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Existence and asymptotic of traveling wave fronts for the delayed Volterra-type cooperative system with spatial diffusion

Yanling Meng[1], Weiguo Zhang[2*] and Zhixian Yu[2]

*Correspondence:
zwgzss@163.com;
zwgzwm@126.com
[2]College of Science, University of
Shanghai for Science and
Technology, Shanghai, China
Full list of author information is
available at the end of the article

Abstract

In the paper, we are concerned with the existence and the exponential asymptotic behavior of traveling waves for the delayed Volterra-type cooperative system with nonquasimonotone condition

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = D_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t)[1 - a_1 u_1(x,t) - b_1 u_1(x, t-\tau_1) + c_1 u_2(x, t-\tau_2)], \\ \frac{\partial u_2(x,t)}{\partial t} = D_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t)[1 - a_2 u_2(x,t) - b_2 u_2(x, t-\tau_3) + c_2 u_1(x, t-\tau_4)], \end{cases}$$

modeling the variation of the populations. Here, the major contribution to population model is the introduction of population self-regulation depending on not only the populations at time t, but also on the earlier population time $t - \tau_i$ ($i = 1, 3$). By constructing a pair of suitable upper and lower solutions we obtain the existence of traveling wave fronts connecting the trivial equilibrium and the positive equilibrium, which indicates that there is a transition zone moving the steady state with no species to the steady state with the coexistence of two species. Furthermore, with the help of Ikehara's theorem, the exponential asymptotic behavior of traveling wave front is exactly derived for this system without quasi-monotone conditions. The results are not only an extension of existing results for the known logistic or cooperative system, but also can extend another type of delayed logistic equation with spatial diffusion

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + r u(x,t)\big[1 - a u(x,t) - b u(x, t-\tau)\big].$$

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1 Introduction

The first model of interacting populations to be considered is a classical one, usually associated with the names of Lotka and Volterra. One of the simplest models of the dynamics of two interacting mutualistic species was in the form of an autonomous system

of ordinary differential equations with quadratic nonlinearities,

$$\begin{cases} \frac{du_1(t)}{dt} = r_1 u_1(t)[1 - b_1 u_1(t) + c_1 u_2(t)], \\ \frac{du_2(t)}{dt} = r_2 u_2(t)[1 - b_2 u_2(t) + c_2 u_1(t)], \end{cases} \tag{1.1}$$

where b_i, c_i, r_i $(i = 1, 2)$ are positive constants, $u_i, i = 1, 2$, are the numbers of individuals of species i, $K_i = \frac{1}{b_i}, i = 1, 2$, are the carrying capacities of species i, $r_i, i = 1, 2$, are the intrinsic rates of increase of species i, $c_1(c_2)$ measures the rate at which an individual of $u_2(u_1)$ benefits the growth rate of population $u_1(u_2)$. We refer to [1, 20, 26, 30, 42] for the details.

Time delays in the growth dynamics of a population or of several interacting species can arise from a great variety of causes and are undoubtedly always present to some extent; see [3, 18, 37, 42]. Certainly, the most famous example is the familiar Volterra-type logistic equation with infinite time delay

$$\frac{du(t)}{dt} = u(t)\left[b - au(t) - d\int_{-\infty}^{t} u(s)k(t - s)\, ds\right], \tag{1.2}$$

which means that the growth rates at time t are effected by population sizes at (possibly all) previous times $s \le t$ distributed (or weighted) in the past by the delay kernel $k(s)$, where a, b, d are positive constants. Cushing [3] investigated the asymptotic behavior (as $t \to +\infty$) of solutions or, more specifically, of topics such as the stability or instability of equilibria and the oscillatory nature of solutions. Seifert [29] obtained asymptotic behavior results by using different methods for a particular case of (1.2)

$$\frac{du(t)}{dt} = u(t)\big[b - au(t) - du(t - 1)\big], \tag{1.3}$$

which arises in models for the variation of the population of species where the death rate depends on not only the population at time t, but also on the population one unit of earlier time. Gopalsamy [7] also studied the asymptotic behavior (as $t \to +\infty$) of solutions for a logistic equation with two delays

$$\frac{du(t)}{dt} = u(t)\big[b - au(t - \tau_1) - du(t - \tau_2)\big], \tag{1.4}$$

where τ_1, τ_2 are positive constants.

To specify an interacting species model, the spatial dispersal interactions often cannot be neglected (see [1, 26]). The theory of traveling wave solutions for the system with spatial dispersal is one of the fast developing areas of modern mathematics. Traveling wave solutions are solutions of special type. They can be usually characterized as the solutions invariant with respect to the translation in space. The most frequently used model for studying the spread of a single population in a homogenous environment is the Fisher equation

$$\frac{\partial u(x, t)}{\partial t} = D\frac{\partial^2 u(x, t)}{\partial x^2} + ru(x, t)\big[1 - au(x, t)\big], \tag{1.5}$$

introduced by Fisher [5], where a, r, D are positive constants. Fisher found that (1.5) has a traveling wave solution $u(x, t) = \phi(x + ct)$ with speed $c \ge 2\sqrt{Dr}$. We refer to [17, 26, 34]

for more detail. In the case where there are more than one species, we can incorporate the interaction between the species to obtain the so-called Lotka–Volterra-type diffusive model. The interaction can be either competitive or cooperative. Some related problems on traveling wave solutions for the Lotka–Volterra competitive system with spatial diffusion

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = D_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t)[1 - a_1 u_1(x,t) - b_1 u_2(x,t)], \\ \frac{\partial u_2(x,t)}{\partial t} = D_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t)[1 - a_2 u_2(x,t) - b_2 u_1(x,t)], \end{cases} \tag{1.6}$$

were investigated in [8–10, 14–16, 31, 33] and the references therein, where a_i, b_i, r_i, D_i ($i = 1, 2$) are positive constants. For the Lotka–Volterra competitive system with delays and spatial diffusion, the existence of traveling wave solutions was also admitted in [4, 21, 22, 25, 40]. Huang and Zou [12] naturally considered the system with two mutualistic species

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = D_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t)[1 - b_1 u_1(x,t-\tau_1) + c_1 u_2(x,t-\tau_2)], \\ \frac{\partial u_2(x,t)}{\partial t} = D_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t)[1 - b_2 u_2(x,t-\tau_3) + c_2 u_1(x,t-\tau_4)], \end{cases} \tag{1.7}$$

where a_i, b_i, r_i, D_i ($i = 1, 2$) and τ_j ($j = 1, 2, 3, 4$) are positive constants. By using the technique developed by Wu and Zou [38], they obtained that, for every $c > 2\max\{\sqrt{D_1 r_1}, \sqrt{D_2 r_2}\}$, (1.7) has a monotone traveling wave solution (i.e., a traveling wave front), which connects $(0, 0)$ to

$$\left(\frac{b_2 + c_1}{b_1 b_2 - c_1 c_2}, \frac{b_1 + c_2}{b_1 b_2 - c_1 c_2} \right),$$

provided that τ_1 and τ_3 are sufficiently small and $b_1 b_2 > c_1 c_2$.

Motivated by the works mentioned, we consider the revised delayed Volterra-type cooperative system with spatial diffusion

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = D_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t)[1 - a_1 u_1(x,t) - b_1 u_1(x,t-\tau_1) + c_1 u_2(x,t-\tau_2)], \\ \frac{\partial u_2(x,t)}{\partial t} = D_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t)[1 - a_2 u_2(x,t) - b_2 u_2(x,t-\tau_3) + c_2 u_1(x,t-\tau_4)], \end{cases}$$

$$\tag{1.8}$$

where $r_i > 0, a_i > 0, b_i > 0, c_i > 0, i = 1, 2, \tau_j > 0, j = 1, 2, 3, 4$. Here, the major contribution to population modeling is the introduction of population self-regulation depending not only on the population at time t, but also on the population at earlier time $t - \tau_i$ ($i = 1, 3$). Thus, it is a very interesting and important mathematical problem to investigate its dynamics on traveling wave solutions. In this paper, we mainly study the existence and exponential asymptotic behavior of traveling wave fronts for (1.8).

To obtain the existence of traveling wave fronts connecting the trivial equilibrium $(0, 0)$ to the positive equilibrium

$$\left(\frac{a_2 + b_2 + c_1}{(a_1 + b_1)(a_2 + b_2) - c_1 c_2}, \frac{a_1 + b_1 + c_2}{(a_1 + b_1)(a_2 + b_2) - c_1 c_2} \right),$$

we still adopt the theory in [13, 38] by constructing a pair of suitable upper and lower solutions. We notice that the result is an extension of the existing results for the known logistic

or cooperative system. For example, letting $a_1 = a_2 = 0$, (1.8) reduces to system (1.7) investigated by Huang and Zou [12]. Especially, taking $c_1 = c_2 = 0$, (1.8) can be decoupled into another type of two delayed logistic scaler equations with spatial diffusion

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + ru(x,t)\big[1 - au(x,t) - bu(x,t-\tau)\big], \tag{1.9}$$

where $a \geq 0$, $b \geq 0$, $D \geq 0$, $r \geq 0$, $\tau > 0$. It is obvious that (1.9) reduces to (1.5) by choosing the parameter $b = 0$ or $\tau = 0$. Taking the parameter $a = 0$, (1.9) reduces to the delayed logistic equation

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + ru(x,t)\big[1 - bu(x,t-\tau)\big], \tag{1.10}$$

which was widely studied in [6, 19, 28, 38].

To our knowledge, the exponential asymptotic behavior at negative infinity of traveling wave fronts for (1.7) cannot be obtained by construction of the upper and lower solutions; see [12]. Notice that the exponential asymptotic behavior of traveling wave fronts for kinds of equations with quasimonotone conditions is widely investigated (see [2, 11, 35, 39, 41]). For a nonquasimonotone equation, Pan [28] obtained the exponential asymptotic behavior at negative infinity of traveling wave fronts. However, for nonquasimonotone systems, the exponential asymptotic behavior of traveling wave fronts is still open. In this paper, we also answer this problem for system (1.9) with the help of Ikehara's theorem.

The rest of our paper is organized as follow. Section 2 is devoted to recalling a known abstract result. In Sect. 3, we establish the existence of traveling wave fronts for (1.8) by applying a known result. In Sect. 4, we investigate the exponential asymptotic behavior at negative infinity of any traveling wave fronts for (1.7)–(1.9).

2 An abstract theory

In the remainder of this paper, we use the usual notation for the standard ordering in $\mathbb{R}^n$, that is, for $u = (u_1, \ldots, u_n)^T$ and $v = (v_1, \ldots, v_n)^T$, we denote $u \leq v$ if $u_i \leq v_i$, $i = 1, 2, \ldots, n$, and $u < v$ if $u \leq v$ but $u \neq v$. In particular, if $u \leq v$, then we denote $(u, v) = \{w \in \mathbb{R}^n : u < w \leq v\}$, $[u, v) = \{w \in \mathbb{R}^n : u \leq w < v\}$, and $[u, v] = \{w \in \mathbb{R}^n : u \leq w \leq v\}$. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ denote the supremum norm in $C([-\tau, 0], \mathbb{R}^n)$.

We first consider the following reaction–diffusion system with time delay:

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + f\big(u_t(x)\big), \tag{2.1}$$

where $t \geq 0$, $x \in \mathbb{R}$, $u \in \mathbb{R}^n$, $D = \text{diag}(D_1, D_2, \ldots, D_n)$ with $D_i > 0$ $(i = 1, \ldots, n)$, $f : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is continuous, and $u_t(x)$ is parameterized in $C([-\tau, 0]; \mathbb{R}^n)$ by $x \in \mathbb{R}$ and given by

$$u_t(x)(\theta) = u(x, t+\theta), \quad \theta \in [-\tau, 0].$$

A traveling wave solution of (2.1) is a special translation invariant solution of the form $u(x,t) = \phi(x+ct)$. Here $\phi \in C^2(\mathbb{R}, \mathbb{R}^n)$ is the wave profile that propagates through the one-dimensional spatial domain at constant velocity $c > 0$. Substituting $\phi(x+ct)$ into (2.1) and

letting $\xi = x + ct$, we have the corresponding wave profile equation

$$D\phi''(\xi) - c\phi'(\xi) + f_c(\phi_\xi) = \mathbf{0} \quad \text{for } \xi \in \mathbb{R}, \tag{2.2}$$

where $f_c : X_c = C([-c\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ is defined by

$$f_c(\phi) = f(\phi^c), \qquad \phi^c(s) = \phi(cs), \quad s \in [-\tau, 0].$$

If (2.2) has a monotone solution ϕ defined on $\mathbb{R}$ for $c > 0$ and the limits

$$\lim_{\xi \to -\infty} \phi(\xi) = \mathbf{0}, \qquad \lim_{\xi \to +\infty} \phi(\xi) = \mathbf{K} \tag{2.3}$$

exist, where $0 := (0, \ldots, 0)$ and $K := (k_1, \ldots, k_n)$ are two equilibria, then $u(x, t) = \phi(x + ct)$ is called a traveling wave front of (2.1) with wave speed c. Therefore, (1.8) has a traveling wave front if and only if (2.2) has a solution on $\mathbb{R}$ satisfying the asymptotic boundary condition (2.3).

To tackle the existence of traveling wave fronts of (1.8), suppose that the reaction term f satisfies the following exponential quasimonotonicity condition:

(QM*) There exists a matrix $\beta = \text{diag}(\beta_1, \ldots, \beta_n)$ with $\beta_i \geq 0$, $i = 1, \ldots, n$, such that

$$f(\phi) - f(\psi) + \beta[\phi(0) - \psi(0)] \geq \mathbf{0}$$

for $\phi := (\phi_1, \ldots, \phi_n)$, $\psi := (\psi_1, \ldots, \psi_n) \in X = C([-\tau, 0]; \mathbb{R}^n)$ with (i) $\mathbf{0} \leq \psi(s) \leq \phi(s) \leq \mathbf{K}$ for $s \in [-\tau, 0]$, (ii) $e^{\beta s}[\phi(s) - \psi(s)]$ is nondecreasing in $s \in [-\tau, 0]$, that is, $e^{\beta_i s}(\phi_i(s) - \psi_i(s))$ is nondecreasing in $s \in [-\tau, 0]$, $i = 1, 2, \ldots, n$.

To state a known result, we introduce the following hypotheses:

(H1) $f(\hat{\mathbf{0}}) = f(\hat{\mathbf{K}}) = \mathbf{0}$ with $\mathbf{0} < \mathbf{K}$, where, for any $u \in \mathbb{R}^n$, $\hat{u}$ denotes the constant value function on $[-\tau, 0]$ taking the value u.

(H2) There is a constant $L > 0$ such that

$$\left| f(\phi) - f(\psi) \right| \leq L \| \phi - \psi \|$$

for $\phi, \psi \in C([-\tau, 0], \mathbb{R}^n)$ with $\mathbf{0} \leq \phi(s), \psi(s) \leq \mathbf{K}$, $s \in [-\tau, 0]$.

Now we define upper and lower solutions for (2.2).

Definition 2.1 A continuous function $\phi : \mathbb{R} \to \mathbb{R}^n$ is called an upper solution of (2.2) if ϕ' and ϕ'' exist almost everywhere (*a.e.*) in $\mathbb{R}$, they are essentially bounded on $\mathbb{R}$, and the following inequality holds:

$$D\phi''(\xi) - c\phi'(\xi) + f_c(\phi_\xi) \leq \mathbf{0} \quad \text{a.e. in } \mathbb{R}. \tag{2.4}$$

A lower solution of (2.2) is defined in a similar way by reversing the inequality in (2.4).

In what follows, we assume that (2.2) has an upper solution $\overline{\Phi}$ and a lower solution $\underline{\Phi}$ satisfying the following hypotheses:

(P1) $\mathbf{0} \leq \underline{\Phi}(\xi) \leq \overline{\Phi}(\xi) \leq \mathbf{K}$ and $\sup_{t \leq \xi} \underline{\Phi}(t) \leq \overline{\Phi}(\xi)$ for $\xi \in \mathbb{R}$;

(P2) $f(\hat{u}) \neq \mathbf{0}$ for $\hat{u} \in (\mathbf{0}, \inf_{\xi \in \mathbb{R}} \overline{\Phi}(\xi)] \cup [\sup_{\xi \in \mathbb{R}} \underline{\Phi}(\xi), \mathbf{K})$;

(P3) $e^{\beta \xi}[\overline{\Phi}(\xi) - \underline{\Phi}(\xi)]$ are nondecreasing in $\xi \in \mathbb{R}$.

Define

$$C_{[\mathbf{0},\mathbf{K}]}(\mathbb{R}, \mathbb{R}^n) = \{\phi : \phi \in C(\mathbb{R}, \mathbb{R}^n) | \mathbf{0} \leq \phi \leq \mathbf{K}\}$$

and the profile set

$$\Gamma^*(\underline{\Phi}, \overline{\Phi}) = \left\{ \Phi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^n) : \begin{array}{ll} \text{(i)} & \underline{\Phi}(\xi) \leq \Phi(\xi) \leq \overline{\Phi}(\xi) \text{ and is nondecreasing} \\ & \text{in } \xi \in \mathbb{R}. \\ \text{(ii)} & e^{\beta \xi}[\overline{\Phi}(\xi) - \Phi(\xi)], e^{\beta \xi}[\Phi(\xi) - \underline{\Phi}(\xi)] \\ & \text{are nondecreasing in } \xi \in \mathbb{R}. \\ \text{(iii)} & e^{\beta \xi}[\Phi(\xi + s) - \Phi(\xi)] \text{ is nondecreasing in} \\ & \xi \in \mathbb{R} \text{ for every } s > 0. \end{array} \right\}$$

Now we recall the following result in [13].

Theorem 2.1 *Assume that* (H1)–(H2) *and* (QM*) *hold. If* (2.2) *has an upper solution* $\overline{\Phi}(\xi)$ *in* $\Gamma^*(\underline{\Phi}, \overline{\Phi})$ *and a lower solution* $\underline{\Phi}(\xi)$ *(which is not necessarily in* $\Gamma^*(\underline{\Phi}, \overline{\Phi})$*) satisfying hypotheses* (P1)–(P3), *then* (2.2)–(2.3) *have a solution* $\Phi \in \Gamma^*(\underline{\Phi}, \overline{\Phi})$, *that is,* (2.1) *has a traveling wave front* $\Phi(\xi)$ *satisfying* $\Phi(-\infty) = \mathbf{0}$ *and* $\Phi(+\infty) = \mathbf{K}$.

Remark 2.1 According to the proof in [23, 27], we can obtain that the traveling wave front $\Phi(\xi)$ satisfies $\underline{\Phi}(\xi) \leq \Phi(\xi) \leq \overline{\Phi}(\xi)$ for all $\xi \in \mathbb{R}$.

3 Existence of traveling wave fronts

Assume that $(a_1 + b_1)(a_2 + b_2) - c_1 c_2 > 0$, (1.8) has four equilibria: $(0,0), (0, \frac{1}{a_2+b_2}), (\frac{1}{a_1+b_1}, 0)$, and

$$(k_1, k_2) := \left(\frac{a_2 + b_2 + c_1}{(a_1 + b_1)(a_2 + b_2) - c_1 c_2}, \frac{a_1 + b_1 + c_2}{(a_1 + b_1)(a_2 + b_2) - c_1 c_2} \right).$$

We are mainly interested in the existence of traveling wave fronts connecting the trivial equilibrium $\mathbf{0} := (0, 0)$ to the positive equilibrium $\mathbf{K} := (k_1, k_2)$, which indicates that there is a transition zone moving the steady state with no species to the steady state with the coexistence of two species. Substituting $u_1(x, t) = \phi_1(x + ct)$ and $u_2(x, t) = \phi_2(x + ct)$ into (1.8) and denoting the moving variable $x + ct$ by ξ, we can reduce (1.8) to the following wave profile equations:

$$\begin{cases} D_1 \phi_1''(\xi) - c\phi_1'(\xi) + r_1 \phi_1(\xi)[1 - a_1 \phi_1(\xi) - b_1 \phi_1(\xi - c\tau_1) + c_1 \phi_2(\xi - c\tau_2)] = 0, \\ D_2 \phi_2''(\xi) - c\phi_2'(\xi) + r_2 \phi_2(\xi)[1 - a_2 \phi_2(\xi) - b_2 \phi_2(\xi - c\tau_3) + c_2 \phi_1(\xi - c\tau_4)] = 0. \end{cases} \quad (3.1)$$

Thus, we will tackle the existence of solutions to (3.1) with the asymptotic boundary conditions

$$\begin{cases} \lim_{\xi \to -\infty} \phi_1(\xi) = 0, & \lim_{\xi \to +\infty} \phi_1(\xi) = k_1, \\ \lim_{\xi \to -\infty} \phi_2(\xi) = 0, & \lim_{\xi \to +\infty} \phi_2(\xi) = k_2. \end{cases} \tag{3.2}$$

Define $f_c(\phi) = (f_{c1}(\phi), f_{c2}(\phi))^T : X_\tau = C([-c\tau, 0]; \mathbb{R}^2) \to \mathbb{R}^2$ by

$$f_{c1}(\phi) = r_1 \phi_1(0)\big[1 - a_1 \phi_1(0) - b_1 \phi_1(-c\tau_1) + c_1 \phi_2(-c\tau_2)\big],$$
$$f_{c2}(\phi) = r_2 \phi_2(0)\big[1 - a_2 \phi_2(0) - b_2 \phi_2(-c\tau_3) + c_2 \phi_1(-c\tau_4)\big],$$

where $\tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$.

Now, we prove that $f_c(\phi)$ satisfies (QM*).

Lemma 3.1 *For any $c > c^* = 2\max\{\sqrt{D_1 r_1}, \sqrt{D_2 r_2}\}$, $f_c(\phi)$ satisfies (QM*) if τ_1 and τ_3 are sufficiently small.*

Proof For any $\phi = (\phi_1, \phi_2)$, $\psi = (\psi_1, \psi_2) \in X_\tau = C([-c\tau, 0]; \mathbb{R}^2)$ with (i) $(0,0) \le \psi(s) \le \phi(s) \le (k_1, k_2)$ for $s \in [-c\tau, 0]$ and (ii) $e^{\beta_i s}[\phi_i(s) - \psi_i(s)]$ nondecreasing in $s \in [-c\tau, 0]$, we have

$$\begin{aligned}
& f_{c1}(\phi) - f_{c1}(\psi) \\
&\quad = r_1 \phi_1(0)\big[1 - a_1 \phi_1(0) - b_1 \phi_1(-c\tau_1) + c_1 \phi_2(-c\tau_2)\big] \\
&\qquad - r_1 \psi_1(0)\big[1 - a_1 \psi_1(0) - b_1 \psi_1(-c\tau_1) + c_1 \psi_2(-c\tau_2)\big] \\
&\quad = r_1\big[\phi_1(0) - \psi_1(0)\big] - r_1 a_1\big[\phi_1^2(0) - \psi_1^2(0)\big] \\
&\qquad - r_1 b_1\big[\phi_1(0)\phi_1(-c\tau_1) - \psi_1(0)\psi_1(-c\tau_1)\big] \\
&\qquad + r_1 c_1[\phi_1(0)\phi_2(-c\tau_2) - \psi_1(0)\psi_2(-c\tau_2) \\
&\quad = r_1\big[\phi_1(0) - \psi_1(0)\big] - r_1 a_1\big[\phi_1(0) - \psi_1(0)\big]\big[\phi_1(0) + \psi_1(0)\big] \\
&\qquad - r_1 b_1\big[\phi_1(0)\phi_1(-c\tau_1) - \phi_1(0)\psi_1(-c\tau_1) + \phi_1(0)\psi_1(-c\tau_1) - \psi_1(0)\psi_1(-c\tau_1)\big] \\
&\qquad + r_1 c_1\big[\phi_1(0)\phi_2(-c\tau_2) - \psi_1(0)\phi_2(-c\tau_2) + \psi_1(0)\phi_2(-c\tau_2) - \psi_1(0)\psi_2(-c\tau_2)\big] \\
&\quad = r_1\big[1 - a_1\big(\phi_1(0) + \psi_1(0)\big) - r_1 b_1 \psi_1(-c\tau_1) + r_1 c_1 \phi_2(-c\tau_2)\big]\big[\phi_1(0) - \psi_1(0)\big] \\
&\qquad - r_1 b_1 \phi_1(0)\big[\phi_1(-c\tau_1) - \psi_1(-c\tau_1)\big] + r_1 c_1 \psi_1(0)\big[\phi_2(-c\tau_2) - \psi_2(-c\tau_2)\big] \\
&\quad \ge r_1\big[1 + c_1 \phi_2(-c\tau_2) - a_1\big(\phi_1(0) + \psi_1(0)\big) - b_1 \psi_1(-c\tau_1) \\
&\qquad - b_1 \psi_1(0)e^{c\beta_1\tau_1}\big]\big[\phi_1(0) - \psi_1(0)\big] \\
&\quad \ge r_1\big[1 - a_1\big(\phi_1(0) + \psi_1(0)\big) - b_1 k_1 - b_1 k_1 e^{\beta_1 c\tau_1}\big]\big[\phi_1(0) - \psi_1(0)\big].
\end{aligned}$$

Hence we get

$$\begin{aligned}
& f_{c1}(\phi) - f_{c1}(\psi) + \beta_1\big[\phi_1(0) - \psi_1(0)\big] \\
&\quad \ge \big[\beta_1 + r_1 - a_1 r_1\big(\phi_1(0) + \psi_1(0)\big) - r_1 b_1 k_1 - r_1 b_1 k_1 e^{c\beta_1\tau_1}\big]\big[\phi_1(0) - \psi_1(0)\big] \\
&\quad \ge \big[\beta_1 + r_1 - 2r_1 a_1 k_1 - r_1 b_1 k_1 - r_1 b_1 k_1 e^{\beta_1 c\tau_1}\big]\big[\phi_1(0) - \psi_1(0)\big].
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
& f_{c2}(\phi) - f_{c2}(\psi) + \beta_2\big[\phi_2(0) - \psi_2(0)\big] \\
& \qquad \geq \big[\beta_2 + r_2 - 2r_2 a_2 k_2 - r_2 b_2 k_2 - r_2 b_2 k_2 e^{\beta_2 c \tau_3}\big]\big[\phi_2(0) - \psi_2(0)\big].
\end{aligned}
$$

Choose $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$
\begin{cases}
\beta_1 > 2r_1 b_1 k_1 + 2r_1 a_1 k_1 - r_1, \\
\beta_2 > 2r_2 b_2 k_2 + 2r_2 a_2 k_2 - r_2.
\end{cases}
\tag{3.3}
$$

Thus we have

$$
\begin{cases}
\beta_1 + r_1 - r_1 b_1 k_1 - 2r_1 a_1 k_1 - r_1 b_1 k_1 e^{\beta_1 c \tau_1} \geq 0, \\
\beta_2 + r_2 - r_2 b_2 k_2 - 2r_2 a_2 k_2 - r_2 b_2 k_2 e^{\beta_2 c \tau_3} \geq 0
\end{cases}
\tag{3.4}
$$

if τ_1 and τ_3 are sufficiently small. Therefore $f_c(\phi) = (f_{c1}(\phi), f_{c2}(\phi))$ satisfies (QM*). This completes the proof. $\qquad\square$

Next, we construct a pair of suitable upper and lower solutions. For $c > c^* = 2\max\{\sqrt{D_1 r_1}, \sqrt{D_2 r_2}\}$, define

$$
\Delta_1(\lambda, c) = D_1 \lambda^2 - c\lambda + r_1, \qquad \Delta_2(\lambda, c) = D_2 \lambda^2 - c\lambda + r_2.
$$

Thus we have

$$
\Delta_1(\lambda, c)
\begin{cases}
> 0 & \text{for } \lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), +\infty), \\
< 0 & \text{for } \lambda \in (\lambda_1(c), \lambda_2(c)),
\end{cases}
$$

and

$$
\Delta_2(\lambda, c)
\begin{cases}
> 0 & \text{for } \lambda \in (0, \lambda_3(c)) \cup (\lambda_4(c), +\infty), \\
< 0 & \text{for } \lambda \in (\lambda_3(c), \lambda_4(c)),
\end{cases}
$$

where

$$
\lambda_1(c) := \frac{c - \sqrt{c^2 - 4D_1 r_1}}{2D_1}, \qquad \lambda_2(c) := \frac{c + \sqrt{c^2 - 4D_1 r_1}}{2D_1},
$$

$$
\lambda_3(c) := \frac{c - \sqrt{c^2 - 4D_2 r_2}}{2D_2}, \qquad \lambda_4(c) := \frac{c + \sqrt{c^2 - 4D_2 r_2}}{2D_2}.
$$

Let

$$
\eta \in \left(1, \min\left\{\frac{\lambda_2(c)}{\lambda_1(c)}, \frac{\lambda_4(c)}{\lambda_3(c)}, \frac{\lambda_1(c) + \lambda_3(c)}{\lambda_1(c)}, \frac{\lambda_1(c) + \lambda_3(c)}{\lambda_3(c)}\right\}\right)
$$

and $q > \max\{q_1, q_2, q_3\}$, where

$$
q_1 = \max\left\{\left(\frac{-4r_1}{\Delta_1(\eta \lambda_1(c), c)}\right)^{\frac{\eta \lambda_1(c)}{\lambda_3}}, \left(\frac{-4r_2}{\Delta_2(\eta \lambda_3(c), c)}\right)^{\frac{\eta \lambda_3(c)}{\lambda_1}}, 1\right\},
$$

$$q_2 = \max\left\{ \frac{4r_1}{-\Delta_1(\eta\lambda_1(c),c)}, \frac{4r_2}{-\Delta_2(\eta\lambda_3(c),c)}, 1 \right\},$$

and

$$q_3 = \max\left\{ \frac{2r_1(a_1 + b_1)k_1}{-\Delta_1(\eta\lambda_1(c),c)}, \frac{2r_2(a_2 + b_2)k_2}{-\Delta_2(\eta\lambda_3(c),c)}, 1 \right\}.$$

For these constants, we define the following continuous functions:

$$\overline{\phi}_1(\xi) = \begin{cases} k_1, & \xi \geq \xi_1, \\ k_1(e^{\lambda_1(c)\xi} + qe^{\eta\lambda_1(c)\xi}), & \xi < \xi_1, \end{cases}$$

$$\overline{\phi}_2(\xi) = \begin{cases} k_2, & \xi \geq \xi_2, \\ k_2(e^{\lambda_3(c)\xi} + qe^{\eta\lambda_3(c)\xi}), & \xi < \xi_2, \end{cases}$$

and

$$\underline{\phi}_1(\xi) = \begin{cases} 0, & \xi \geq \xi_3, \\ k_1(e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi}), & \xi < \xi_3, \end{cases}$$

$$\underline{\phi}_2(\xi) = \begin{cases} 0, & \xi \geq \xi_4, \\ k_2(e^{\lambda_3(c)\xi} - qe^{\eta\lambda_3(c)\xi}), & \xi < \xi_4. \end{cases}$$

If q is large enough, then we easily see that $-\xi_i > 0$ ($i = 1, 2, 3, 4$) are large enough.

By choosing

$$\beta_1 > \frac{2\lambda_2 e^{\lambda_2\xi_1}}{1 - 2e^{\lambda_2\xi_1}} \quad \text{and} \quad \beta_2 > \frac{2\lambda_4 e^{\lambda_4\xi_2}}{1 - 2e^{\lambda_4\xi_2}} \tag{3.5}$$

we easily check that $\overline{\Phi}(\xi) \in \Gamma^*(\underline{\Phi}, \overline{\Phi})$ and $\overline{\Phi}(\xi), \underline{\Phi}(\xi)$ satisfy (P1)–(P3). In view of (3.3) and (3.5), we choose

$$\beta_1 > \max\left\{ 2r_1 b_1 k_1 + 2r_1 a_1 k_1 - r_1, \frac{2\lambda_2 e^{\lambda_2\xi_1}}{1 - 2e^{\lambda_2\xi_1}} \right\}$$

and

$$\beta_2 > \max\left\{ 2r_2 b_2 k_2 + 2r_2 a_2 k_2 - r_2, \frac{2\lambda_4 e^{\lambda_4\xi_2}}{1 - 2e^{\lambda_4\xi_2}} \right\}.$$

Next, it suffices to prove that $\overline{\Phi}(\xi)$ and $\underline{\Phi}(\xi)$ are a pair of upper and lower solutions of (3.1), respectively.

Lemma 3.2 *If τ_1 and τ_3 are sufficiently small, then $\overline{\Phi}(\xi) = (\overline{\phi}_1(\xi), \overline{\phi}_2(\xi))$ is an upper solution of* (3.1).

Proof For $\overline{\phi}_1(\xi)$, it suffices to prove that

$$D_1 \overline{\phi}_1''(\xi) - c\overline{\phi}_1'(\xi) + r_1 \overline{\phi}_1(\xi)\left[1 - a_1\overline{\phi}_1(\xi) - b_1\overline{\phi}_1(\xi - c\tau_1) + c_1\overline{\phi}_2(\xi - c\tau_2)\right] \leq 0 \tag{3.6}$$

for $\xi \in \mathbb{R} \setminus \xi_1$.

If $\xi > \xi_1 + c\tau_1$, then $\overline{\phi}_1(\xi) = \overline{\phi}_1(\xi - c\tau_1) = k_1$, $\overline{\phi}_2(\xi - c\tau_2) \le k_2$. Thus we get

$$D_1\overline{\phi}_1''(\xi) - c\overline{\phi}_1'(\xi) + r_1\overline{\phi}_1(\xi)\big[1 - a_1\overline{\phi}_1(\xi) - b_1\overline{\phi}_1(\xi - c\tau_1) + c_1\overline{\phi}_2(\xi - c\tau_2)\big]$$
$$\le r_1 k_1\big[1 - (a_1 + b_1)k_1 + c_1\overline{\phi}_2(\xi - c\tau_2)\big]$$
$$\le r_1 k_1\big[1 - (a_1 + b_1)k_1 + c_1 k_2\big] = 0.$$

If $\xi_1 < \xi < \xi_1 + c\tau_1$, then $\overline{\phi}_1(\xi) = k_1$, $\overline{\phi}_1(\xi - c\tau_1) \le k_1(e^{\lambda_1(c)(\xi - c\tau_1)} + qe^{\eta\lambda_1(c)(\xi - c\tau_1)})$, $\overline{\phi}_2(\xi - c\tau_2) \le k_2$. Thus

$$D_1\overline{\phi}_1''(\xi) - c\overline{\phi}_1'(\xi) + r_1\overline{\phi}_1(\xi)\big[1 - a_1\overline{\phi}_1(\xi) - b_1\overline{\phi}_1(\xi - c\tau_1) + c_1\overline{\phi}_2(\xi - c\tau_2)\big]$$
$$\le r_1 k_1\big[1 - a_1 k_1 - b_1 k_1\big(e^{\lambda_1(c)(\xi - c\tau_1)} + qe^{\eta\lambda_1(c)(\xi - c\tau_1)}\big) + c_1 k_2\big] \le 0$$

for $\xi = \xi_1 + c\tau_1$. Since $\overline{\phi}_1''(\xi)$, $\overline{\phi}_1'(\xi)$, $\overline{\phi}_1(\xi)$, and $\overline{\phi}_2(\xi)$ are uniformly bounded and uniformly continuous for $\xi \in \mathbb{R} \setminus \{\xi_1, \xi_2\}$, (3.6) still holds for $\xi_1 < \xi < \xi_1 + c\tau_1$ and small enough τ_1.

If $\xi < \xi_1$, then $\overline{\phi}_1(\xi) = k_1(e^{\lambda_1(c)\xi} + qe^{\eta\lambda_1(c)\xi})$, $\overline{\phi}_1(\xi - c\tau_1) = k_1(e^{\lambda_1(c)(\xi - c\tau_1)} + qe^{\eta\lambda_1(c)(\xi - c\tau_1)})$, $\overline{\phi}_2(\xi - c\tau_2) \le k_2(e^{\lambda_3(c)(\xi - c\tau_2)} + qe^{\eta\lambda_3(c)(\xi - c\tau_2)})$, which yields

$$D_1\overline{\phi}_1''(\xi) - c\overline{\phi}_1'(\xi) + r_1\overline{\phi}_1(\xi)\big[1 - a_1\overline{\phi}_1(\xi) - b_1\overline{\phi}_1(\xi - c\tau_1) + c_1\overline{\phi}_2(\xi - c\tau_2)\big]$$
$$\le D_1 k_1\big[\lambda_1^2 e^{\lambda_1\xi} + q(\eta\lambda_1)^2 e^{\eta\lambda_1\xi}\big] - ck_1\big[\lambda_1 e^{\lambda_1\xi} + q\eta\lambda_1 e^{\eta\lambda_1\xi}\big]$$
$$\quad + r_1 k_1\big(e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}\big)\big\{1 - a_1 k_1\big(e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}\big)$$
$$\quad - b_1 k_1\big(e^{\lambda_1(\xi - c\tau_1)} + qe^{\eta\lambda_1(\xi - c\tau_1)}\big) + c_1 k_2\big(e^{\lambda_3(\xi - c\tau_2)} + qe^{\eta\lambda_3(\xi - c\tau_2)}\big)\big\}$$
$$= k_1 e^{\lambda_1\xi}\big(D_1\lambda_1^2 - c\lambda_1 + r_1\big) + qk_1 e^{\eta\lambda_1\xi}\big(D_1(\eta\lambda_1)^2 - c\eta\lambda_1 + r_1\big)$$
$$\quad + r_1 k_1\big(e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}\big)\big\{-a_1 k_1\big(e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}\big) - b_1 k_1\big(e^{\lambda_1(\xi - c\tau_1)} + qe^{\eta\lambda_1(\xi - c\tau_1)}\big)$$
$$\quad + c_1 k_2\big(e^{\lambda_3(\xi - c\tau_2)} + qe^{\eta\lambda_3(\xi - c\tau_2)}\big)\big\}$$
$$= \Delta_1(\eta\lambda_1, c)qk_1 e^{\eta\lambda_1\xi} + r_1 k_1\big(e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}\big)\big\{-a_1 k_1\big(e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}\big)$$
$$\quad - b_1 k_1\big(e^{\lambda_1(\xi - c\tau_1)} + qe^{\eta\lambda_1(\xi - c\tau_1)}\big) + c_1 k_2\big(e^{\lambda_3(\xi - c\tau_2)} + qe^{\eta\lambda_3(\xi - c\tau_2)}\big)\big\}.$$

We divide this inequality into two cases.

Case 1. Assume that $\lambda_1 > \lambda_3$ and τ_2 is large enough. We easily see that

$$-a_1 k_1\big(e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}\big) - b_1 k_1\big(e^{\lambda_1(\xi - c\tau_1)} + qe^{\eta\lambda_1(\xi - c\tau_1)}\big)$$
$$\quad + c_1 k_2\big(e^{\lambda_3(\xi - c\tau_2)} + qe^{\eta\lambda_3(\xi - c\tau_2)}\big) \le 2e^{\lambda_3\xi}$$

for all $\xi < \xi_1$. According to the definition of q, we have

$$-\Delta_1(\eta\lambda_1, c)qe^{\eta\lambda_1\xi} \ge 4r_1 e^{\lambda_3\xi}e^{\lambda_1\xi}, \qquad -\Delta_1(\eta\lambda_1, c) \ge 4r_1 e^{\lambda_3\xi}, \tag{3.7}$$

which implies that

$$\Delta_1(\eta\lambda_1, c)qk_1 e^{\eta\lambda_1\xi} + 2r_1 k_1 e^{\lambda_3\xi}\big(e^{\lambda_1\xi + qe^{\eta\lambda_1\xi}}\big) \le 0 \quad \text{for } \xi < \xi_1. \tag{3.8}$$

Then (3.6) holds.

Case 2. Assume that $\lambda_1 \leq \lambda_3$ and τ_2 is large enough. Then, for $\xi < \xi_1$, we have

$$a_1 k_1 \left(e^{\lambda_1 \xi} + q e^{\eta \lambda_1 \xi} \right) + b_1 k_1 \left(e^{\lambda_1 (\xi - c\tau_1)} + q e^{\eta \lambda_1 (\xi - c\tau_1)} \right) \geq c_1 k_2 \left(e^{\lambda_3 (\xi - c\tau_2)} + q e^{\eta \lambda_3 (\xi - c\tau_2)} \right).$$

Thus, (3.6) still holds.

Similarly, when τ_3 is sufficiently small and τ_4 is large enough, it follows that

$$D_2 \overline{\phi}_2''(\xi) - c \overline{\phi}_2'(\xi) + r_2 \overline{\phi}_2(\xi) \left[1 - a_2 \overline{\phi}_2(\xi) - b_2 \overline{\phi}_2(\xi - c\tau_3) + c_2 \overline{\phi}_1(\xi - c\tau_4) \right] \leq 0$$

for $\xi \in \mathbb{R}$. Therefore $\overline{\Phi}(\xi) = (\overline{\phi}_1(\xi), \overline{\phi}_2(\xi))$ is an upper solution of (3.1). This completes the proof. $\qquad\square$

Lemma 3.3 *If τ_1 and τ_3 are sufficiently small, then $\underline{\Phi}(\xi) = (\underline{\phi}_1(\xi), \underline{\phi}_2(\xi))$ is a lower solution of* (3.1).

Proof For $\underline{\phi}_1(\xi)$, we need to prove the following inequality:

$$D_1 \underline{\phi}_1''(\xi) - c \underline{\phi}_1'(\xi) + r_1 \underline{\phi}_1(\xi) \left[1 - a_1 \underline{\phi}_1(\xi) - b_1 \underline{\phi}_1(\xi - c\tau_1) + c_1 \underline{\phi}_2(\xi - c\tau_2) \right] \geq 0. \qquad (3.9)$$

If $\xi > \xi_3$, then we easily see that $\underline{\phi}_1(\xi) = 0$ and (3.9) holds.

If $\xi < \xi_3$, then we know that $\underline{\phi}_1(\xi) = k_1 (e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi})$, $\underline{\phi}_1(\xi - c\tau_1) = k_1 (e^{\lambda_1 (\xi - c\tau_1)} - q e^{\eta \lambda_1 (\xi - c\tau_1)})$ and $\underline{\phi}_2(\xi - c\tau_2) \geq 0$. Thus

$$
\begin{aligned}
&D_1 \underline{\phi}_1''(\xi) - c \underline{\phi}_1'(\xi) + r_1 \underline{\phi}_1(\xi) \left[1 - a_1 \underline{\phi}_1(\xi) - b_1 \underline{\phi}_1(\xi - c\tau_1) + c_1 \underline{\phi}_2(\xi - c\tau_2) \right] \\
&\quad \geq D_1 k_1 \left(\lambda_1^2 e^{\lambda_1 \xi} - q(\eta \lambda_1)^2 e^{\eta \lambda_1 \xi} \right) - c k_1 \left(\lambda_1 e^{\lambda_1 \xi} - q(\eta \lambda_1) e^{\eta \lambda_1 \xi} \right) + r_1 k_1 \left(e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi} \right) \\
&\qquad + r_1 k_1 \left(e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi} \right) \left\{ -a_1 k_1 \left(e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi} \right) - b_1 k_1 \left(e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi} \right) \right\} \\
&\quad = -\Delta_1(\eta \lambda_1, c) q k_1 e^{\eta \lambda_1 \xi} - a_1 k_1^2 r_1 \left(e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi} \right)^2 \\
&\qquad - r_1 b_1 k_1^2 \left(e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi} \right) \left(e^{\lambda_1 (\xi - c\tau_1)} - q e^{\eta \lambda_1 (\xi - c\tau_1)} \right) \\
&\quad \geq -\Delta_1(\eta \lambda_1, c) q k_1 e^{\eta \lambda_1 \xi} - 2 r_1 (a_1 + b_1) k_1^2 \left(e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi} \right)^2 \\
&\quad > -\Delta_1(\eta \lambda_1, c) q k_1 e^{\eta \lambda_1 \xi} - 2 r_1 (a_1 + b_1) k_1^2 e^{2 \lambda_1 \xi}.
\end{aligned}
$$

By the definition of q_3 we easily see that $-\Delta_1(\eta \lambda_1, c) q k_1 e^{\eta \lambda_1 \xi} - 2 r_1 (a_1 + b_1) k_1^2 e^{2 \lambda_1 \xi} > 0$. Then we obtain

$$D_1 \underline{\phi}_1''(\xi) - c \underline{\phi}_1'(\xi) + r_1 \underline{\phi}_1(\xi) \left[1 - a_1 \underline{\phi}_1(\xi) - b_1 \underline{\phi}_1(\xi - c\tau_1) + c_1 \underline{\phi}_2(\xi - c\tau_2) \right] \geq 0, \quad \xi \in \mathbb{R}.$$

Similarly, we can also prove that

$$D_2 \underline{\phi}_2''(\xi) - c \underline{\phi}_2'(\xi) + r_2 \underline{\phi}_2(\xi) \left[1 - a_2 \underline{\phi}_2(\xi) - b_2 \underline{\phi}_2(\xi - c\tau_3) + c_2 \underline{\phi}_1(\xi - c\tau_4) \right] \geq 0, \quad \xi \in \mathbb{R}.$$

This completes the proof. $\qquad\square$

According to Lemmas 3.1–3.3, Theorem 2.1, and Remark 2.1, we have the following result.

Theorem 3.1 (Existence) *Assume that $(a_1 + b_1)(a_2 + b_2) - c_1c_2 > 0$. If τ_1 and τ_3 are sufficiently small, then for every $c > 2\max\{\sqrt{D_1r_1}, \sqrt{D_2r_2}\}$, equation (1.8) has a nondecreasing traveling wave front $(\phi_1(x + ct), \phi_2(x + ct))$ connecting $(0,0)$ to*

$$(k_1, k_2) \triangleq \left(\frac{a_2 + b_2 + c_1}{(a_1 + b_1)(a_2 + b_2) - c_1c_2}, \frac{a_1 + b_1 + c_2}{(a_1 + b_1)(a_2 + b_2) - c_1c_2} \right)$$

and satisfying $(0,0) < (\phi_1(\xi), \phi_2(\xi)) < (k_1, k_2)$ for all $\xi \in \mathbb{R}$.

From Theorem 3.1 we obtain the following result for (1.9).

Corollary 3.1 *If τ is sufficiently small, then for every $c > 2\sqrt{Dr}$, equation (1.9) has a nondecreasing traveling wave front $\phi(x + ct)$ connecting 0 to $\frac{1}{a+b}$ and satisfying $0 < \phi(\xi) < \frac{1}{a+b}$ for all $\xi \in \mathbb{R}$.*

Remark 3.1 By choosing suitable parameters we can extend our results to Theorem 5.1.5 in [38] and Theorem 3.1 in [12].

Remark 3.2 By adopting the limiting arguments in [24, 32] Corollary 3.1 still holds for $c = 2\sqrt{Dr}$. In further work, we will discuss necessary and sufficient conditions for (1.9) to have monotone traveling waves.

4 Asymptotic behavior of traveling wave fronts

In this section, the a priori asymptotic behavior of any nondecreasing traveling wave front is exactly derived with the help of Ikehara's theorem. We assume that $(a_1 + b_1)(a_2 + b_2) > c_1c_2$ and $(\phi_1(x + ct), \phi_2(x + ct))$ is any traveling wave front of (1.8) connecting $\mathbf{0} := (0,0)$ to $\mathbf{K} := (k_1, k_2)$ with wave speed $c > 2\max\{\sqrt{D_1r_1}, \sqrt{D_2r_2}\}$ and satisfying $(0,0) < (\phi_1(\xi), \phi_2(\xi)) < (k_1, k_2)$ for all $\xi \in \mathbb{R}$.

By the monotonicity of traveling wave fronts the following result is clear.

Lemma 4.1 *We have $\lim_{\xi \to -\infty}(\phi_1'(\xi), \phi_2'(\xi)) = (0,0)$ for all $\xi \in \mathbb{R}$.*

Lemma 4.2 *For all $\xi \in \mathbb{R}$, $\int_{-\infty}^{\xi} \phi_i(s)\,ds < \infty$ and $\int_{-\infty}^{\xi} \int_{-\infty}^{s} \phi_i(t)\,dt\,ds < \infty$, $i = 1, 2$.*

Proof Since $\lim_{\xi \to -\infty}(\phi_1(\xi), \phi_2(\xi)) = (0,0)$, there exists a negative large number $\xi' < 0$ such that, for all $\xi < \xi'$,

$$D_1\phi_1''(\xi) - c\phi_1(\xi) + \frac{r_1}{2}\phi_1(\xi)$$
$$\leq D_1\phi_1''(\xi) - c\phi_1(\xi) + r_1\phi_1(\xi)\left[1 - a_1\phi_1(\xi) - b_1\phi_1(\xi - c\tau_1) + c_1\phi_2(\xi - c\tau_2)\right]$$
$$= 0,$$

which implies that

$$0 \leq \frac{r_1}{2}\phi_1(\xi) \leq -D_1\phi_1''(\xi) + c\phi_1'(\xi) \quad \text{for } \xi < \xi'. \tag{4.1}$$

By Lemma 4.1 it is easy to see that $\lim_{\xi \to -\infty}(\phi_1'(\xi), \phi_2'(\xi)) = (0, 0)$. Integrating (4.1) from $-\infty$ to ξ ($\xi < \xi'$), we have

$$0 \le \frac{r_1}{2} \int_{-\infty}^{\xi} \phi_1(t)\, dt \le -D_1 \phi_1'(\xi) + c\phi_1(\xi) \quad \text{for } \xi < \xi'. \tag{4.2}$$

Thus, $\int_{-\infty}^{\xi} \phi_1(t)\, dt < \infty$ for all $\xi < \xi'$.

Integrating (4.2) from $-\infty$ to ξ ($\xi < \xi'$), we obtain

$$\begin{aligned}
0 &\le \frac{r_1}{2} \int_{-\infty}^{\xi} \int_{-\infty}^{s} \phi_1(t)\, dt\, ds \\
&\le -D_1 \phi_1(\xi) + c \int_{-\infty}^{\xi} \phi_1(t)\, dt \\
&\le c \int_{-\infty}^{\xi} \phi_1(t)\, dt \quad \text{for } \xi < \xi',
\end{aligned} \tag{4.3}$$

which implies that $\int_{-\infty}^{\xi} \int_{-\infty}^{s} \phi_1(t)\, dt\, ds < \infty$ for $\xi < \xi'$.

Thus, for any given ξ, $\int_{-\infty}^{\xi} \phi_1(t)\, dt < \infty$ and $\int_{-\infty}^{\xi} \int_{-\infty}^{s} \phi_1(t)\, dt\, ds < \infty$.

Similarly, we can prove that $\int_{-\infty}^{\xi} \phi_2(t)\, dt < \infty$ and $\int_{-\infty}^{\xi} \int_{-\infty}^{s} \phi_2(t)\, dt\, ds < \infty$ for any given ξ. This completes the proof. □

Given a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, define the two-side Laplace transform as follows:

$$L(\lambda, \phi) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} \phi(\xi)\, d\xi.$$

We have the following lemma.

Lemma 4.3 *Assume that $(a_1 + b_1)(a_2 + b_2) > c_1 c_2$ and $(\phi_1(\xi), \phi_2(\xi))$ is any solution of* (3.1) *and* (3.2) *satisfying $(0, 0) < (\phi_1(\xi), \phi_2(\xi)) < (k_1, k_2)$. Then:*

(i) *there exists $\lambda' > 0$ such that $\sup_{\xi \in \mathbb{R}} \phi_1(\xi)e^{-\lambda'\xi} < \infty$ and $|L(\lambda, \phi_1)| < \infty$ for all $\Re\lambda \in (0, \lambda')$;*

(ii) *there exists $\lambda'' > 0$ such that $\sup_{\xi \in \mathbb{R}} \phi_2(\xi)e^{-\lambda''\xi} < \infty$ and $|L(\lambda, \phi_2)| < \infty$ for all $\Re\lambda \in (0, \lambda'')$.*

Here and in what follows, $\Re\lambda$ denotes the real part of λ.

Proof We only need to prove (i). Similarly, we can prove (ii). To do this, we first show that there exists $\lambda' > 0$ such that $\sup_{\xi \in \mathbb{R}} \phi_1(\xi)e^{-\lambda'\xi} < \infty$. Letting $w(\xi) = \int_{-\infty}^{\xi} \phi_1(t)\, dt$, it follows that $0 < w(\xi) < \infty$ for all $\xi \in \mathbb{R}$, and $w(\xi)$ is nondecreasing on ξ. According to (4.3), for any $l > 0$, we have

$$0 \le r_1 l w(\xi - l) \le r_1 \int_{\xi-l}^{\xi} w(t)\, dt \le r_1 \int_{-\infty}^{\xi} w(t)\, dt \le 2cw(\xi)$$

for $\xi < \xi'$, where ξ' is given in Lemma 4.2. Therefore, there exists a constant $l_0 > 0$ such that $0 < \frac{2c}{l_0 r_1} < 1$ and $w(\xi - l_0) < \frac{2c}{l_0 r_1} w(\xi)$ for $\xi < \xi'$.

Let $m(\xi) = w(\xi)e^{-\lambda'\xi}$, where $\lambda' = \frac{1}{l_0} \ln \frac{l_0 r_1}{2c} > 0$. Then

$$m(\xi - l_0) = w(\xi - l_0)e^{-\lambda'(\xi - l_0)} = w(\xi - l_0)e^{-\lambda'\xi}e^{\lambda' l_0}$$

$$= \frac{l_0 r_1}{2c}w(\xi - l_0)e^{-\lambda'\xi} \leq w(\xi)e^{-\lambda'\xi} = m(\xi) \quad \text{for } \xi < \xi'. \tag{4.4}$$

Note that $m(\xi)$ is bounded for $\xi \in [\xi' - l_0, \xi']$. Then (4.4) implies that $m(\xi)$ is bounded for all $\xi \leq \xi'$. Therefore, it follows from the continuity of $m(\xi)$ that $m(\xi)$ is bounded for all $\xi \leq 0$. Integrating $\phi_1(\xi)$ from 0 to ξ for $\xi > 0$, we have

$$w(\xi) - w(0) = \int_0^\xi \phi_1(s)\, ds \leq k_1 \xi,$$

which implies that $0 \leq w(\xi) \leq w(0) + k_1\xi$. Since $\lim_{\xi \to +\infty} \xi e^{-\lambda'\xi} = 0$, we have $\lim_{\xi \to +\infty} w(\xi)e^{-\lambda'\xi} = 0$. Again, since $m(\xi)$ is bounded for all $\xi \leq 0$, it follows that $0 < \sup_{\xi \in \mathbb{R}} w(\xi)e^{-\lambda'\xi} < \infty$. According to (4.3), we have

$$0 \leq D_1\phi_1(\xi)e^{-\lambda'\xi} \leq cw(\xi)e^{-\lambda'\xi}$$

for all $\xi < \xi'$. On the other hand, it is easy to see that $\lim_{\xi \to +\infty} \phi_1(\xi)e^{-\lambda'\xi} = 0$, since $\phi_1(\xi)$ is bounded. Therefore $0 < \sup_{\xi \in \mathbb{R}} \phi_1(\xi)e^{-\lambda'\xi} < \infty$.

Next, we prove that $|L(\lambda, \phi_1)| < \infty$ for all $\Re\lambda \in (0, \lambda')$. Indeed, letting $M = \sup_{\xi \in \mathbb{R}} \phi_1(\xi) \times e^{-\lambda'\xi}$, we have $0 < \phi_1(\xi) \leq Me^{\lambda'\xi}$ for all $\xi \in \mathbb{R}$. Since

$$0 < \left| \int_0^{+\infty} \phi_1(\xi)e^{-\lambda\xi}\, d\xi \right| < k_1 \int_0^{+\infty} e^{-\Re\lambda\xi}\, d\xi = \frac{k_1}{\Re\lambda}$$

and

$$\left| \int_{-\infty}^0 \phi_1(\xi)e^{-\lambda\xi}\, d\xi \right| \leq \int_{-\infty}^0 Me^{(\lambda' - \Re\lambda)\xi}\, d\xi = \frac{M}{\lambda' - \Re\lambda}$$

for all $\Re\lambda \in (0, \lambda')$, we get

$$\left| L(\lambda, \phi_1) \right| = \left| \int_0^{+\infty} \phi_1(\xi)e^{-\lambda\xi}\, d\xi + \int_{-\infty}^0 \phi_1(\xi)e^{-\lambda\xi}\, d\xi \right| < \infty.$$

Similarly, we can prove that there is a positive number $\lambda'' > 0$ such that $|L(\lambda, \phi_2)| < \infty$ for all $\Re\lambda \in (0, \lambda'')$. This completes the proof. □

To study the asymptotic behavior of the solutions $(\phi_1(\xi), \phi_2(\xi))$ at the minus infinity, we need the following version of Ikehara's theorem [2].

Proposition 4.1 (Ikehara's theorem) *Let $F(\lambda) = \int_0^{+\infty} \phi(\xi)e^{-\lambda\xi}\, d\xi$ with positive decreasing function $\phi(\xi)$. Assume that $F(\lambda)$ has the representation*

$$F(\lambda) = \frac{H(\lambda)}{(\lambda + \alpha)^{k+1}},$$

where $k > -1$, $\alpha > 0$, and $H(\lambda)$ is analytic in the strip $-\alpha \le \Re\lambda < 0$. Then

$$\lim_{\xi \to -\infty} \frac{\phi(\xi)}{|\xi|^k e^{-\alpha\xi}} = \frac{H(-\alpha)}{\Gamma(\alpha + 1)}.$$

Theorem 4.1 (Asymptotic behavior) *Assume that $(a_1 + b_1)(a_2 + b_2) > c_1 c_2$. If for $c > 2\max\{\sqrt{D_1 r_1}, \sqrt{D_2 r_2}\}$, $(\phi_1(\xi), \phi_2(\xi))$ is any traveling wave front of (1.8) connecting $(0, 0)$ to (k_1, k_2) and satisfying $(0, 0) < (\phi_1(\xi), \phi_2(\xi)) < (k_1, k_2)$, then there exist two constants h_1 and h_2 such that*

$$\lim_{\xi \to -\infty} \left(\phi_1(\xi) e^{-\lambda_1(\xi + h_1)}, \phi_2(\xi) e^{-\lambda_3(\xi + h_2)} \right) = (1, 1),$$

where λ_1 and λ_3 are defined in Sect. 3, that is,

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4D_1 r_1}}{2D_1} \quad and \quad \lambda_3 = \frac{c - \sqrt{c^2 - 4D_2 r_2}}{2D_2}.$$

Proof According to the wave profile equations (3.1), it follows that

$$\left(D_1 \lambda^2 - c\lambda + r_1 \right) L_1(\lambda, \phi_1)$$
$$= r_1 \int_{-\infty}^{\infty} e^{-\lambda\xi} \left[a_1 \phi_1^2(\xi) + b_1 \phi_1(\xi)\phi_1(\xi - c\tau_1) - c_1 \phi_1(\xi)\phi_2(\xi - c\tau_2) \right] d\xi$$
$$=: r_1 \int_{-\infty}^{\infty} e^{-\lambda\xi} \mathcal{Q}_1(\phi_1, \phi_2)(\xi) \, d\xi \tag{4.5}$$

and

$$\left(D_2 \lambda^2 - c\lambda + r_2 \right) L_2(\lambda, \phi_2)$$
$$= r_2 \int_{-\infty}^{\infty} e^{-\lambda\xi} \left[a_2 \phi_2^2(\xi) + b_2 \phi_2(\xi)\phi_2(\xi - c\tau_3) - c_2 \phi_2(\xi)\phi_1(\xi - c\tau_4) \right] d\xi$$
$$=: r_2 \int_{-\infty}^{\infty} e^{-\lambda\xi} \mathcal{Q}_2(\phi_1, \phi_2)(\xi) \, d\xi. \tag{4.6}$$

It is easily seen that the left-hand sides of (4.5) and (4.6) are analytic for $\Re\lambda \in (0, \lambda')$ and $\Re\lambda \in (0, \lambda'')$, respectively. Note that

$$\left| \int_{-\infty}^{\infty} e^{-\lambda\xi} \phi_1^2(\xi) \, d\xi \right| \le \int_{-\infty}^{\infty} e^{-\Re\lambda\xi} \phi_1^2(\xi) \, d\xi \le \sup_{\xi \in \mathbb{R}} \phi_1(\xi) e^{-\lambda'\xi} \int_{-\infty}^{\infty} e^{-(\Re\lambda - \lambda')\xi} \phi_1(\xi) \, d\xi < \infty \tag{4.7}$$

for $\Re\lambda \in (0, 2\lambda')$. Similarly, we can obtain

$$\left| \int_{-\infty}^{\infty} e^{-\lambda\xi} \phi_1(\xi)\phi_1(\xi - c\tau_1) \, d\xi \right| < \infty \quad \text{for } \Re\lambda \in \left(0, 2\lambda' \right) \tag{4.8}$$

and

$$\left| \int_{-\infty}^{\infty} e^{-\lambda\xi} \phi_1(\xi)\phi_2(\xi - c\tau_2) \, d\xi \right| < \infty \quad \text{for } \Re\lambda \in \left(0, \lambda' + \lambda'' \right). \tag{4.9}$$

According to (4.7)–(4.9), it follows that

$$\left| \int_{-\infty}^{\infty} e^{-\lambda \xi} \mathcal{Q}_1(\phi_1, \phi_2)(\xi) \, d\xi \right| < \infty \quad \text{for } \Re\lambda \in \left(0, \min\{2\lambda', \lambda' + \lambda''\}\right). \tag{4.10}$$

Similarly, we have

$$\left| \int_{-\infty}^{\infty} e^{-\lambda \xi} \mathcal{Q}_2(\phi_1, \phi_2)(\xi) \, d\xi \right| < \infty \quad \text{for } \Re\lambda \in \left(0, \min\{2\lambda'', \lambda' + \lambda''\}\right). \tag{4.11}$$

We now use a property of the Laplace transform [36]. Since $\phi_1(\xi) > 0$, there exists a real number κ such that $L(\lambda, \phi_1)$ is analytic for $0 < \Re\lambda < \kappa$ and $L(\lambda, \phi_1)$ has a singularity at $\mu = \kappa$. Hence, according to (4.5) and $D_1\lambda_1^2 - c\lambda_1 + r_1 = 0$, we have $\kappa = \lambda_1$, that is, $L(\lambda, \phi_1)$ is analytic for $0 < \Re\lambda < \lambda_1$, and $L(\lambda, \phi_1)$ has a singularity at $\lambda = \lambda_1$. Similarly, we can prove that $L(\lambda, \phi_2)$ is analytic for $0 < \Re\lambda < \lambda_3$ and $L(\lambda, \phi_2)$ has a singularity at $\lambda = \lambda_3$.

To apply Proposition 4.1 (Ikehara's theorem), we need to check that ϕ_1 and ϕ_2 satisfy all conditions of Ikehara's theorem. For $c > 2\max\{\sqrt{D_1 r_1}, \sqrt{D_2 r_2}\}$, we rewrite (4.5) as

$$
\begin{aligned}
&\int_{-\infty}^{0} \phi_1(\xi) e^{-\lambda \xi} \, d\xi \\
&= \frac{r_1 \int_{-\infty}^{\infty} e^{-\lambda \xi} \mathcal{Q}_1(\phi_1, \phi_2)(\xi) \, d\xi}{D_1\lambda^2 - c\lambda + r_1} - \int_{0}^{+\infty} \phi_1(\xi) e^{-\lambda \xi} \, d\xi \quad \text{for } \lambda \in (0, \lambda_1).
\end{aligned}
\tag{4.12}
$$

Note that $\int_{0}^{+\infty} \phi_1(\xi) e^{-\lambda \xi} \, d\xi$ is analytic for $\Re\lambda > 0$ and $D_1\lambda^2 - c\lambda + r_1 = 0$ does not have any positive roots with $\Re\lambda = \lambda_1$ other than $\lambda = \lambda_1$.

Let $u_i(\xi) = \phi_i(-\xi)$, $i = 1, 2$, and

$$\mathcal{Q}_1(\phi_1, \phi_2)(\xi) = a_1 u_1^2(-\xi) + b_1 u_1(-\xi)u_1(-\xi + c\tau_1) - c_1 u_1(-\xi)u_2(-\xi + c\tau_2)$$

$$:= \mathbf{T}(u_1, u_2)(-\xi).$$

It is clear that $u(\xi)$ is decreasing in ξ,

$$
\begin{aligned}
\int_{0}^{+\infty} e^{\lambda \theta} u_1(\theta) \, d\theta &= \frac{\int_{-\infty}^{+\infty} e^{\lambda \xi} \mathbf{T}(u_1, u_2)(\theta) \, d\theta}{D_1\lambda^2 - c\lambda + r_1} - \int_{0}^{+\infty} e^{-\lambda \xi} \phi_1(\xi) \, d\xi \\
&=: \frac{H(\lambda)}{\lambda - \lambda_1} \neq 0,
\end{aligned}
$$

and

$$H(\lambda) = \frac{(\lambda - \lambda_1) \int_{-\infty}^{\infty} e^{\lambda \theta} \mathbf{T}(u_1, u_2)(\theta) \, d\theta}{D_1\lambda^2 - c\lambda + r_1} - (\lambda - \lambda_1) \int_{0}^{+\infty} e^{-\lambda \xi} \phi_1(\xi) \, d\xi.$$

We easily check that $\lim_{\lambda \to \lambda_1} H(\lambda)$ exists. Therefore $H(\lambda)$ is analytic for all $0 < \Re\lambda \le \lambda_1$. According to Proposition 4.1, we obtain

$$\lim_{\xi \to +\infty} \frac{u_1(\xi)}{e^{-\lambda_1 \xi}} \quad \text{exists, i.e.,} \quad \lim_{\xi \to -\infty} \frac{\phi_1(\xi)}{e^{\lambda_1 \xi}} \quad \text{exists.}$$

Similarly, we can also verify that

$$\lim_{\xi \to -\infty} \frac{\phi_2(\xi)}{e^{\lambda_3 \xi}} \quad \text{exists.}$$

Therefore there exist two constants h_1 and h_2 such that

$$\lim_{\xi \to -\infty} \left(\phi_1(\xi) e^{-\lambda_1(\xi + h_1)}, \phi_2(\xi) e^{-\lambda_3(\xi + h_2)} \right) = (1, 1).$$

This completes the proof. □

According to Theorem 4.1, we can obtain the following results for (1.7) and (1.9), respectively. For (1.7), taking $a_1 = a_2 = 0$ in (1.8), we have the following:

Corollary 4.1 (Asymptotic behavior) *Assume that $b_1 b_2 > c_1 c_2$. If for $c > 2 \max\{\sqrt{D_1 r_1}, \sqrt{D_2 r_2}\}$, $(\phi_1(x + ct), \phi_2(x + ct))$ is any traveling wave front of (1.7) connecting $(0, 0)$ to (k_1, k_2) and satisfying $(0, 0) < (\phi_1(\xi), \phi_2(\xi)) < (k_1, k_2)$, then there exist two constants h_1 and h_2 such that*

$$\lim_{\xi \to -\infty} \left(\phi_1(\xi) e^{-\lambda_1(\xi + h_1)}, \phi_2(\xi) e^{-\lambda_3(\xi + h_2)} \right) = (1, 1),$$

where

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4 D_1 r_1}}{2 D_1} \quad \text{and} \quad \lambda_3 = \frac{c - \sqrt{c^2 - 4 D_2 r_2}}{2 D_2}.$$

For (1.9), we have the following:

Corollary 4.2 (Asymptotic behavior) *If for $c > 2\sqrt{Dr}$, $\phi(\xi)$ is any traveling wave front of (1.9) connecting 0 to $\frac{1}{a+b}$ and satisfying $0 < \phi(\xi) < \frac{1}{a+b}$, then there exists a constant h such that*

$$\lim_{\xi \to -\infty} \phi(\xi) e^{-\mu(\xi + h)} = 1,$$

where

$$\mu = \frac{c - \sqrt{c^2 - 4Dr}}{2D}.$$

Remark 4.1 Letting $a = 0$ and $b = 1$ in (1.9), Corollary 4.2 reduces to Theorem 2.6 in [28].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equal contributions. All authors read and approved the final manuscript.

Author details
[1]Business School, University of Shanghai for Science and Technology, Shanghai, P.R. China. [2]College of Science, University of Shanghai for Science and Technology, Shanghai, China.

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