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Oscillation results for a fractional order dynamic equation on time scales with conformable fractional derivative

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Abstract

In this paper, we investigate oscillatory and asymptotic properties for a class of fractional order dynamic equations on time scales, where the fractional derivative is defined in the sense of the conformable fractional derivative. Based on the properties of conformable fractional differential and integral, some new oscillatory and asymptotic criteria are established. Applications of the established results show that they can be used to research oscillation for fractional order equations in various time scales such as fractional order differential equations, fractional order difference equations, and so on.

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1 Introduction

In the last few decades, research into oscillation of various equations, including differential equations, difference equations, has been a hot topic in the literature, and much effort has been put in to establish new oscillatory criteria for these equations so far (for example, see [1–12] and the references therein). In [13], Hilger initiated the theory of time scale trying to treat continuous and discrete analysis in a consistent way. Since then, the theory of time scale has received a lot of attention in recent years, and various investigations have been done by many authors [14–28]. Among these investigations, some authors have taken research in oscillation of dynamic equations on time scales (see [29–46] and the references therein). In these investigations for oscillation of dynamic equations on time scales, we notice that most of the results are concerned with dynamic equations involving derivatives of integer order, while none attention has been paid to the research into oscillation of fractional order dynamic equations on time scales so far in the literature.

A time scale is an arbitrary nonempty closed subset of real numbers. $\mathbb{T}$ denotes an arbitrary time scale. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T},\mathbb{T})$ and $\rho \in (\mathbb{T},\mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$.

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf\mathbb{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup\mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The set $\mathbb{T}^\kappa$ is defined to be

$\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum.

A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous at right-dense points and if the left-sided limits exist at left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f | f \in \mathfrak{R}, 1 + \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}$.

Definition 1.1 For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T}, \mathbb{R})$, the *delta derivative* of f at t is denoted by $f^\Delta(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying

$$\left|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)\big(\sigma(t) - s\big)\right| \le \varepsilon\left|\sigma(t) - s\right| \quad \text{for all } s \in \mathfrak{U}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$, which represents the forward difference.

Definition 1.2 For $p \in \mathfrak{R}$, the *exponentialfunction* is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big)\Delta\tau\right) \quad \text{for } s, t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{R}$, then

$$e_p(t,s) = \exp\left(\int_s^t p(\tau)\,d\tau\right) \quad \text{for } s, t \in \mathbb{R}.$$

If $\mathbb{T} = \mathbb{Z}$, then

$$e_p(t,s) = \prod_{\tau=s}^{t-1}\big[1 + p(\tau)\big] \quad \text{for } s, t \in \mathbb{Z}, s < t.$$

According to [47, Theorem 5.2], if $p \in \mathfrak{R}^+$, then $e_p(t,s) > 0$ for $\forall s, t \in \mathbb{T}$.

Recently, Benkhettou et al. developed a conformable fractional calculus theory on arbitrary time scales [48], and established the basic tools for fractional differentiation and fractional integration on time scales.

Definition 1.3 ([48, Definition 1]) For $t \in \mathbb{T}^\kappa$, $\alpha \in (0,1]$, and a function $f \in (\mathbb{T}, \mathbb{R})$, the fractional derivative of α order for f at t is denoted by $f^{(\alpha)}(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying

$$\left|\big[f\big(\sigma(t)\big) - f(s)\big]t^{1-\alpha} - f^{(\alpha)}(t)\big(\sigma(t) - s\big)\right| \le \varepsilon\left|\sigma(t) - s\right| \quad \text{for all } s \in \mathfrak{U}.$$

Definition 1.4 ([48, Definition 28]) If $F^{(\alpha)}(t) = f(t)$, $t \in \mathbb{T}^\kappa$, then F is called an α-order *antiderivative* of f, and the Cauchy α-fractional integral of f is defined by

$$\int_a^b f(t)\Delta^\alpha t = \int_a^b f(t)t^{\alpha-1}\Delta t = F(b) - F(a), \quad \text{where } a, b \in \mathbb{T}.$$

Theorem 1.5 ([48, Theorem 4]) *For $t \in \mathbb{T}^{\kappa}$, $\alpha \in (0,1]$, and a function $f \in (\mathbb{T},\mathbb{R})$, the following conclusions hold*:

(i) *If f is conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is conformable fractional differentiable of order α at t with $f^{(\alpha)}(t) = \frac{f(\sigma(t))-f(t)}{\sigma(t)-t}t^{1-\alpha} = \frac{f(\sigma(t))-f(t)}{\mu(t)}t^{1-\alpha}$.*

(iii) *If t is right-dense, then f is conformable fractional differentiable of order α at t if, and only if, the limit $\lim_{s\to t}\frac{f(s)-f(t)}{s-t}t^{1-\alpha}$ exists as a finite number. In this case, $f^{(\alpha)}(t) = \lim_{s\to t}\frac{f(s)-f(t)}{s-t}t^{1-\alpha}$.*

(iv) *If f is fractional differentiable of order α at t, then $f(\sigma(t)) = f(t) + \mu(t)t^{1-\alpha}f^{(\alpha)}(t)$.*

Corollary 1.6 *According to the definition of the conformable fractional differentiable of order α, it holds that $f^{(\alpha)}(t) = t^{1-\alpha}f^{\Delta}(t)$, where $f^{\Delta}(t)$ is the usual Δ derivative in the case $\alpha = 1$. Furthermore, if $f^{(\alpha)}(t) > 0$ (< 0) for $t > 0$, then f is increasing* (*decreasing*) *for $t > 0$.*

Theorem 1.7 *Let $\widetilde{p}(t) = t^{\alpha-1}p(t)$, $\alpha \in (0,1]$. If $\widetilde{p} \in \mathfrak{R}$, and fix $t_0 \in \mathbb{T}$, then the* exponential function $e_{\widetilde{p}}(t,t_0)$ *is the unique solution of the following initial value problem*:

$$\begin{cases} y^{(\alpha)}(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

Proof By [47, Theorem 5.1], if $p \in \mathfrak{R}$, and fix $t_0 \in \mathbb{T}$, then the *exponential function $e_p(t,t_0)$* is the unique solution of the following initial value problem:

$$\begin{cases} y^{\Delta}(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

So, according to Corollary 1.6, one has

$$\bigl(e_{\widetilde{p}}(t,t_0)\bigr)^{(\alpha)} = t^{1-\alpha}\bigl(e_{\widetilde{p}}(t,t_0)\bigr)^{\Delta} = t^{1-\alpha}\widetilde{p}(t)e_{\widetilde{p}}(t,t_0) = p(t)e_{\widetilde{p}}(t,t_0),$$

which confirms the proof. □

Theorem 1.8 ([48, Theorem 15]) *Assume that $f, g \in (\mathbb{T},\mathbb{R})$ are conformable fractional differentiable of order α. Then*

(i) $(f+g)^{(\alpha)}(t) = f^{(\alpha)}(t) + g^{(\alpha)}(t)$.

(ii) $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t) = f^{(\alpha)}(t)g(\sigma(t)) + f(t)g^{(\alpha)}(t)$.

(iii) $(\frac{1}{f})^{(\alpha)}(t) = -\frac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))}$.

(iv) $(\frac{f}{g})^{(\alpha)}(t) = \frac{f^{(\alpha)}(t)g(t)-f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}$.

Theorem 1.9 *Let $\alpha \in (0,1]$, f, g be two rd-continuous functions. Then*

$$\int_a^b f^{(\alpha)}(t)g(t)\Delta^{\alpha}t = \bigl[f(t)g(t)\bigr]_a^b - \int_a^b f\bigl(\sigma(t)\bigr)g^{(\alpha)}(t)\Delta^{\alpha}t.$$

The proof of Theorem 1.9 can be reached by fulfilling α-fractional integral for the first equality in Theorem 1.8(ii).

For more details about the calculus of time scales, we refer to [49].

In this paper, we investigate oscillatory and asymptotic behavior of solutions of the following fractional order dynamic equation on time scales:

$$\bigl(a(t)\bigl(\bigl[r(t)x^{(\alpha)}(t)\bigr]^{(\alpha)}\bigr)^{\gamma}\bigr)^{(\alpha)} + p(t)\bigl(\bigl[r(t)x^{(\alpha)}(t)\bigr]^{(\alpha)}\bigr)^{\gamma} + q(t)f\bigl(x(t)\bigr) = 0, \quad t \in \mathbb{T}_0, \tag{1.1}$$

where $\mathbb{T}$ is an arbitrary time scale, $\mathbb{T}_0 = [t_0,\infty) \cap \mathbb{T}$, $a,r,p,q \in C_{\mathrm{rd}}(\mathbb{T}_0,\mathbb{R}_+)$, $f \in C(\mathbb{R},\mathbb{R})$ satisfying $xf(x) > 0$, $\frac{f(x)}{x^{\gamma}} \geq L > 0$ for $x \neq 0$, and $\gamma \geq 1$ is a quotient of two odd positive integers.

A solution of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory in case all its solutions are oscillatory.

We will establish some new criteria of oscillatory and asymptotic behavior for Eq. (1.1) based on the properties of conformable fractional differential and integral together with a generalized Riccati transformation technique in Sect. 2, and present some applications for the established results in Sect. 3. Some conclusions are presented in Sect. 4. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = (0,\infty)$, while $\mathbb{Z}$ denotes the set of integers. $\widetilde{p}(t) = t^{\alpha-1}p(t)$, $t_i \in \mathbb{T}$, $[t_i,\infty)_{\mathbb{T}} = [t_i,\infty) \cap \mathbb{T}$, $i = 0,1,\dots,5$. $\vartheta_1(t,a) = \int_a^t \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta^{\alpha}s$, $\vartheta_2(t,a) = \int_a^t \frac{\vartheta_1(s,a)}{r(s)}\Delta^{\alpha}s$.

2 Main results

Lemma 2.1 *Suppose $-\frac{\widetilde{p}}{a} \in \Re_+$, and assume that*

$$\int_{t_0}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta^{\alpha}s = \infty, \tag{2.1}$$

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta^{\alpha}s = \infty, \tag{2.2}$$

and Eq. (1.1) *has an eventually positive solution x. Then the following conclusions hold.*

(i) *There exists a sufficiently large $T_1^* \in \mathbb{T}$ such that*

$$\left(\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}\right)^{(\alpha)} < 0, \qquad \bigl[r(t)x^{(\alpha)}(t)\bigr]^{(\alpha)} > 0 \quad on\ \bigl[T_1^*,\infty\bigr)_{\mathbb{T}}.$$

(ii) *If we assume that*

$$\limsup_{t\to\infty} \int_{t_0}^{t}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s\right)^{\frac{1}{\gamma}}\Delta^{\alpha}\tau\right]\Delta^{\alpha}\xi$$
$$= \infty, \tag{2.3}$$

then either there exists a sufficiently large $T_2^* \in \mathbb{T}$ such that $x^{(\alpha)}(t) > 0$ on $[T_2^*,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$.

Proof of (i) According to $-\frac{\widetilde{p}}{a} \in \Re_+$, one has $e_{-\frac{\widetilde{p}}{a}}(t,t_0) > 0$. Since x is eventually a positive solution of (1.1), there exists a sufficiently large t_1 such that $x(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, and for

$t \in [t_1,\infty)_{\mathbb{T}}$, by use of Theorem 1.8(iv) and Theorem 1.7, one can deduce that

$$\begin{aligned}
&\left(\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)}\right)^{(\alpha)} \\
&\quad= \frac{e_{-\frac{\tilde{p}}{a}}(t,t_0)(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma})^{(\alpha)} - (e_{-\frac{\tilde{p}}{a}}(t,t_0))^{(\alpha)}a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} \\
&\quad= \frac{(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma})^{(\alpha)} + p(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} = \frac{-q(t)f(x(t))}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} < 0. \qquad (2.4)
\end{aligned}$$

So by Corollary 1.6 one can see that $\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)}$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, and considering $a(t) > 0$, $e_{-\frac{\tilde{p}}{a}}(t,t_0) > 0$, one can deduce that $[r(t)x^{(\alpha)}(t)]^{(\alpha)}$ is eventually of one sign.

Now we claim $[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Otherwise, assume there exists sufficiently large $t_3 > t_2$ such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)} < 0$ on $[t_3,\infty)_{\mathbb{T}}$. Then by Corollary 1.6 one can see $r(t)x^{(\alpha)}(t)$ is strictly decreasing, and due to Definition 1.4 it holds that

$$\begin{aligned}
r(t)x^{(\alpha)}(t) - r(t_3)x^{(\alpha)}(t_3) &= \int_{t_3}^{t} \frac{[e_{-\frac{\tilde{p}}{a}}(s,t_0)a(s)]^{\frac{1}{\gamma}}[r(s)x^{(\alpha)}(s)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(s,t_0)a(s)]^{\frac{1}{\gamma}}}\Delta^{\alpha}s \\
&\leq \frac{a^{\frac{1}{\gamma}}(t_3)[r(t_3)x^{(\alpha)}(t_3)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t_3,t_0)]^{\frac{1}{\gamma}}}\int_{t_3}^{t}\frac{[e_{-\frac{\tilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta^{\alpha}s. \qquad (2.5)
\end{aligned}$$

From (2.1) one can see that $\lim_{t\to\infty} r(t)x^{(\alpha)}(t) = -\infty$, and thus there exists a sufficiently large $t_4 \in [t_3,\infty)_{\mathbb{T}}$ such that $r(t)x^{(\alpha)}(t) < 0$ on $[t_4,\infty)_{\mathbb{T}}$. Furthermore,

$$x(t) - x(t_4) = \int_{t_4}^{t}\frac{r(s)x^{(\alpha)}(s)}{r(s)}\Delta^{\alpha}s \leq r(t_4)x^{(\alpha)}(t_4)\int_{t_4}^{t}\frac{1}{r(s)}\Delta^{\alpha}s.$$

Using (2.2) one has $\lim_{t\to\infty} x(t) = -\infty$, which leads to a contradiction. So it holds that $[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_2,\infty)_{\mathbb{T}}$, and the proof is complete after setting $T_1^* = t_2$. □

Proof of (ii) According to (i), one can obtain that $x^{(\alpha)}(t)$ is eventually of one sign. So there exists a sufficiently large $t_5 > t_4$ such that either $x^{(\alpha)}(t) > 0$ or $x^{(\alpha)}(t) < 0$ on $[t_5,\infty)_{\mathbb{T}}$, where t_4 is defined as in Lemma 2.1.

If $x^{(\alpha)}(t) < 0$, considering $x(t)$ is an eventually positive solution of Eq. (1.1), one can obtain that $\lim_{t\to\infty} x(t) = \beta_1 \geq 0$ and $\lim_{t\to\infty} r(t)x^{(\alpha)}(t) = \beta_2 \leq 0$.

We claim $\beta_1 = 0$. Otherwise, if we assume $\beta_1 > 0$, then $x(t) \geq \beta_1$ on $[t_5,\infty)_{\mathbb{T}}$, and for $t \in [t_5,\infty)\cap\mathbb{T}$, fulfilling α-fractional integral for (2.4) from t to ∞, considering $\frac{f(x)}{x^{\gamma}} \geq L > 0$, we get that

$$\begin{aligned}
-\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)} &= -\lim_{t\to\infty}\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)} + \int_{t}^{\infty}\frac{-q(s)f(x(s))}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s \\
&\leq -\lim_{t\to\infty}\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)} + \int_{t}^{\infty}\frac{-Lq(s)x^{\gamma}(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s \\
&\leq -L\int_{t}^{\infty}\frac{q(s)x^{\gamma}(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s \leq -L\beta_1^{\gamma}\int_{t}^{\infty}\frac{q(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s,
\end{aligned}$$

which means

$$-\big[r(t)x^{(\alpha)}(t)\big]^{(\alpha)} \le -\left\{L\beta_1^{\gamma}\left[\frac{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}{a(t)}\int_t^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s\right]\right\}^{\frac{1}{\gamma}}. \tag{2.6}$$

Substituting t with τ in (2.6), fulfilling α-fractional integral for (2.6) with respect to τ from t to ∞ yields

$$\begin{aligned} r(t)x^{(\alpha)}(t) &= \lim_{t\to\infty} r(t)x^{(\alpha)}(t) - \beta_1 L^{\frac{1}{\gamma}}\int_t^{\infty}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s\right)^{\frac{1}{\gamma}}\Delta^{\alpha}\tau \\ &= \beta_2 - \beta_1 L^{\frac{1}{\gamma}}\int_t^{\infty}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s\right)^{\frac{1}{\gamma}}\Delta^{\alpha}\tau \\ &\le -\beta_1 L^{\frac{1}{\gamma}}\int_t^{\infty}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s\right)^{\frac{1}{\gamma}}\Delta^{\alpha}\tau, \end{aligned}$$

which implies

$$x^{(\alpha)}(t) \le -\beta_1 L^{\frac{1}{\gamma}}\frac{1}{r(t)}\int_t^{\infty}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s\right)^{\frac{1}{\gamma}}\Delta^{\alpha}\tau. \tag{2.7}$$

Substituting t with ξ in (2.7), fulfilling α-fractional integral for (2.7) with respect to ξ from t_5 to t yields

$$\begin{aligned} &x(t) - x(t_5) \\ &\quad\le -\beta_1 L^{\frac{1}{\gamma}}\int_{t_5}^{t}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^{\alpha}s\right)^{\frac{1}{\gamma}}\Delta^{\alpha}\tau\right]\Delta^{\alpha}\xi. \end{aligned} \tag{2.8}$$

By (2.8) and (2.3) one can deduce that $\lim_{t\to\infty} x(t) = -\infty$, which leads to a contradiction. So we have $\beta_1 = 0$, and the proof is complete. □

Lemma 2.2 *If* $-\frac{\widetilde{p}}{a} \in \Re_+$, *and* x *is a positive solution of Eq.* (1.1) *such that*

$$\big[r(t)x^{(\alpha)}(t)\big]^{(\alpha)} > 0, \qquad x^{(\alpha)}(t) > 0 \quad \text{on } \big[T_3^*,\infty\big)_{\mathbb{T}},$$

where $T_3^* \in \mathbb{T}$ *is sufficiently large, then for* $t \in [T_3^*,\infty)_{\mathbb{T}}$ *it holds that*

$$x^{(\alpha)}(t) \ge \frac{\vartheta_1(t,T_3^*)}{r(t)}\left\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\widetilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\right\}$$

and

$$x(t) \ge \vartheta_2\big(t,T_3^*\big)\left\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\widetilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\right\}.$$

Proof Take $T_3^* > \max(T_1^*, T_2^*)$, where T_1^*, T_2^* are defined as in Lemma 2.1. By Lemma 2.1 one can see that $\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^\gamma}{e_{-\frac{\tilde{p}}{a}}(t,t_0)}$ is decreasing on $[T_3^*, \infty)$. Then

$$\begin{aligned} r(t)x^{(\alpha)}(t) &\geq r(t)x^{(\alpha)}(t) - r\bigl(T_3^*\bigr)x^{(\alpha)}\bigl(T_3^*\bigr) = \int_{T_3^*}^t \frac{[e_{-\frac{\tilde{p}}{a}}(s,t_0)a(s)]^{\frac{1}{\gamma}}[r(s)x^{(\alpha)}(s)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(s,t_0)a(s)]^{\frac{1}{\gamma}}}\Delta^\alpha s \\ &\geq \frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\int_{T_3^*}^t \frac{[e_{-\frac{\tilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta^\alpha s = \vartheta_1\bigl(t, T_3^*\bigr)\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}, \end{aligned}$$

and

$$x^{(\alpha)}(t) \geq \frac{\vartheta_1(t,T_3^*)}{r(t)}\left\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\right\}.$$

Furthermore,

$$\begin{aligned} x(t) &\geq x(t) - x\bigl(T_3^*\bigr) = \int_{T_3^*}^t x^{(\alpha)}(s)\Delta^\alpha s \geq \int_{T_3^*}^t \frac{\vartheta_1(s,T_3^*)}{r(s)}\left\{\frac{a^{\frac{1}{\gamma}}(s)[r(s)x^{(\alpha)}(s)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}\right\}\Delta^\alpha s \\ &\geq \left\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\right\}\int_{T_3^*}^t \frac{\vartheta_1(s,T_3^*)}{r(s)}\Delta^\alpha s = \vartheta_2\bigl(t,T_3^*\bigr)\left\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\right\}. \end{aligned}$$

The proof is complete. □

Lemma 2.3 ([50, Theorem 41]) *Assume that X and Y are nonnegative real numbers. Then*

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda-1)Y^\lambda \quad \textit{for all } \lambda > 1.$$

Theorem 2.4 *Suppose* $-\frac{\tilde{p}}{a} \in \mathfrak{R}_+$, *and assume that* (2.1)–(2.3) *hold, and for all sufficiently large* T,

$$\begin{aligned} \limsup_{t\to\infty}\biggl\{\int_T^t\biggl\{ & L\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} + \frac{\varsigma(s)\vartheta_1(s,T)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \\ & - \left[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,T)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\biggr\}\Delta^\alpha s\biggr\} = \infty, \end{aligned} \tag{2.9}$$

where ς, η *are two given nonnegative functions on* $\mathbb{T}$ *with* $\varsigma(t) > 0$. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero.*

Proof Assume that (1.1) has a nonoscillatory solution $x(t)$ on $\mathbb{T}_0$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, where t_1 is sufficiently large. According to Lemma 2.1, there exists sufficiently large t_2 such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_2, \infty)_{\mathbb{T}}$, and either $x^{(\alpha)}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$.

Now we assume $x^{(\alpha)}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Define a generalized Riccati function:

$$\omega(t) = \varsigma(t)a(t)\left[\frac{([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{x^{\gamma}(t)e_{-\frac{\tilde{p}}{a}}(t,t_0)} + \eta(t)\right].$$

Then, for $t \in [t_2, \infty)_{\mathbb{T}}$, by Theorems 1.7–1.8 one has

$$\begin{aligned}
\omega^{(\alpha)}(t) = {} & \frac{\varsigma(t)}{x^{\gamma}(t)}\left\{\frac{a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)}\right\}^{(\alpha)} + \left[\frac{\varsigma(t)}{x^{\gamma}(t)}\right]^{(\alpha)} \frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} \\
& + \varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} + \varsigma^{(\alpha)}(t)a\big(\sigma(t)\big)\eta\big(\sigma(t)\big) \\
= {} & \frac{\varsigma(t)}{x^{\gamma}(t)}\left\{\frac{e_{-\frac{\tilde{p}}{a}}(t,t_0)(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma})^{(\alpha)} - (e_{-\frac{\tilde{p}}{a}}(t,t_0))^{(\alpha)}a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}\right\} \\
& + \left[\frac{x^{\gamma}(t)\varsigma^{(\alpha)}(t) - (x^{\gamma}(t))^{(\alpha)}\varsigma(t)}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\right]\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} \\
& + \varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} + \varsigma^{(\alpha)}(t)a\big(\sigma(t)\big)\eta\big(\sigma(t)\big) \\
= {} & \frac{\varsigma(t)}{x^{\gamma}(t)}\left[\frac{(a(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma})^{(\alpha)} + p(t)([r(t)x^{(\alpha)}(t)]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}\right] + \frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\big(\sigma(t)\big) \\
& - \left[\frac{\varsigma(t)(x^{\gamma}(t))^{(\alpha)}}{x^{\gamma}(t)}\right]\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{x^{\gamma}(\sigma(t))e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} \\
= {} & -\frac{\varsigma(t)}{x^{\gamma}(t)}\left[\frac{q(t)f(x(t))}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}\right] + \frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\big(\sigma(t)\big) \\
& - \left[\frac{\varsigma(t)(x^{\gamma}(t))^{(\alpha)}}{x^{\gamma}(t)}\right]\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{x^{\gamma}(\sigma(t))e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} \\
\leq {} & -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\big(\sigma(t)\big) \\
& - \left[\frac{\varsigma(t)(x^{\gamma}(t))^{(\alpha)}}{x^{\gamma}(t)}\right]\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{x^{\gamma}(\sigma(t))e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)}.
\end{aligned}$$

By [49, Theorems 1.87 and 1.93], we have $(x^{\gamma}(t))^{\Delta} \geq \gamma x^{\gamma-1}(t)x^{\Delta}(t)$. So by Corollary 1.6 it holds that $(x^{\gamma}(t))^{(\alpha)} = t^{1-\alpha}(x^{\gamma}(t))^{\Delta} \geq t^{1-\alpha}\gamma x^{\gamma-1}(t)x^{\Delta}(t) = \gamma x^{\gamma-1}(t)x^{(\alpha)}(t)$, which implies

$$\begin{aligned}
\omega^{(\alpha)}(t) \leq {} & -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\big(\sigma(t)\big) \\
& - \varsigma(t)\left[\frac{\gamma x^{\gamma-1}(t)x^{(\alpha)}(t)}{x^{\gamma}(t)}\right]\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{x^{\gamma}(\sigma(t))e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} \\
\leq {} & -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\big(\sigma(t)\big) \\
& - \left[\frac{\gamma\varsigma(t)}{x(\sigma(t))}\right]\left\{\frac{\vartheta_1(t,t_2)}{r(t)}\left[\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{x^{\gamma}(\sigma(t))e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \varsigma(t)\bigl[a(t)\eta(t)\bigr]^{(\alpha)} \\
\le{}& -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\bigl(\sigma(t)\bigr) \\
&- \gamma\frac{\varsigma(t)\vartheta_1(t,t_2)}{r(t)}\biggl[\frac{\omega(\sigma(t))}{\varsigma(\sigma(t))} - a\bigl(\sigma(t)\bigr)\eta\bigl(\sigma(t)\bigr)\biggr]^{1+\frac{1}{\gamma}} \\
&+ \varsigma(t)\bigl[a(t)\eta(t)\bigr]^{(\alpha)},
\end{aligned}
\tag{2.10}
$$

where Lemma 2.2 has been used in the last two steps.

On the other hand, by use of the following inequality (see [51, Eq. (2.17)]):

$$
(u-v)^{1+\frac{1}{\gamma}} \ge u^{1+\frac{1}{\gamma}} + \frac{1}{\gamma}v^{1+\frac{1}{\gamma}} - \left(1+\frac{1}{\gamma}\right)v^{\frac{1}{\gamma}}u,
$$

where u, v are constants and $\gamma \ge 1$ is a quotient of two odd positive integers, one can obtain that

$$
\begin{aligned}
&\biggl[\frac{\omega(\sigma(t))}{\varsigma(\sigma(t))} - a\bigl(\sigma(t)\bigr)\eta\bigl(\sigma(t)\bigr)\biggr]^{1+\frac{1}{\gamma}} \\
&\quad\ge \frac{\omega^{1+\frac{1}{\gamma}}(\sigma(t))}{\varsigma^{1+\frac{1}{\gamma}}(\sigma(t))} + \frac{1}{\gamma}\bigl[a\bigl(\sigma(t)\bigr)\eta\bigl(\sigma(t)\bigr)\bigr]^{1+\frac{1}{\gamma}} \\
&\qquad - \left(1+\frac{1}{\gamma}\right)\frac{[a(\sigma(t))\eta(\sigma(t))]^{\frac{1}{\gamma}}\omega(\sigma(t))}{\varsigma(\sigma(t))}.
\end{aligned}
\tag{2.11}
$$

So, by a combination of (2.10) and (2.11), one can deduce that

$$
\begin{aligned}
\omega^{(\alpha)}(t) \le{}& -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \varsigma(t)\bigl[a(t)\eta(t)\bigr]^{(\alpha)} \\
&- \frac{\varsigma(t)\vartheta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{1+\frac{1}{\gamma}}}{r(t)} \\
&+ \frac{r(t)\varsigma^{(\alpha)}(t) + (\gamma+1)\varsigma(t)\vartheta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{\frac{1}{\gamma}}}{r(t)\varsigma(\sigma(t))}\omega\bigl(\sigma(t)\bigr) \\
&- \gamma\frac{\varsigma(t)\vartheta_1(t,t_2)}{r(t)}\frac{\omega^{1+\frac{1}{\gamma}}(\sigma(t))}{\varsigma^{1+\frac{1}{\gamma}}(\sigma(t))}.
\end{aligned}
\tag{2.12}
$$

Setting

$$
\begin{aligned}
&\lambda = 1+\frac{1}{\gamma}, \qquad X^{\lambda} = \gamma\frac{\varsigma(t)\vartheta_1(t,t_2)}{r(t)}\frac{\omega^{1+\frac{1}{\gamma}}(\sigma(t))}{\varsigma^{1+\frac{1}{\gamma}}(\sigma(t))}, \\
&Y^{\lambda-1} = \gamma^{\frac{1}{\gamma+1}}\biggl[\frac{r(t)\varsigma^{(\alpha)}(t) + (\gamma+1)\varsigma(t)\vartheta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(t)\varsigma^{\frac{\gamma}{\gamma+1}}(t)\vartheta_1^{\frac{\gamma}{\gamma+1}}(t,t_2)}\biggr],
\end{aligned}
$$

an application of Lemma 2.3 to (2.12) yields that

$$\begin{aligned}\omega^{(\alpha)}(t) \le{}& -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)} + \varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} - \frac{\varsigma(t)\vartheta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{1+\frac{1}{\gamma}}}{r(t)} \\ &+ \left[\frac{r(t)\varsigma^{(\alpha)}(t) + (\gamma+1)\varsigma(t)\vartheta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(t)\varsigma^{\frac{\gamma}{\gamma+1}}(t)\vartheta_1^{\frac{\gamma}{\gamma+1}}(t,t_2)}\right]^{\gamma+1}. \end{aligned}\tag{2.13}$$

Substituting t with s in (2.13), fulfilling α-fractional integral for (2.13) with respect to s from t_2 to t yields

$$\begin{aligned}&\int_{t_2}^{t}\Bigg\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\big[a(s)\eta(s)\big]^{(\alpha)} + \frac{\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \\ &\quad - \left[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,t_2)}\right]^{\gamma+1}\Bigg\}\Delta^\alpha s \\ &\le \omega(t_2) - \omega(t) \le \omega(t_2) < \infty,\end{aligned}$$

which contradicts (2.9). The proof is complete. □

Theorem 2.5 *If* $-\frac{\tilde{p}}{a} \in \Re_+$, (2.1)–(2.3) *hold, and for all sufficiently large* T,

$$\begin{aligned}\lim_{t\to\infty}\sup\Bigg\{\int_T^t\Bigg\{&L\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(s)\big[a(s)\eta(s)\big]^{(\alpha)} \\ &+ \frac{\gamma\varsigma(s)\vartheta_1(s,T)\vartheta_2^{\gamma-1}(\sigma(s),T)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)} \\ &- \frac{[r(s)\varsigma^{(\alpha)}(s) + 2\gamma\varsigma(s)\vartheta_1(s,T)\vartheta_2^{\gamma-1}(\sigma(s),T)a(\sigma(s))\eta(\sigma(s))]^2}{4\gamma r(s)\varsigma(s)\vartheta_1(s,T)\vartheta_2^{\gamma-1}(\sigma(s),T)}\Bigg\}\Delta^\alpha s\Bigg\} = \infty,\end{aligned}\tag{2.14}$$

where ς, η *are defined as in Theorem* 2.4. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero.*

Proof Assume that (1.1) has a nonoscillatory solution x on $\mathbb{T}_0$. Similar to Theorem 2.4, we may assume $x(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. By Lemma 2.1, there exists sufficiently large t_2 such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{(\alpha)}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$.

Now we assume $x^{(\alpha)}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Let $\omega(t)$ be defined as in Theorem 2.4. By Lemma 2.2, one has the following observation:

$$\begin{aligned}\frac{x^{(\alpha)}(t)}{x(t)} &\ge \frac{\vartheta_1(t,t_2)}{r(t)}\left\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}}\right\}\frac{1}{x(\sigma(t))} \\ &= \frac{\vartheta_1(t,t_2)}{r(t)}\left\{\frac{a^{\frac{1}{\gamma}}(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(t,t_0)]^{\frac{1}{\gamma}}x^{\gamma}(\sigma(t))}\right\}x^{\gamma-1}\big(\sigma(t)\big)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\vartheta_1(t,t_2)}{r(t)}\left\{\frac{a^{\frac{1}{\gamma}}(\sigma(t))[r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)]^{\frac{1}{\gamma}}x^{\gamma}(\sigma(t))}\right\}x^{\gamma-1}\big(\sigma(t)\big) \\
&\geq \frac{\vartheta_1(t,t_2)}{r(t)}\left\{\frac{a^{\frac{1}{\gamma}}(\sigma(t))[r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)]^{\frac{1}{\gamma}}x^{\gamma}(\sigma(t))}\right\} \\
&\quad\times \vartheta_2^{\gamma-1}\big(\sigma(t),t_2\big)\left\{\frac{a^{\frac{1}{\gamma}}(\sigma(t))[r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)}}{[e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)]^{\frac{1}{\gamma}}}\right\}^{\gamma-1} \\
&= \frac{\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)}{r(t)}\left\{\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)x^{\gamma}(\sigma(t))}\right\}.
\end{aligned}
\tag{2.15}
$$

Using (2.15) in (2.10) one can deduce that

$$
\begin{aligned}
\omega^{(\alpha)}(t) &\leq -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}+\frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\big(\sigma(t)\big) \\
&\quad-\varsigma(t)\left[\frac{\gamma x^{(\alpha)}(t)}{x(t)}\right]\frac{a(\sigma(t))[(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}]^{\gamma}}{x^{\gamma}(\sigma(t))e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}+\varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} \\
&\leq -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}+\frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\big(\sigma(t)\big) \\
&\quad-\gamma\varsigma(t)\frac{\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)}{r(t)}\left\{\frac{a(\sigma(t))([r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)})^{\gamma}}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)x^{\gamma}(\sigma(t))}\right\}^2 \\
&\quad+\varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} \\
&= -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}+\frac{\varsigma^{(\alpha)}(t)}{\varsigma(\sigma(t))}\omega\big(\sigma(t)\big) \\
&\quad-\frac{\gamma\varsigma(t)\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)}{r(t)}\left[\frac{\omega(\sigma(t))}{\varsigma(\sigma(t))}-a\big(\sigma(t)\big)\eta\big(\sigma(t)\big)\right]^2 \\
&\quad+\varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} \\
&= -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}+\varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} \\
&\quad-\frac{\gamma\varsigma(t)\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)a^2(\sigma(t))\eta^2(\sigma(t))}{r(t)} \\
&\quad+\left[\frac{r(t)\varsigma^{(\alpha)}(t)+2\gamma\varsigma(t)\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)a(\sigma(t))\eta(\sigma(t))}{r(t)\varsigma(\sigma(t))}\right]\omega\big(\sigma(t)\big) \\
&\quad-\frac{\gamma\varsigma(t)\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)}{r(t)\varsigma^2(\sigma(t))}\omega^2\big(\sigma(t)\big) \\
&\leq -L\frac{q(t)\varsigma(t)}{e_{-\frac{\tilde{p}}{a}}(\sigma(t),t_0)}+\varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)} \\
&\quad-\frac{\gamma\varsigma(t)\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)a^2(\sigma(t))\eta^2(\sigma(t))}{r(t)} \\
&\quad+\frac{[r(t)\varsigma^{(\alpha)}(t)+2\gamma\varsigma(t)\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)a(\sigma(t))\eta(\sigma(t))]^2}{4\gamma r(t)\varsigma(t)\vartheta_1(t,t_2)\vartheta_2^{\gamma-1}(\sigma(t),t_2)}.
\end{aligned}
\tag{2.16}
$$

Substituting t with s in (2.16), fulfilling α-fractional integral for (2.16) with respect to s from t_2 to t one can get that

$$\begin{aligned}&\int_{t_2}^{t}\Big\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varsigma(s)\big[a(s)\eta(s)\big]^{(\alpha)}+\frac{\gamma\varsigma(s)\vartheta_1(s,t_2)\vartheta_2^{\gamma-1}(\sigma(s),t_2)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}\\&\quad-\frac{[r(s)\varsigma^{(\alpha)}(s)+2\gamma\varsigma(s)\vartheta_1(s,t_2)\vartheta_2^{\gamma-1}(\sigma(s),t_2)a(\sigma(s))\eta(\sigma(s))]^2}{4\gamma r(s)\varsigma(s)\vartheta_1(s,t_2)\vartheta_2^{\gamma-1}(\sigma(s),t_2)}\Big\}\Delta^\alpha s\\&\quad\le\omega(t_2)-\omega(t)\le\omega(t_2)<\infty,\end{aligned}$$

which contradicts (2.14). The proof is complete. □

Theorem 2.6 *Suppose* $-\frac{\widetilde{p}}{a}\in\Re_+$, (2.1)–(2.3) *hold, and define* $\mathbb{D}=\{(t,s)|t\ge s\ge t_0\}$. *If there exists a function* $H\in C_{\mathrm{rd}}(\mathbb{D},\mathbb{R})$ *such that*

$$H(t,t)=0\quad\text{for } t\ge t_0,\qquad H(t,s)>0\quad\text{for } t>s\ge t_0,\tag{2.17}$$

and H *has a nonpositive continuous* α*-partial fractional derivative* $H_s^{(\alpha)}(t,s)$ *with respect to the second variable, and furthermore, for all sufficiently large* T,

$$\begin{aligned}&\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\Big\{\int_{t_0}^{t}H(t,s)\Big\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\varsigma(t)\big[a(s)\eta(s)\big]^{(\alpha)}\\&\quad+\frac{\varsigma(s)\vartheta_1(s,T)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)}\\&\quad-\Big[\frac{r(s)\varsigma^{(\alpha)}(s)+(\gamma+1)\varsigma(s)\vartheta_1(s,T)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\Big]^{\gamma+1}\Big\}\Delta^\alpha s\Big\}=\infty,\end{aligned}\tag{2.18}$$

where ς, η *are defined as in Theorem* 2.4. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero.*

Proof Assume that (1.1) has a nonoscillatory solution x on $\mathbb{T}_0$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$, where t_1 is sufficiently large. According to Lemma 2.1, there exists sufficiently large t_2 such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0$ on $[t_2,\infty)_{\mathbb{T}}$, and either $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty}x(t)=0$. Now we assume $x^{(\alpha)}(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$. Let $\omega(t)$ be defined as in Theorem 2.4. By (2.13) we have

$$\begin{aligned}&L\frac{q(t)\varsigma(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}-\varsigma(t)\big[a(t)\eta(t)\big]^{(\alpha)}+\frac{\varsigma(t)\vartheta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{1+\frac{1}{\gamma}}}{r(t)}\\&\quad-\Big[\frac{r(t)\varsigma^{(\alpha)}(t)+(\gamma+1)\varsigma(t)\vartheta_1(t,t_2)[a(\sigma(t))\eta(\sigma(t))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(t)\varsigma^{\frac{\gamma}{\gamma+1}}(t)\vartheta_1^{\frac{\gamma}{\gamma+1}}(t,t_2)}\Big]^{\gamma+1}\le-\omega^{(\alpha)}(t).\end{aligned}\tag{2.19}$$

Substituting t with s in (2.19), multiplying both sides by $H(t,s)$ and then fulfilling α-fractional integral with respect to s from t_2 to t, together with the use of Theorem 1.9, yields that

$$\begin{aligned}
&\int_{t_2}^{t} H(t,s)\Biggl\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} + \frac{\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \\
&\quad - \biggl[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,t_2)}\biggr]^{\gamma+1}\Biggr\}\Delta^\alpha s \\
&\le -\int_{t_2}^{t} H(t,s)\omega^{(\alpha)}(s)\Delta^\alpha s = H(t,t_2)\omega(t_2) + \int_{t_2}^{t} H_s^{(\alpha)}(t,s)\omega\bigl(\sigma(s)\bigr)\Delta^\alpha s \\
&\le H(t,t_2)\omega(t_2) \le H(t,t_0)\omega(t_2).
\end{aligned}$$

So

$$\begin{aligned}
&\int_{t_0}^{t} H(t,s)\Biggl\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} + \frac{\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \\
&\quad - \biggl[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,t_2)}\biggr]^{\gamma+1}\Biggr\}\Delta^\alpha s \\
&= \int_{t_0}^{t_2} H(t,s)\Biggl\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} + \frac{\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \\
&\quad - \biggl[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,t_2)}\biggr]^{\gamma+1}\Biggr\}\Delta^\alpha s \\
&\quad + \int_{t_2}^{t} H(t,s)\Biggl\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} + \frac{\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \\
&\quad - \biggl[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,t_2)}\biggr]^{\gamma+1}\Biggr\}\Delta^\alpha s \\
&\le H(t,t_0)\omega(t_2) + H(t,t_0)\int_{t_0}^{t_2}\Biggl|L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} \\
&\quad + \frac{\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \\
&\quad - \biggl[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,t_2)}\biggr]^{\gamma+1}\Biggr|\Delta^\alpha s.
\end{aligned}$$

So

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\Biggl\{\int_{t_0}^{t} H(t,s)\Biggl\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} \\
&\quad + \frac{\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \\
&\quad - \biggl[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,t_2)}\biggr]^{\gamma+1}\Biggr\}\Delta^\alpha s\Biggr\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \omega(t_2) + \int_{t_0}^{t_2} \left| L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(t)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} + \frac{\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{1+\frac{1}{\gamma}}}{r(s)} \right.\\
&\quad \left. - \left[\frac{r(s)\varsigma^{(\alpha)}(s) + (\gamma+1)\varsigma(s)\vartheta_1(s,t_2)[a(\sigma(s))\eta(\sigma(s))]^{\frac{1}{\gamma}}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,t_2)}\right]^{\gamma+1} \right| \Delta^\alpha s < \infty,
\end{aligned}
$$

which contradicts (2.18). So the proof is complete. □

Based on (2.16) and the deduction process in Theorem 2.6, one can easily prove the following theorem.

Theorem 2.7 *Suppose* $-\frac{\widetilde{p}}{a} \in \Re_+$, *and* (2.1)–(2.3) *hold. Let* H *be defined as in Theorem* 2.6, *and for all sufficiently large* T,

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \left\{ \int_{t_0}^{t} H(t,s) \left\{ L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \varsigma(s)\bigl[a(s)\eta(s)\bigr]^{(\alpha)} \right.\right.\\
&\quad + \frac{\gamma\varsigma(s)\vartheta_1(s,T)\vartheta_2^{\gamma-1}(\sigma(s),T)a^2(\sigma(s))\eta^2(\sigma(s))}{r(s)}\\
&\quad \left.\left. - \frac{[r(s)\varsigma^{(\alpha)}(s) + 2\gamma\varsigma(s)\vartheta_1(s,T)\vartheta_2^{\gamma-1}(\sigma(s),T)a(\sigma(s))\eta(\sigma(s))]^2}{4\gamma r(s)\varsigma(s)\vartheta_1(s,T)\vartheta_2^{\gamma-1}(\sigma(s),T)} \right\} \Delta^\alpha s \right\} = \infty,
\end{aligned}
\tag{2.20}
$$

where ς, η *are defined as in Theorem* 2.4. *Then every solution of Eq.* (1.1) *is oscillatory or tends to zero.*

Remark 2.8 If we set $\alpha = 1$, then the established results above reduce to the case of dynamic equations on time scales of integer order derivative, and the latter is an extension of [51, Theorems 2.1, 2.4] except that the latter is related to time delay.

Remark 2.9 In Theorems 2.4–2.7, if we take $\mathbb{T}$ for some special time scales, such as $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = q^{\mathbb{Z}}$, we can obtain corresponding oscillation criteria for fractional differential equations, fractional difference equations, fractional q-difference equations and so on.

3 Applications

In this section, we will present some applications for the established results above.

Example 1 We consider the following fractional order differential equation with damping term:

$$
\begin{aligned}
&\left\{t^{\frac{\gamma}{2}}\left[\left(t^{-\frac{1}{2}}x^{(\frac{1}{2})}(t)\right)^{(\frac{1}{2})}\right]^\gamma\right\}^{(\frac{1}{2})} + \frac{1}{t^{\gamma+1.5}}\left[\left(t^{-\frac{1}{2}}x^{(\frac{1}{2})}(t)\right)^{(\frac{1}{2})}\right]^\gamma + \frac{1}{t^{\gamma+0.5}}x^\gamma(t)\bigl[e^{x(t)}+1\bigr]\\
&\quad = 0, \quad t \in [2,\infty),
\end{aligned}
\tag{3.1}
$$

where $\gamma \geq 1$ is a quotient of two odd positive integers.

Related to (1.1), one has $\mathbb{T} = \mathbb{R}$, $\alpha = \frac{1}{2}$, $a(t) = t^{\frac{\gamma}{2}}$, $p(t) = \frac{1}{t^{\gamma+1.5}}$, $\widetilde{p}(t) = t^{\alpha-1}p(t) = \frac{1}{t^{\gamma+2}}$, $q(t) = \frac{1}{t^{\gamma+0.5}}$, $f(x) = x^\gamma[e^x+1]$, $r(t) = t^{-\frac{1}{2}}$, $t_0 = 2$. So $\frac{f(x)}{x^\gamma} \geq 1 = L$, $\mu(t) = \sigma(t) - t = 0$, which implies

$-\frac{\widetilde{p}}{a} \in \Re_+$. Then $e_{-\frac{\widetilde{p}}{a}}(t,t_0) = e_{-\frac{\widetilde{p}}{a}}(t,2) = \exp(-\int_2^t \frac{\widetilde{p}(s)}{a(s)}\,ds)$. Furthermore,

$$\begin{aligned} 1 &> \exp\left(-\int_2^t \frac{\widetilde{p}(s)}{a(s)}\,ds\right) \geq 1 - \int_2^t \frac{\widetilde{p}(s)}{a(s)}\,ds = 1 - \int_2^t \frac{1}{s^{\frac{3\gamma}{2}+2}}\,ds \\ &\geq 1 - \int_2^t \frac{1}{s^{\frac{3\gamma}{2}+1}}\,ds = 1 + \frac{2}{3\gamma}\left[t^{-\frac{3}{2}\gamma} - 2^{-\frac{3}{2}\gamma}\right] > \frac{1}{3}. \end{aligned}$$

Now we check the conditions (2.1)–(2.3). To this end, one has the following observations:

$$\begin{aligned} \int_{t_0}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta^\alpha s &= \int_{t_0}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1}\Delta s \\ &= \int_{t_0}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1}\,ds > \frac{1}{3^{\frac{1}{\gamma}}}\int_2^\infty \frac{1}{s}\,ds = \infty, \end{aligned}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta^\alpha s = \int_{t_0}^{\infty} \frac{1}{r(s)} s^{\alpha-1}\Delta s = \int_{t_0}^{\infty} \frac{1}{r(s)} s^{\alpha-1}\,ds = \int_{t_0}^{\infty} 1\,ds = \infty.$$

Furthermore,

$$\begin{aligned} &\int_{t_0}^{\infty}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\right)^{\frac{1}{\gamma}}\Delta^\alpha\tau\right]\Delta^\alpha\xi \\ &= \int_{t_0}^{\infty}\xi^{\alpha-1}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}s^{\alpha-1}\Delta s\right)^{\frac{1}{\gamma}}\Delta\tau\right]\Delta\xi \\ &= \int_{t_0}^{\infty}\xi^{\alpha-1}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}s^{\alpha-1}\,ds\right)^{\frac{1}{\gamma}}d\tau\right]d\xi \\ &= \int_{2}^{\infty}\left[\int_{\xi}^{\infty}\tau^{-\frac{1}{2}}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,2)}{\tau^{\frac{\gamma}{2}}}\int_{\tau}^{\infty}\frac{1}{s^{\gamma+1}e_{-\frac{\widetilde{p}}{a}}(s,2)}\,ds\right)^{\frac{1}{\gamma}}d\tau\right]d\xi \\ &> \frac{1}{3^{\frac{1}{\gamma}}}\int_{2}^{\infty}\left[\int_{\xi}^{\infty}\tau^{-\frac{1}{2}}\left(\frac{1}{\tau^{\frac{\gamma}{2}}}\int_{\tau}^{\infty}\frac{1}{s^{\gamma+1}}\,ds\right)^{\frac{1}{\gamma}}d\tau\right]d\xi \\ &= \frac{1}{(3\gamma)^{\frac{1}{\gamma}}}\int_{2}^{\infty}\left[\int_{\xi}^{\infty}\frac{1}{\tau^2}\,d\tau\right]d\xi = \frac{1}{(3\gamma)^{\frac{1}{\gamma}}}\int_{2}^{\infty}\frac{1}{\xi}\,d\xi = \infty. \end{aligned}$$

So (2.1)–(2.3) all hold. On the other hand, for a sufficiently large T, when $t \to \infty$, one has

$$\begin{aligned} \vartheta_1(t,T) &= \int_{T}^{t} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta^\alpha s = \int_{T}^{t} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1}\Delta s \\ &= \int_{T}^{t} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1}\,ds > \frac{1}{3^{\frac{1}{\gamma}}}\int_{T}^{t}\frac{1}{s}\,ds \to \infty. \end{aligned}$$

So there exists a sufficiently large $T^* > T$ such that $\vartheta_1(t,T) > 1$ for $t \in [T^*, \infty)$. Taking $\varsigma(t) = t^\gamma$, $\eta(t) = 0$ in (2.9), one can obtain that

$$\begin{aligned}
&\lim_{t\to\infty} \sup\left\{\int_T^t \left\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \left[\frac{r(s)\varsigma^{(\alpha)}(s)}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}\Delta^\alpha s\right\} \\
&= \lim_{t\to\infty} \sup\left\{\int_T^t \left\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)} - \left[\frac{r(s)\varsigma^{\Delta}(s)s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\Delta s\right\} \\
&= \lim_{t\to\infty} \sup\left\{\int_T^t \left\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)} - \left[\frac{r(s)\varsigma'(s)s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\,ds\right\} \\
&= \lim_{t\to\infty} \sup\left\{\int_T^{T^*} \left\{\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)} - \left[\frac{r(s)\varsigma'(s)s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\,ds\right. \\
&\quad \left. + \int_{T^*}^t \left\{\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)} - \left[\frac{r(s)\varsigma'(s)s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\,ds\right\} \\
&> \lim_{t\to\infty} \sup\left\{\int_T^{T^*} \left\{\frac{q(s)\varsigma(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)} - \left[\frac{r(s)\varsigma'(s)s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\,ds\right. \\
&\quad \left. + \int_{T^*}^t \left[1 - \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\right]\frac{1}{s}\,ds\right\} \to \infty \quad (t\to\infty).
\end{aligned}$$

So (2.9) also holds, and by Theorem 2.4 one can deduce that every solution of Eq. (3.1) is oscillatory or tends to zero.

Example 2 Consider the following fractional order difference equation:

$$\begin{aligned}
&\Delta^{(\frac{1}{2})}\left\{t^{\frac{\gamma}{2}}\left[\Delta^{(\frac{1}{2})}\left(t^{-\frac{1}{2}}\Delta^{(\frac{1}{2})}x(t)\right)\right]^\gamma\right\} + \frac{1}{t^{\gamma+1.5}}\left[\Delta^{(\frac{1}{2})}\left(t^{-\frac{1}{2}}\Delta^{(\frac{1}{2})}x(t)\right)\right]^\gamma + \frac{M}{t^{\gamma+0.5}}x^\gamma(t) \\
&= 0, \quad t \in [2,\infty)_{\mathbb{Z}},
\end{aligned} \tag{3.2}$$

where $\Delta^{(\frac{1}{2})}$ denotes the fractional difference operator of order $\frac{1}{2}$, $M > 0$ is a constant, and $\gamma \geq 1$ is a quotient of two odd positive integers.

Related to (1.1), one has $\mathbb{T} = \mathbb{Z}$, $\alpha = \frac{1}{2}$, $a(t) = t^{\frac{\gamma}{2}}$, $p(t) = \frac{1}{t^{\gamma+1.5}}$, $\widetilde{p}(t) = t^{\alpha-1}p(t) = \frac{1}{t^{\gamma+2}}$, $q(t) = \frac{1}{t^{\gamma+0.5}}$, $f(x) = Mx^\gamma$, $r(t) = t^{-\frac{1}{2}}$, $t_0 = 2$. So $\frac{f(x)}{x^\gamma} \geq M = L$, $\mu(t) = \sigma(t) - t = 1$, and

$$1 - \mu(t)\frac{\widetilde{p}(t)}{a(t)} = 1 - \frac{1}{t^{\frac{3\gamma}{2}+1}} \geq 1 - \frac{1}{2} > 0,$$

which means $-\frac{\widetilde{p}}{a} \in \mathfrak{R}_+$. So by [52, Lemma 2] one can obtain

$$\begin{aligned}
e_{-\frac{\widetilde{p}}{a}}(t,t_0) = e_{-\frac{\widetilde{p}}{a}}(t,2) &\geq 1 - \int_2^t \frac{\widetilde{p}(s)}{a(s)}\Delta s = 1 - \int_2^t \frac{1}{s^{\frac{3\gamma}{2}+2}}\Delta s \geq 1 - \int_2^t \frac{1}{s^{\frac{3\gamma}{2}+1}}\Delta s \\
&= 1 - \sum_{s=2}^{t-1}\frac{1}{s^{\frac{3\gamma}{2}+1}} \geq 1 - \int_1^{t-1}\frac{1}{s^{\frac{3\gamma}{2}+1}}\,ds = 1 + \frac{2}{3\gamma}\left[(t-1)^{-\frac{3\gamma}{2}} - 1\right] > \frac{1}{3},
\end{aligned}$$

and

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0) \leq \exp\left(-\int_2^t \frac{\widetilde{p}(s)}{a(s)}\Delta s\right) < 1.$$

Then we have

$$\begin{aligned}
\int_{t_0}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta^\alpha s &= \int_{t_0}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1}\Delta s = \sum_{s=t_0}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1} \\
&= \sum_{s=2}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,2)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1} = \sum_{s=2}^{\infty} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,2)]^{\frac{1}{\gamma}}}{s} > \frac{1}{3^{\frac{1}{\gamma}}}\sum_{s=2}^{\infty}\frac{1}{s} = \infty,
\end{aligned}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta^\alpha s = \int_{t_0}^{\infty} \frac{1}{r(s)} s^{\alpha-1}\Delta s = \sum_{s=t_0}^{\infty} \frac{1}{r(s)} s^{\alpha-1} = \sum_{s=2}^{\infty} 1 = \infty.$$

Furthermore,

$$\begin{aligned}
&\int_{t_0}^{\infty}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\right)^{\frac{1}{\gamma}}\Delta^\alpha\tau\right]\Delta^\alpha\xi \\
&= \int_{t_0}^{\infty}\xi^{\alpha-1}\left[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}s^{\alpha-1}\Delta s\right)^{\frac{1}{\gamma}}\Delta\tau\right]\Delta\xi \\
&= \sum_{\xi=t_0}^{\infty}\left[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)}\right)^{\frac{1}{\gamma}}\right] \\
&= \sum_{\xi=2}^{\infty}\left[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}\left(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,2)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,2)}\right)^{\frac{1}{\gamma}}\right] \\
&> \frac{1}{3^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}\left[\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}\left(\frac{1}{\tau^{\frac{\gamma}{2}}}\sum_{s=\tau}^{\infty}\frac{1}{s^{\gamma+1}}\right)^{\frac{1}{\gamma}}\right] \\
&\geq \frac{1}{3^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}\left[\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}\left(\frac{1}{\tau^{\frac{\gamma}{2}}}\int_{\tau}^{\infty}\frac{1}{s^{\gamma+1}}\,ds\right)^{\frac{1}{\gamma}}\right] = \frac{1}{(3\gamma)^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}\left[\sum_{\tau=\xi}^{\infty}\frac{1}{\tau^2}\right] \\
&> \frac{1}{(3\gamma)^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\frac{1}{\tau(\tau+1)} = \frac{1}{(3\gamma)^{\frac{1}{\gamma}}}\sum_{\xi=2}^{\infty}\frac{1}{\xi} = \infty.
\end{aligned}$$

On the other hand, for sufficiently large $T > 1$, when $t \to \infty$, one has

$$\begin{aligned}
\vartheta_1(t,T) &= \int_T^t \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\Delta^\alpha s = \int_T^t \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1}\Delta s \\
&= \sum_{s=T}^{t-1} \frac{[e_{-\frac{\widetilde{p}}{a}}(s,t_0)]^{\frac{1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)} s^{\alpha-1} > \frac{1}{3^{\frac{1}{\gamma}}}\sum_{s=T}^{t-1}\frac{1}{s} \to \infty.
\end{aligned}$$

So there exists $T^* > T$ such that $\vartheta_1(s,T) > 1$ for $t \in [T^*,\infty)_{\mathbb{Z}}$. Setting $\varsigma(t) = t^\gamma$, $\eta(t) = 0$ in (2.9), by use of the inequality $(t+1)^\gamma - t^\gamma \le \gamma(t+1)^{\gamma-1} < \gamma 2^{\gamma-1}t^{\gamma-1}$, $t \ge T^*$, we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\sup\left\{\int_T^t\left\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)} - \left[\frac{r(s)\varsigma^{(\alpha)}(s)}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}\Delta^\alpha s\right\}\\
&= \lim_{t\to\infty}\sup\left\{\int_T^t\left\{L\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(s,t_0)} - \left[\frac{r(s)\varsigma^{\Delta}(s)s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\Delta s\right\}\\
&= \lim_{t\to\infty}\sup\left\{\sum_{s=T}^{t-1}\left\{M\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(s+1,t_0)} - \left[\frac{r(s)(\varsigma(s+1)-\varsigma(s))s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\right\}\\
&= \lim_{t\to\infty}\sup\left\{\sum_{s=T}^{T^*}\left\{M\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(s+1,t_0)} - \left[\frac{r(s)(\varsigma(s+1)-\varsigma(s))s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\right.\\
&\qquad\left.+\sum_{s=T^*}^{t-1}\left\{M\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(s+1,t_0)} - \left[\frac{r(s)(\varsigma(s+1)-\varsigma(s))s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\right\}\\
&> \lim_{t\to\infty}\sup\left\{\sum_{s=T}^{T^*}\left\{M\frac{q(s)\varsigma(s)}{e_{-\frac{\tilde{p}}{a}}(s+1,t_0)} - \left[\frac{r(s)(\varsigma(s+1)-\varsigma(s))s^{1-\alpha}}{(\gamma+1)r^{\frac{1}{\gamma+1}}(s)\varsigma^{\frac{\gamma}{\gamma+1}}(s)\vartheta_1^{\frac{\gamma}{\gamma+1}}(s,T)}\right]^{\gamma+1}\right\}s^{\alpha-1}\right.\\
&\qquad\left.+\sum_{s=T^*}^{t-1}\left[M-\left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}2^{\gamma^2-1}\right]\frac{1}{s}\right\}\to\infty\quad(t\to\infty),
\end{aligned}$$

provided that $M > (\frac{\gamma}{\gamma+1})^{\gamma+1}2^{\gamma^2-1}$. So (2.1)–(2.3) and (2.9) all hold, and by Theorem 2.4 one can obtain that every solution of Eq. (3.2) is oscillatory or tends to zero under the condition $M > (\frac{\gamma}{\gamma+1})^{\gamma+1}2^{\gamma^2-1}$.

4 Conclusions

Based on the properties of conformable fractional calculus, we have established some new oscillatory and asymptotic criteria for a class of fractional order dynamic equation on time scales, which extend the oscillation results for corresponding dynamic equations on time scales involving integer order derivative. For illustrating the validity of the present results, some examples have been proposed. We note that this approach can be applied to research oscillatory and asymptotic properties of other types of fractional order dynamic equation on time scales.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QF carried out the main part of this article. All authors read and approved the final manuscript.

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