. + \frac{Mb^\mu N_1}{\mu\Gamma(\mu)}(1 + bLL_1) + \frac{mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu) + \mu)} \right\} \|x_1(t) - x_2(t)\| \\
 & \leq \chi_3 \|x_1(t) - x_2(t)\|.
 \end{aligned}$$

Therefore,

$$\sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Psi x_1)(t) - (\Psi x_2)(t)\| \leq \chi_3 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|x_1(t) - x_2(t)\|.$$

This implies that

$$\|\Psi x_1 - \Psi x_2\|_Y \leq \chi_3 \|x_1 - x_2\|_Y.$$

Then, Ψ is a contraction mapping on B_r . From the Banach fixed point theorem, Ψ has a unique fixed point $x(t)$ on J . Therefore system (4.1) has a unique mild solution on J .

Second, we will study the approximate controllability for Sobolev-type neutral Hilfer fractional differential equation with impulsive condition of the form

$$\begin{cases}
 D_{0+}^{v,\mu} [Zx(t) + G(t, x(\gamma_1(t)))] \\
 = Ax(t) + Bu(t) + f(t, x(\gamma_2(t)), \int_0^t h(t, s)g(s, x(\gamma_3(s))) ds), \\
 t \in J = (0, b], t \neq t_k, \\
 \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m \\
 I_{0+}^{(1-\nu)(1-\mu)} x(0) = x_0,
 \end{cases} \tag{4.4}$$

where the control function $u(\cdot)$ is given in $L^2(J, U)$, the Banach space of admissible control functions with U a Banach space. The symbol B stands for a bounded linear from U into E . □

Definition 4.2 We say that $x \in Y$ is a mild solution of system (4.4) if the function $AZ^{-1}P_\mu(t - s)G(s, x(\gamma_1(s)), s \in (0, b)$ is integrable on $(0, b)$ and the following integral equation is verified:

$$\begin{aligned}
 x(t) &= Z^{-1}S_{v,\mu}(t)[Zx(0) + G(0, x(0))] - Z^{-1}G(t, x(\gamma_1(t))) \\
 &+ \int_0^t Z^{-1}AZ^{-1}P_\mu(t - s)G(s, x(\gamma_1(s))) ds + \int_0^t Z^{-1}P_\mu(t - s)Bu(s) ds \\
 &+ \int_0^t Z^{-1}P_\mu(t - s)f\left(s, x(\gamma_2(s)), \int_0^s h(s, \tau)g(\tau, x(\gamma_3(\tau))) d\tau\right) ds \\
 &+ Z^{-1} \sum_{0 < t_k < t} S_{v,\mu}(t - t_k)I_k(x(t_k^-)), \quad t \in J.
 \end{aligned} \tag{4.5}$$

In order to study the approximate controllability for system (4.4), we introduce the following Sobolev-type linear fractional differential system:

$$\begin{cases} D_{0+}^{v,\mu} Zx(t) = Ax(t) + Bu(t), & t \in J \\ I_{0+}^{(1-v)(1-\mu)} x(0) = x_0. \end{cases} \tag{4.6}$$

It is convenient at this point to introduce the operators associated with (4.6) as follows:

$$\Gamma_0^b = \int_0^b (b-s)^{\mu-1} T_\mu(b-s) BB^* T_\mu^*(b-s) ds,$$

$$R(\lambda, \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1}, \quad \lambda > 0.$$

Let $x(b; x_0, u)$ be the state value of (4.4) at terminal state b , corresponding to the control u and the initial value x_0 . Denote by $R(b, x_0) = \{x(b; x_0, u) : u \in L^2(J, U)\}$ the reachable set of system (4.4) at terminal time b , its closure in X is denoted by $\overline{R(b, x_0)}$.

Definition 4.3 System (4.4) is said to be approximately controllable on the interval J if $\overline{R(b, x_0)} = E$.

Lemma 4.3 (see [28]) *The linear system (4.6) is approximate controllable on J if and only if the operator $\lambda R(\lambda, \Gamma_0^b) = \lambda(\lambda I + \Gamma_0^b)^{-1} \rightarrow 0$ as $\lambda \rightarrow 0^+$ in the strong operator topology.*

We formulate sufficient conditions for the approximate controllability of system (4.4). For this purpose, we first prove the existence of a mild solution for system (4.4). Second we prove that system (4.4) is approximately controllable under certain assumptions.

Theorem 4.3 *If hypotheses (H1)–(H2) and (H6)–(H11) are satisfied, then system (4.4) has a mild solution on J provided that*

$$\begin{aligned} \chi_4 := & 6q \|Z^{-1}\| \left\{ M_0 K_1 + \frac{C_{1-\beta} b^{\beta\mu} K_1 \Gamma(1+\beta)}{\Gamma(1+\mu\beta)} + \frac{Mb^\mu N_1}{\mu\Gamma(\mu)} (1 + bLL_1) + \frac{mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu)+\mu)} \right\} \\ & \times \left\{ 1 + \frac{\|Z^{-1}\| M^2 \|B\| \|B^*\| b^\mu}{\lambda\mu\Gamma^2(\mu)} \right\} < 1. \end{aligned}$$

Proof Consider the operator Ψ^* on Y defined as follows:

$$\begin{aligned} (\Psi^*x)(t) = & Z^{-1} S_{v,\mu}(t) [Zx(0) + G(0, x(0))] - Z^{-1} G(t, x(\gamma_1(t))) \\ & + \int_0^t Z^{-1} A Z^{-1} P_\mu(t-s) G(s, x(\gamma_1(s))) ds \\ & + \int_0^t Z^{-1} P_\mu(t-s) B u(s) ds \\ & + \int_0^t Z^{-1} P_\mu(t-s) f\left(s, x(\gamma_2(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_3(\tau))) d\tau\right) ds \\ & + Z^{-1} \sum_{0 < t_k < t} S_{v,\mu}(t-t_k) I_k(x(t_k^-)), \end{aligned}$$

where

$$\begin{aligned}
 u(t) = & B^* T_\mu^*(b-t)(\lambda I + \Gamma_0^b)^{-1} \left\{ \bar{h} - Z^{-1} S_{v,\mu}(b) [Zx(0) + G(0, x(0))] + Z^{-1} G(b, x(\gamma_1(b))) \right. \\
 & - \int_0^b Z^{-1} A Z^{-1} P_\mu(b-s) G(s, x(\gamma_1(s))) ds - Z^{-1} \sum_{0 < t_k < b} S_{v,\mu}(b-t_k) I_k(x(t_k^-)) \\
 & \left. - \int_0^b Z^{-1} P_\mu(b-s) f\left(s, x(\gamma_2(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_3(\tau))) d\tau\right) ds \right\}.
 \end{aligned}$$

It will be shown that the operator Ψ^* has a fixed point. This fixed point is then a mild solution of system (4.4). We show that Ψ^* maps B_r into itself. For $x \in B_r$,

$$\begin{aligned}
 \|\Psi^* x\|_Y &= \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Psi^* x)(t)\| \\
 &\leq 6 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \left\{ \|Z^{-1} S_{v,\mu}(t) [Zx(0) + G(0, x(0))]\| + \|Z^{-1} G(t, x(\gamma_1(t)))\| \right. \\
 &\quad + \int_0^t \|Z^{-1} A Z^{-1} P_\mu(t-s) G(s, x(\gamma_1(s)))\| ds \\
 &\quad + \int_0^t \|Z^{-1} P_\mu(t-s)\| \|B\| \|u(s)\| ds \\
 &\quad + \int_0^t \|Z^{-1} P_\mu(t-s)\| \left\| f\left(s, x(\gamma_1(s)), \int_0^s h(s, \tau) g(\tau, x(\gamma_2(\tau))) d\tau\right)\right\| ds \\
 &\quad \left. + \sum_{0 < t_k < t} \|Z^{-1} S_{v,\mu}(t-t_k) I_k(x(t_k^-))\| \right\} \\
 &\leq \frac{6M \|Z^{-1}\|}{\Gamma(\nu(1-\mu) + \mu)} [\|Z\| \|x_0\| + M_0(K_1 \|x_0\| + K_2) + m(L_3 r + L_4)] \\
 &\quad + 6M_0 b^{(1-\nu)(1-\mu)} \|Z^{-1}\| [K_1 r + K_2] \\
 &\quad + \frac{6 \|Z^{-1}\| C_{1-\beta} b^{\beta\mu + (1-\nu)(1-\mu)} \Gamma(1 + \beta)}{\beta \Gamma(1 + \mu\beta)} [K_1 r + K_2] + \frac{6M^2 b^\mu \|Z^{-1}\| \|B\| \|B^*\|}{\lambda \mu \Gamma^2(\mu)} \\
 &\quad \times \left\{ b^{(1-\nu)(1-\mu)} \|\bar{h}\| \right. \\
 &\quad + \frac{M \|Z^{-1}\|}{\Gamma(\nu(1-\mu) + \mu)} [\|Z\| \|x_0\| + M_0(K_1 \|x_0\| + K_2) + m(L_3 r + L_4)] \\
 &\quad + \|Z^{-1}\| b^{(1-\nu)(1-\mu)} (K_1 r + K_2) \left[M_0 + \frac{C_{1-\beta} b^{\beta\mu} \Gamma(1 + \beta)}{\beta \Gamma(1 + \mu\beta)} \right] \\
 &\quad \left. + \frac{\|Z^{-1}\| M b^{1-\nu(1-\mu)}}{\mu \Gamma(\mu)} [N_1(r + bL(L_1 r + L_2)) + N_2] \right\} \\
 &\quad + \frac{6 \|Z^{-1}\| M b^{1-\nu(1-\mu)}}{\mu \Gamma(\mu)} (N_1(r + bL(L_1 r + L_2)) + N_2) \\
 &= 6 \|Z^{-1}\| \left\{ [1 + \varrho_4] \frac{M \varrho_1}{\Gamma(\nu(1-\mu) + \mu)} \right. \\
 &\quad \left. + b^{(1-\nu)(1-\mu)} \left[M_0 + \frac{C_{1-\beta} b^{\beta\mu} \Gamma(1 + \beta)}{\beta \Gamma(1 + \mu\beta)} \right] \varrho_2 + \frac{M b^{1-\nu(1-\mu)} \varrho_3}{\mu \Gamma(\mu)} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &+ 6b^{(1-\nu)(1-\mu)} \|\bar{h}\|_{\mathcal{Q}_4} \\
 &\leq r.
 \end{aligned}$$

Thus Ψ^* maps B_r into itself.

Next, for $x_1, x_2 \in B_r$, we obtain

$$\begin{aligned}
 &\|(\Psi^* x_1)(t) - (\Psi^* x_2)(t)\| \\
 &\leq 6 \|Z^{-1}\| \left\{ \|(AZ^{-1})^{-\beta}\| \|(AZ^{-1})^\beta G(t, x_1(\gamma_1(t))) - (AZ^{-1})^\beta G(t, x_2(\gamma_1(t)))\| \right. \\
 &\quad + \int_0^t (t-s)^{\mu-1} \|(AZ^{-1})^{1-\beta} T_\mu(t-s)\| \|(AZ^{-1})^\beta G(s, x_1(\gamma_1(s))) \\
 &\quad - (AZ^{-1})^\beta G(s, x_2(\gamma_1(s)))\| ds \\
 &\quad + \int_0^t \|P_\mu(t-s)\| \left\| f\left(s, x_1(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x_1(\gamma_2(\tau))) d\tau\right) \right. \\
 &\quad \left. - f\left(s, x_2(\gamma_1(s)), \int_0^s h(s, \tau)g(\tau, x_2(\gamma_2(\tau))) d\tau\right) \right\| ds \\
 &\quad + \sum_{0 < t_k < t} \|S_{v,\mu}(t-t_k)\| \|I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\| \left. \right\} \\
 &\quad + 6 \|Z^{-1}\| \int_0^t \|P_\mu(t-s)\| \|B\| \|B^*\| \|T_\mu^*(b-s)\| (\lambda I + \Gamma_0^b)^{-1} \left[\|(AZ^{-1})^{-\beta}\| \right. \\
 &\quad \times \|(AZ^{-1})^\beta G(b, x_1(\gamma_1(b))) - (AZ^{-1})^\beta G(b, x_2(\gamma_1(b)))\| \\
 &\quad + \int_0^b (b-s)^{\mu-1} \|(AZ^{-1})^{1-\beta} T_\mu(b-s)\| \\
 &\quad \times \|(AZ^{-1})^\beta G(s, x_1(\gamma_1(s))) - (AZ^{-1})^\beta G(s, x_2(\gamma_1(s)))\| ds + \int_0^b \|P_\mu(b-\tau)\| \\
 &\quad \times \left\| f\left(\tau, x_1(\gamma_1(\tau)), \int_0^\tau h(\tau, \eta)g(\eta, x_1(\gamma_2(\eta))) d\eta\right) \right. \\
 &\quad \left. - f\left(\tau, x_2(\gamma_1(\tau)), \int_0^\tau h(\tau, \eta)g(\eta, x_2(\gamma_2(\eta))) d\eta\right) \right\| d\tau \\
 &\quad + \sum_{0 < t_k < b} \|S_{v,\mu}(b-t_k)\| \|I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\| \left. \right] ds \\
 &\leq 6q \|Z^{-1}\| \left\{ M_0 K_1 + \frac{C_{1-\beta} b^{\beta\mu} K_1 \Gamma(1+\beta)}{\Gamma(1+\mu\beta)} + \frac{Mb^\mu N_1}{\mu\Gamma(\mu)} (1+bLL_1) \right. \\
 &\quad \left. + \frac{mMb^{(v-1)(1-\mu)}}{\Gamma(v(1-\mu)+\mu)} \right\} \\
 &\quad \times \left\{ 1 + \frac{\|Z^{-1}\| M^2 \|B\| \|B^*\| b^\mu}{\lambda\mu\Gamma^2(\mu)} \right\} \|x_1(t) - x_2(t)\| \\
 &\leq \chi_4 \|x_1(t) - x_2(t)\|.
 \end{aligned}$$

Therefore,

$$\sup_{t \in J} t^{(1-\nu)(1-\mu)} \|(\Psi^* x_1)(t) - (\Psi^* x_2)(t)\| \leq \chi_4 \sup_{t \in J} t^{(1-\nu)(1-\mu)} \|x_1(t) - x_2(t)\|.$$

This implies that

$$\|\Psi^* x_1 - \Psi^* x_2\|_Y \leq \chi_4 \|x_1 - x_2\|_Y.$$

Then Ψ^* is a contraction mapping and hence there exists a unique fixed point $x \in B_r$ such that $\Psi^* x(t) = x(t)$. Hence, any fixed point of Ψ^* is a mild solution of (4.4) on J . \square

Theorem 4.4 *Assume that hypotheses (H1)–(H2) and (H6)–(H11) hold. Further, if the functions $f : J \times E \times E \rightarrow E$, $G : J \times E \rightarrow E$ are uniformly bounded and $\{S(t), t > 0\}$ is compact, then system (4.4) is approximately controllable on J .*

Proof Let $x^\lambda(\cdot)$ be a fixed point of Ψ^* in B_r . Any fixed point of Ψ^* is a mild solution of (4.4) on J under the control

$$u^\lambda(t) = B^* T_\mu^*(b - t)R(\lambda, \Gamma_0^b)p(x^\lambda),$$

where

$$\begin{aligned} p(x^\lambda) &= h - Z^{-1}S_{\nu,\mu}(b)[Zx(0) + G(0, x(0))] + Z^{-1}G(b, x^\lambda(\gamma_1(b))) \\ &\quad - \int_0^b Z^{-1}AZ^{-1}P_\mu(b - s)G(s, x^\lambda(\gamma_1(s))) ds \\ &\quad - Z^{-1} \sum_{0 < t_k < b} S_{\nu,\mu}(b - t_k)I_k(x^\lambda(t_k^-)) \\ &\quad - \int_0^b Z^{-1}P_\mu(b - s)f\left(s, x^\lambda(\gamma_2(s)), \int_0^s h(s, \tau)g(\tau, x^\lambda(\gamma_3(\tau))) d\tau\right) ds, \end{aligned}$$

and satisfies

$$x^\lambda(b) = h - \lambda(\lambda I + \Gamma_0^b)^{-1}p(x^\lambda). \tag{4.7}$$

It follows from the assumption on f and G that there exists $D > 0$ such that

$$\begin{aligned} \left\| f\left(s, x^\lambda(\gamma_2(s)), \int_0^s k(s, \tau)g(\tau, x^\lambda(\gamma_3(\tau))) d\tau\right) \right\| &\leq D, \\ \|G(s, x^\lambda(\gamma_1(s)))\| &\leq D. \end{aligned}$$

Consequently, the sequence $\{G(s, x^\lambda(\gamma_1(s))), f(s, x^\lambda(\gamma_2(s)), \int_0^s k(s, \tau)g(\tau, x^\lambda(\gamma_3(\tau))) d\tau)\}$ is bounded in $L^2(J, E)$.

Thus there is a subsequence, still denoted by $\{G(s, x^\lambda(\gamma_1(s))), f(s, x^\lambda(\gamma_2(s)), \int_0^s k(s, \tau) \times g(\tau, x^\lambda(\gamma_3(\tau))) d\tau)\}$, that converges to say $\{G(s), f(s)\}$. On the other hand, by Lemma 4.3, the operator $\lambda(\lambda I + \Gamma_0^b)^{-1} \rightarrow 0$ strongly as $\lambda \rightarrow 0^+$ for all $0 < s \leq b$, and, moreover, $\|\lambda(\lambda I + \Gamma_0^b)^{-1}\| \leq 1$.

Thus, the Lebesgue dominated convergence theorem and the compactness of $P_\mu(t)$ yield

$$\begin{aligned} & \|x^\lambda(b) - h\| \\ & \leq \left\| \lambda(\lambda I + \Gamma_0^b)^{-1} [h - Z^{-1}S_{v,\mu}(b)(Zx(0) + G(0, x(0)))] \right\| \\ & \quad + \left\| \lambda(\lambda I + \Gamma_0^b)^{-1} Z^{-1}G(b, x^\lambda(\gamma_1(b))) \right\| \\ & \quad + \int_0^b \left\| \lambda(\lambda I + \Gamma_0^b)^{-1} Z^{-1}AZ^{-1}P_\mu(b-s)G(s, x^\lambda(\gamma_1(s))) \right\| ds \\ & \quad + \int_0^b \left\| \lambda(\lambda I + \Gamma_0^b)^{-1} Z^{-1}P_\mu(b-s) f \left(s, x^\lambda(\gamma_2(s)), \int_0^s k(s, \tau)g(\tau, x^\lambda(\gamma_3(\tau))) d\tau \right) \right\| ds \\ & \quad + \left\| \lambda(\lambda I + \Gamma_0^b)^{-1} \right\| \left\| Z^{-1} \sum_{0 < t_k < b} S_{v,\mu}(b-t_k)I_k(x^\lambda(t_k^-)) \right\| \rightarrow 0, \quad \text{as } \lambda \rightarrow 0^+. \end{aligned}$$

This gives the approximate controllability of (4.4), the proof is complete. □

5 Application

Consider the following Sobolev-type Hilfer fractional delay partial differential equation with impulsive condition:

$$\begin{cases} D_{0+}^{v, \frac{3}{5}} [x(t, y) - x_{yy}(t, y) + \tilde{G}(t, x(t - \rho, y))] \\ \quad = \frac{\partial^2 x(t, y)}{\partial y^2} + \chi(t, y) + x(t - \rho, y) + \int_0^t \sin x(s - \rho, y) ds, \\ \quad 0 \leq y \leq \pi, t \in J = (0, b], t \neq t_k, \\ x(t, 0) = x(t, \pi) = 0, \quad t \in J, \\ x(t_k^+, y) - x(t_k^-, y) = I_k(x(t_k^-, y)), \quad k = 1, 2, \dots, m, \\ I_{0+}^{\frac{2}{5}(1-v)}(x(0, y)) = x_0(y), \quad 0 \leq y \leq \pi, \end{cases} \tag{5.1}$$

where $D_{0+}^{v, \frac{3}{5}}$ is a Hilfer fractional derivative of order $0 \leq v \leq 1, \mu = \frac{3}{5}$.

Let $E = U = L^2([0, \pi])$, define the operators $Z : D(Z) \subset E \rightarrow E$ and $A : D(A) \subset E \rightarrow E$ by $Zx = x - x_{yy}$ and $Ax = x_{yy}$, where the domains $D(Z)$ and $D(A)$ are given by $\{x \in E : x, x_y \text{ are absolutely continuous, } x_{yy} \in E, x(0) = x(\pi) = 0\}$.

Then A and Z can be written as

$$Ax = \sum_{n=1}^\infty n^2(x, x_n)x_n, \quad x \in D(A), \quad Zx = \sum_{n=1}^\infty (1 + n^2)(x, x_n)x_n, \quad x \in D(Z).$$

Furthermore, for $x \in E$, we have

$$Z^{-1}x = \sum_{n=1}^\infty \frac{1}{1 + n^2}(x, x_n)x_n, \quad AZ^{-1}x = \sum_{n=1}^\infty \frac{n^2}{1 + n^2}(x, x_n)x_n.$$

It is known that AZ^{-1} is self-adjoint and has the eigenvalues $\lambda_n = -n^2\pi^2, n \in N$, with the corresponding normalized eigenvectors $e_n(\xi) = \sqrt{2} \sin(n\pi \xi)$. Furthermore, AZ^{-1} generates a uniformly strongly continuous semigroup of bounded linear operators $S(t), t > 0$, on

E which is given by

$$S(t)\xi = \sum_{n=1}^{\infty} (\xi_n, e_n)e_n = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \sin(n\pi y) \int_0^1 \sin(n\pi \tau)\xi(\tau) d\tau, \quad \xi \in E.$$

We define the bounded operator $B : U \rightarrow E$ by $Bu = \chi(t, y), 0 \leq y \leq \pi, u \in U$.

Also, we define the following functions:

$$\begin{aligned} x(t)y &= x(t, y), G(t, x(\gamma_1(t)))(y) = \tilde{G}(t, x(t - \rho, y)), \\ \int_0^t h(t, s)g(s, x(\gamma_3(s)))(y) ds &= \int_0^t \sin x(s - \rho, y) ds, \\ f\left(t, x(\gamma_2(t)), \int_0^t h(t, s)g(s, x(\gamma_2(s))) ds\right)(y) &= x(t - \rho, y) + \int_0^t \sin x(s - \rho, y) ds, \end{aligned}$$

where $h(t, s) = 1$. Choose b and other constants such that conditions (H1)–(H2) and (H6)–(H11) are satisfied.

Hence, all the hypotheses of Theorem 4.3 and Theorem 4.4 are satisfied and

$$\begin{aligned} \chi_4 := 6q \|Z^{-1}\| &\left\{ M_0 K_1 + \frac{C_{1-\beta} b^{\beta\mu} K_1 \Gamma(1 + \beta)}{\Gamma(1 + \mu\beta)} + \frac{M b^\mu N_1}{\mu \Gamma(\mu)} (1 + bLL_1) + \frac{m M b^{(v-1)(1-\mu)}}{\Gamma(v(1 - \mu) + \mu)} \right\} \\ &\times \left\{ 1 + \frac{\|Z^{-1}\| M^2 \|B\| \|B^*\| b^\mu}{\lambda \mu \Gamma^2(\mu)} \right\} < 1. \end{aligned}$$

So the Sobolev-type Hilfer fractional delay partial differential equation with impulsive condition (5.1) is approximately controllable on J .

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