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Existence and attractivity of solutions for fractional difference equations

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Abstract

In this paper, we study a kind of difference equations with Riemann–Liouville-like fractional difference. The results on existence and attractivity are obtained by using the Picard iteration method and Schauder's fixed point theorem. Examples are provided to illustrate the main results.

MSC: 39A06; 39A13

Keywords: Fractional difference equations; Existence; Attractivity

1 Introduction

In this paper, we study the fractional difference equation

$$\begin{cases} \Delta_{\alpha-1}^{\alpha} u(t) = f(t+\alpha, u(t+\alpha)), & t \in \mathbb{N}_0, \alpha \in (0,1], \\ \Delta_{\alpha-1}^{\alpha-1} u(t)|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where Δ^α is the Riemann–Liouville-like fractional difference, which is defined in later sections, $f : \mathbb{N}_\alpha \times \mathbb{R} \to \mathbb{R}$, $f(t,u)$ is continuous with respect to t and u, and $\mathbb{N}_\alpha = \{\alpha, \alpha+1, \alpha+2, \ldots\}$.

In the last decade, fractional differential equations have been recognized as valuable tools to describe many phenomena in various fields of engineering, physics, science, and so on. A huge number of results focused on fractional differential equations; see the monographs of Kilbas et al. [12] and Zhou [17, 18], the papers [1, 10], and the references therein. Within the past ten years, however, there has been more interest in developing discrete fractional equations, that is, fractional difference equations. This development has demonstrated that fractional difference equations have a number of unexpected difficulties and technical complications [2–6, 9, 13–15].

Motivated by the works mentioned, in this paper, we investigate the existence and attractivity of solutions for fractional difference equations. In Sect. 2, we describe the discrete fractional difference calculus and some properties. The main results are obtained in Sect. 3. Using the Picard iteration method, we prove the existence of equation (1.1) in Sect. 3.1. In Sect. 3.2, we also obtain the existence of attractive solutions for fractional difference equations. Section 3.3 is devoted to inducing the existence and attractivity of equation (1.1) by Schauder's fixed point theorem according to the introduced weighted space. Finally, we provide three examples to illustrate our results.

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2 Preliminaries

This section is devoted to some preliminary facts.

Definition 2.1 (see [5, 11]) Let $\nu > 0$. The νth fractional sum is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \bigl(t-\sigma(s)\bigr)^{(\nu-1)} f(s), \tag{2.1}$$

where f is defined for $s = a \mod (1)$, and $\Delta_a^{-\nu} f$ is defined for $t = a + \nu \mod (1)$, $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$, $\sigma(s) = s+1$, and Γ is the gamma function. The fractional sum $\Delta^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$.

Definition 2.2 (see [5, 11]) Let $\mu > 0$ and $m-1 \le \mu \le m$, where m is a positive integer. The μth fractional difference is defined as

$$\Delta_a^{\mu} f(t) = \Delta^m \bigl(\Delta_a^{-(m-\mu)} f(t)\bigr). \tag{2.2}$$

Lemma 2.1 (see [4]) *Assume that $\mu + 1$ is not a nonpositive integer. Then*

$$\Delta_a^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}. \tag{2.3}$$

Lemma 2.2 (see [4]) *Assume that the following factorial functions are well defined. Then*:

(i) $\Delta t^{(\mu)} = \mu t^{(\mu-1)}$;
(ii) $(t-\mu) t^{(\mu)} = t^{(\mu+1)}$;
(iii) $\mu^{(\mu)} = \Gamma(\mu+1)$;
(iv) $t^{(\mu+\nu)} = (t-\nu)^{(\mu)} t^{(\nu)}$.

We indicate the following properties of the gamma function (see [7, 12]):

(i) for all $x \in \mathbb{R}$, excluding $x = 0, -1, -2, \ldots$,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)};$$

(ii) the gamma function is logarithmically convex function on the positive real axis, that is, $\log\Gamma(x)$ is convex.

By the log-convexity property of the gamma function we have that

$$\Gamma\bigl(\lambda x + (1-\lambda) y\bigr) \le \Gamma^{\lambda}(x)\Gamma^{1-\lambda}(y), \quad \lambda \in (0,1), x, y > 0.$$

Lemma 2.3

(i) *If* $0 < \alpha < 1$, *then* $(t^{(\nu)})^{\alpha} \le t^{(\alpha\nu)}$;
(ii) *If* $\alpha > 1$ *and* $\nu > 0$, *then*

$$\bigl(t^{(-\nu)}\bigr)^{\alpha} \le \frac{\Gamma(t-[t]+\alpha\nu)}{\Gamma^{\alpha}(t-[t]+\nu)} t^{(-\alpha\nu)} \quad \textit{for } t \in \mathbb{R}^+.$$

Here $[t]$ *means the integer part of* t.

Proof The proof of (i) is given in [4]. Chen [8] proved (ii) for $t \in \mathbb{N}_1$. We prove (ii) for all $t \in \mathbb{R}^+$:

$$\begin{aligned}
&\Gamma^{\alpha-1}(t+1)\Gamma(t+\alpha\nu+1)\Gamma^{\alpha}\big(t-[t]+\nu\big)\\
&\quad= t^{\alpha-1}(t-1)^{\alpha-1}\cdots\big(t-[t]\big)^{\alpha-1}(t+\alpha\nu)(t-1+\alpha\nu)\cdots\big(t-[t]+\alpha\nu\big)\\
&\qquad\times\Gamma\big(t-[t]+\alpha\nu\big)\Gamma^{\alpha}\big(t-[t]+\nu\big)\\
&\quad=\left[t^{\alpha}\left(1+\frac{\alpha\nu}{t}\right)\right]\left[(t-1)^{\alpha}\left(1+\frac{\alpha\nu}{t-1}\right)\right]\cdots\left[\big(t-[t]\big)^{\alpha}\left(1+\frac{\alpha\nu}{(t-[t])}\right)\right]\\
&\qquad\times\Gamma\big(t-[t]+\alpha\nu\big)\Gamma^{\alpha}\big(t-[t]+\nu\big)\\
&\quad<\left[t^{\alpha}\left(1+\frac{\nu}{t}\right)^{\alpha}\right]\left[(t-1)^{\alpha}\left(1+\frac{\nu}{t-1}\right)^{\alpha}\right]\cdots\left[\big(t-[t]\big)^{\alpha}\left(1+\frac{\nu}{(t-[t])}\right)^{\alpha}\right]\\
&\qquad\times\Gamma\big(t-[t]+\alpha\nu\big)\Gamma^{\alpha}\big(t-[t]+\nu\big)\\
&\quad=(t+\nu)^{\alpha}(t-1+\nu)^{\alpha}\cdots\big(t-[t]+\nu\big)^{\alpha}\Gamma^{\alpha}\big(t-[t]+\nu\big)\Gamma\big(t-[t]+\alpha\nu\big)\\
&\quad=\Gamma^{\alpha}(t+\nu+1)\Gamma\big(t-[t]+\alpha\nu\big),
\end{aligned}$$

where the inequality $1+\frac{\alpha\nu}{t}<(1+\frac{\nu}{t})^{\alpha}$ is used. Then

$$\frac{\Gamma^{\alpha-1}(t+1)}{\Gamma^{\alpha}(t+\nu+1)}\le\frac{\Gamma(t-[t]+\alpha\nu)}{\Gamma^{\alpha}(t-[t]+\nu)}\cdot\frac{1}{\Gamma(t+\alpha\nu+1)}.$$

Hence

$$\big(t^{(-\nu)}\big)^{\alpha}=\frac{\Gamma^{\alpha}(t+1)}{\Gamma^{\alpha}(t+\nu+1)}\le\frac{\Gamma(t-[t]+\alpha\nu)}{\Gamma^{\alpha}(t-[t]+\nu)}\cdot\frac{\Gamma(t+1)}{\Gamma(t+\alpha\nu+1)}=\frac{\Gamma(t-[t]+\alpha\nu)}{\Gamma^{\alpha}(t-[t]+\nu)}t^{(-\alpha\nu)}$$

for $t \in \mathbb{R}^+$. The proof is completed. □

Remark 2.1 In (ii), if $t-[t]+\alpha\nu \in (0,1)$, then we also have that

$$\big(t^{(-\nu)}\big)^{\alpha}\le t^{(-\alpha\nu)}.$$

Lemma 2.4 (see [12, (1.5.15)]) *The quotient expansion of two gamma functions at infinity is*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=z^{a-b}\left(1+O\left(\frac{1}{z}\right)\right)\quad\big(\big|\arg(z+a)\big|<\pi,|z|\to\infty\big). \tag{2.4}$$

By Lemma 2.4 we can easily get that

$$t^{(-\nu)}\to 0\quad\text{as } t\to\infty\ (\nu>0) \tag{2.5}$$

and

$$\frac{t^{(a)}}{t^{(b)}}\to 0\quad\text{as } t\to\infty\ (a<b). \tag{2.6}$$

Lemma 2.5

(i) *If* $-1 < t < r$, *then* $t^{(-\nu)} \geq r^{(-\nu)}$ *for* $\nu > 0$;

(ii) *If* $\nu - 1 < t < r$, *then* $t^{(\nu)} \leq r^{(\nu)}$ *for* $\nu > 0$;

(iii) *If* $\alpha, \beta \geq 0$, *then* $t^{(\beta)} t^{(-\alpha)} \leq t^{(\beta-\alpha)}$ *for* $t > \beta - 1$.

Proof (i) Since

$$\begin{aligned}\frac{t^{(-\nu)}}{r^{(-\nu)}} &= \frac{\Gamma(t+1)\Gamma(r+\nu+1)}{\Gamma(t+\nu+1)\Gamma(r+1)} \\ &= \frac{\Gamma(t+1)\Gamma(r+\nu+1)}{\Gamma(\lambda(r+\nu+1)+(1-\lambda)(t+1))\Gamma((1-\lambda)(r+\nu+1)+\lambda(t+1))},\end{aligned}$$

where $\lambda = \frac{\nu}{r-t+\nu} \in (0,1)$, by the log-convexity property of the gamma function, we get that

$$\begin{aligned}\frac{t^{(-\nu)}}{r^{(-\nu)}} &= \frac{\Gamma(t+1)\Gamma(r+\nu+1)}{\Gamma(\lambda(r+\nu+1)+(1-\lambda)(t+1))\Gamma((1-\lambda)(r+\nu+1)+\lambda(t+1))} \\ &\geq \frac{\Gamma(t+1)\Gamma(r+\nu+1)}{(\Gamma(r+\nu+1))^{\lambda+1-\lambda}(\Gamma(t+1))^{1-\lambda+\lambda}} \\ &= 1.\end{aligned}$$

Then $t^{(-\nu)} \geq r^{(-\nu)}$ for $-1 < t < r$ and $\nu > 0$.

(ii) Since

$$\frac{t^{(\nu)}}{r^{(\nu)}} = \frac{\Gamma(t+1)\Gamma(r-\nu+1)}{\Gamma(t-\nu+1)\Gamma(r+1)},$$

similarly to (i), we can easily get that $t^{(\nu)} \leq r^{(\nu)}$ for $\nu - 1 < t < r$ and $\nu > 0$.

(iii) By (i) we know that

$$t^{(-\alpha)} \leq (t-\beta)^{(-\alpha)} \quad \text{for } t > \beta - 1.$$

According to (iv) of Lemma 2.2, it is easy to get that

$$t^{(\beta)} t^{(-\alpha)} \leq t^{(\beta)} (t-\beta)^{(-\alpha)} = t^{(\beta-\alpha)} \quad \text{for } t > \beta - 1.$$

The proof is completed. □

Remark 2.2 Chen and Liu [9] obtained (i) of Lemma 2.5. However, the proof holds in the case that $t \to \infty$. Then we give the proof as before.

Lemma 2.6 (see [3]) *Let* $a, \nu \in \mathbb{R}$, $a > \nu > -1$, *and* $a \leq b$. *Then*

$$\sum_{s=a}^{b} s^{(\nu)} = \frac{(b+1)^{(\nu+1)} - a^{(\nu+1)}}{\nu+1}. \tag{2.7}$$

In particular, if $a = \nu$, *then*

$$\sum_{s=\nu}^{b} s^{(\nu)} = \frac{(b+1)^{(\nu+1)}}{\nu+1}. \tag{2.8}$$

Remark 2.3 Let $\nu > 0$ be noninteger, and let $m-1 \le \mu \le m$, where m is a positive integer. Set $\nu = m - \mu$. Then we have

$$\sum_{s=a+\nu}^{t-\mu} \bigl(t-\sigma(s)\bigr)^{(\mu-1)} = \frac{(t-a-\nu)^{(\mu)}}{\mu}. \tag{2.9}$$

Lemma 2.7 (see [4]) *Let f be a real-value function defined on $\mathbb{N}_a$, and let $\mu, \nu > 0$. Then we have the following equalities*:

(i) $\Delta_{a+\mu}^{-\nu}(\Delta_a^{-\mu} f(t)) = \Delta_a^{-(\mu+\nu)} f(t) = \Delta_{a+\nu}^{-\mu}(\Delta_a^{-\nu} f(t))$;

(ii) $\Delta_a^{-\nu}\Delta f(t) = \Delta\Delta_a^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a)$.

Definition 2.3 The discrete Mittag-Leffler function is defined by

$$F_{\alpha,\beta}(\lambda,t) = \sum_{n=0}^{\infty} \lambda^n \frac{t^{(n\alpha)}}{\Gamma(n\alpha+\beta)} \quad \bigl(|\lambda| < 1\bigr),$$

where $\alpha, \beta \in \mathbb{R}^+$ and $\lambda \in \mathbb{C}$.

Notice that the series is absolutely convergent for $|\lambda| < 1$. In fact, since

$$\begin{aligned}
\left|\frac{t^{(n\alpha)}}{\Gamma(n\alpha+\beta)}\right| &= \left|\frac{\Gamma(t+1)}{\Gamma(t-n\alpha+1)\Gamma(n\alpha+\beta)}\right| \\
&= \left|\frac{\Gamma(t+1)\Gamma(n\alpha-t)\sin(\pi(t-n\alpha+1))}{\pi\Gamma(n\alpha+\beta)}\right| \\
&\le \frac{1}{\pi}\frac{\Gamma(t+1)\Gamma(n\alpha-t)}{\Gamma(n\alpha+\beta)} \\
&\le \frac{1}{\pi}\frac{\Gamma(t+1)(\Gamma(t+1))^{\eta}(\Gamma(n\alpha+\beta))^{1-\eta}}{\Gamma(n\alpha+\beta)} \\
&= \frac{1}{\pi}\frac{(\Gamma(t+1))^{1+\eta}}{(\Gamma(n\alpha+\beta))^{\eta}} \\
&\le \frac{(\Gamma(t+1))^{1+\eta}}{\pi}
\end{aligned}$$

for $\eta = \frac{t+1}{n\alpha+\beta-t-1}$ and $n > \frac{2t+1}{\alpha}$, we have

$$\sum_{n>\frac{2t+1}{\alpha}}^{\infty} \left|\lambda^n \frac{t^{(\alpha)}}{\Gamma(n\alpha+\beta)}\right| \le \frac{(\Gamma(t+1))^{1+\eta}}{\pi} \sum_{n>\frac{2t+1}{\alpha}}^{\infty} |\lambda|^n.$$

It is easy to see that the series $\sum_{n=0}^{\infty} \lambda^n \frac{t^{(\alpha)}}{\Gamma(n\alpha+\beta)}$ ($|\lambda| < 1$) is absolutely convergent.

Definition 2.4 A solution u of the fractional difference equation (1.1) is said to be attractive if $u(t) \to 0$ as $t \to \infty$.

The space $\ell_{n_0}^{\infty}$ is the set of real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. It is well known that, under the supremum norm, $\ell_{n_0}^{\infty}$ is a Banach space [16].

Definition 2.5 (see [16]) A set Ω of sequences in $\ell_{n_0}^\infty$ is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$, there exists an integer N such that $|x(i) - x(j)| < \varepsilon$ whenever $i, j > N$ for any $x = \{x(n)\}$ in Ω.

Theorem 2.1 (see [16, Discrete Arzelà–Ascoli's theorem]) *A bounded, uniformly Cauchy subset Ω of $\ell_{n_0}^\infty$ is relatively compact.*

Definition 2.6 Let (X, d) be a metric space. An operator $T : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $\operatorname{Fix} T = \{x^*\}$ and the sequence $\{T^n(x_0)\}_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Theorem 2.2 (Schauder's fixed point theorem) *Let Ω be a closed, convex, and nonempty subset of a Banach space X. Let $T : \Omega \to \Omega$ be a continuous operator such that $T\Omega$ is a relatively compact subset of X. Then T has at least one fixed point in Ω.*

According to Definitions 2.1–2.2, it is suitable to rewrite the fractional difference equations (1.1) in the equivalent summation equation

$$u(t) = \frac{u_0}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t - \sigma(s)\bigr)^{(\alpha-1)} f\bigl(s + \alpha, u(s + \alpha)\bigr), \quad t \in \mathbb{N}_\alpha. \tag{2.10}$$

3 Main results

3.1 Results via Picard iteration method

Theorem 3.1 *Assume that*

(H_1) *$f(t, u)$ satisfies the Lipschitz condition*

$$\bigl|f(t, u_1) - f(t, u_2)\bigr| \le A|u_1 - u_2|, \tag{3.1}$$

where $0 < A < 1$ is independent of t;

(H_2) *there exists a constant $M > 0$ such that $|f(t, u(t))| \le M$ for $t \in \mathbb{N}_\alpha$.*

Then the fractional difference equation (1.1) has at least one solution.

Proof Define the sequence $\{g_n(\cdot) : n \in \mathbb{N}_0\}$ as follows:

$$g_0(t) = \frac{u_0}{\Gamma(\alpha)} t^{(\alpha-1)}, \quad t \in \mathbb{N}_\alpha, \tag{3.2}$$

$$g_n(t) = g_0(t) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t - \sigma(s)\bigr)^{(\alpha-1)} f\bigl(s + \alpha, g_{n-1}(s + \alpha)\bigr), \quad t \in \mathbb{N}_\alpha, n \in \mathbb{N}_0. \tag{3.3}$$

Clearly, by induction we have

$$\bigl|g_n(t) - g_{n-1}(t)\bigr| \le MA^{n-1} \frac{t^{(n\alpha)}}{\Gamma(n\alpha + 1)}.$$

In fact, for $n = 1$, by condition (H_1) we can conclude that

$$\begin{aligned}\left|g_1(t) - g_0(t)\right| &\le \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{(\alpha-1)}M \\ &= M\frac{t^{(\alpha)}}{\Gamma(\alpha+1)}.\end{aligned}$$

Without loss of generality, we assume that

$$\left|g_{n-1}(t) - g_{n-2}(t)\right| \le MA^{n-2}\frac{t^{((n-1)\alpha)}}{\Gamma((n-1)\alpha+1)}.$$

Then

$$\begin{aligned}\left|g_n(t) - g_{n-1}(t)\right| &\le A\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{(\alpha-1)}MA^{n-2}\frac{(s+\alpha)^{((n-1)\alpha)}}{\Gamma((n-1)\alpha+1)} \\ &= \frac{MA^{n-1}}{\Gamma((n-1)\alpha+1)}\Delta^{-\alpha}(t+\alpha)^{((n-1)\alpha)} \\ &= MA^{n-1}\frac{t^{(n\alpha)}}{\Gamma(n\alpha+1)}.\end{aligned}$$

Set

$$g(t) = \lim_{n\to\infty} g_n(t) = \lim_{n\to\infty}\bigl(g_n(t) - g_0(t)\bigr) + g_0(t) = \sum_{k=1}^{\infty}\bigl(g_k(t) - g_{k-1}(t)\bigr) + g_0(t).$$

Since the series $\frac{M}{A}\sum_{k=1}^{\infty} A^k \frac{t^{(k\alpha)}}{\Gamma(k\alpha+1)} = \frac{M}{A}F_{\alpha,\beta}(A,t)$ is absolutely convergent for $0 < A < 1$, the existence of the solution for the fractional difference equation (1.1) is proved. The proof is completed. □

3.2 Results via Schauder's fixed point theorem

In this section, we deal with the existence and attractivity of the solution for fractional difference equations by a fixed point theorem. First, for any $u \in \ell_\alpha^\infty$, we define the operator T as follows:

$$Tu(t) = \frac{u_0}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{(\alpha-1)}f\bigl(s+\alpha, u(s+\alpha)\bigr), \quad t \in \mathbb{N}_\alpha. \tag{3.4}$$

Then the existence of the solution of the fractional difference equation (1.1) is equivalent to that T has a fixed point.

Theorem 3.2 *Assume that*

(H_3) *there exist constants $\beta_1 \in (\alpha, 1)$, $\delta > 0$, and $L_1 \ge 0$ such that*

$$\left|f(t,u)\right| \le L_1(t+1)^{(-\beta_1)}|u|^\delta \quad \text{for } t \in \mathbb{N}_\alpha \text{ and } u \in \mathbb{R}. \tag{3.5}$$

Furthermore, suppose that

$$\frac{|u_0|}{\Gamma(\alpha)}\alpha^{(\alpha-1)} + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\alpha^{(\alpha-\beta_1)} \le 1.$$

Then the fractional difference equation (1.1) *has at least one solution in* S_1. *Further, the solutions of* (1.1) *are attractive.*

Proof Choose $\gamma > 0$ sufficiently small such that

$$\alpha + \gamma - 1 < 0, \qquad 1 - \beta_1 - \gamma\delta > 0, \qquad \alpha + \gamma - \beta_1 - \gamma\delta < 0,$$

and

$$\frac{|u_0|}{\Gamma(\alpha)}(\alpha+\gamma)^{(\alpha+\gamma-1)} + \frac{L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)}(\alpha+\gamma)^{(\alpha-\beta_1+\gamma-\gamma\delta)} \le 1.$$

For $t \in \mathbb{N}_\alpha$, define the closed subset $S_1 \subset \ell_\alpha^\infty$ as follows:

$$S_1 = \bigl\{u \in \ell_\alpha^\infty : \bigl|u(t)\bigr| \le t^{(-\gamma)} \text{ for } t \in \mathbb{N}_\alpha\bigr\}.$$

It is easy to see that S_1 is a closed, bounded, and convex subset of the Banach space ℓ_α^∞.

In the case $\delta > 1$, we can get the following inequality by (ii) of Lemma 2.3 and Remark 2.1:

$$\bigl(t^{(-\gamma)}\bigr)^\delta \le \frac{\Gamma(\alpha+\gamma\delta)}{\Gamma^\delta(\alpha+\gamma)}t^{(-\gamma\delta)} \le t^{(-\gamma\delta)} \quad \text{for } t \in \mathbb{N}_\alpha.$$

Combining this with (i) of Lemma 2.3, we have

$$\bigl(t^{(-\gamma)}\bigr)^\delta \le t^{(-\gamma\delta)} \quad \text{for } \delta > 0, t \in \mathbb{N}_\alpha. \tag{3.6}$$

We first show that T is continuous in S_1 and maps S_1 into S_1. Applying (H_3), (3.4), and (3.6) to any $u \in S_1$, we have

$$\begin{aligned}
\bigl|Tu(t)\bigr| &\le \frac{|u_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{(\alpha-1)}\bigl|f\bigl(s+\alpha, u(s+\alpha)\bigr)\bigr| \\
&\le \frac{|u_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{(\alpha-1)}L_1(s+\alpha+1)^{(-\beta_1)}\bigl|u(s+\alpha)\bigr|^\delta \\
&\le \frac{|u_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{(\alpha-1)}L_1(s+\alpha+\gamma\delta)^{(-\beta_1)}\bigl|u(s+\alpha)\bigr|^\delta \\
&\le \frac{|u_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{L_1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{(\alpha-1)}(s+\alpha+\gamma\delta)^{(-\beta_1)}\bigl((s+\alpha)^{(-\gamma)}\bigr)^\delta \\
&\le \frac{|u_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{L_1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\bigl(t-\sigma(s)\bigr)^{(\alpha-1)}(s+\alpha+\gamma\delta)^{(-\beta_1)}(s+\alpha)^{(-\gamma\delta)}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{|u_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} (s+\alpha)^{(-\beta_1-\gamma\delta)} \\
&= \frac{|u_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} (t+\alpha)^{(\alpha-\beta_1-\gamma\delta)} \\
&\le \frac{|u_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} t^{(\alpha-\beta_1-\gamma\delta)} \\
&= \left(\frac{|u_0|}{\Gamma(\alpha)} (t+\gamma)^{(\alpha+\gamma-1)} + \frac{L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} (t+\gamma)^{(\alpha-\beta_1-\gamma\delta+\gamma)} \right) t^{(-\gamma)} \\
&\le \biggl(\frac{|u_0|}{\Gamma(\alpha)} (\alpha+\gamma)^{(\alpha+\gamma-1)} \\
&\quad + \frac{L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} (\alpha+\gamma)^{(\alpha-\beta_1-\gamma\delta+\gamma)} \biggr) t^{(-\gamma)} \\
&\le t^{(-\gamma)},
\end{aligned}
$$

which implies that T maps S_1 into S_1.

By (2.5) it is easy to see that for all $\varepsilon > 0$, there exists $K_1 \in \mathbb{N}_+$ large enough such that

$$
\frac{L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} t^{(\alpha-\beta_1-\gamma\delta)} \le \frac{\varepsilon}{2} \quad \text{and} \quad \frac{u_0}{\Gamma(\alpha)} t^{(\alpha-1)} \le \frac{\varepsilon}{2} \quad \text{for } t \in \mathbb{N}_{\alpha+K_1}. \tag{3.7}
$$

Let $u_n, u \in S_1$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} u_n = u$. Applying (2.9), for $t \in \{\alpha, \alpha+1, \ldots, \alpha+K_1-1\}$, we have

$$
\begin{aligned}
&\bigl|Tu_n(t) - Tu(t)\bigr| \\
&\quad\le \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} \bigl|f\bigl(s+\alpha, u_n(s+\alpha)\bigr) - f\bigl(s+\alpha, u(s+\alpha)\bigr)\bigr| \\
&\quad\le \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} \max_{s\in[0,1,\ldots,K_1-1]} \bigl|f\bigl(s+\alpha, u_n(s+\alpha)\bigr) - f\bigl(s+\alpha, u(s+\alpha)\bigr)\bigr| \\
&\quad= \frac{t^{(\alpha)}}{\Gamma(\alpha+1)} \max_{s\in[0,1,\ldots,K_1-1]} \bigl|f\bigl(s+\alpha, u_n(s+\alpha)\bigr) - f\bigl(s+\alpha, u(s+\alpha)\bigr)\bigr| \\
&\quad\le \frac{(\alpha+K_1-1)^{(\alpha)}}{\Gamma(\alpha+1)} \max_{s\in[0,1,\ldots,K_1-1]} \bigl|f\bigl(s+\alpha, u_n(s+\alpha)\bigr) - f\bigl(s+\alpha, u(s+\alpha)\bigr)\bigr| \\
&\quad\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

For $t \in \mathbb{N}_{\alpha+K_1}$, by (3.6) we also have

$$
\begin{aligned}
&\bigl|Tu_n(t) - Tu(t)\bigr| \\
&\quad\le \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} \bigl(\bigl|f\bigl(s+\alpha, u_n(s+\alpha)\bigr)\bigr| + \bigl|f\bigl(s+\alpha, u(s+\alpha)\bigr)\bigr|\bigr) \\
&\quad\le \frac{2L_1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} (s+\alpha+\gamma\delta)^{(-\beta_1)} \bigl|u(s+\alpha)\bigr|^{\delta}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{2L_1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \big(t-\sigma(s)\big)^{(\alpha-1)} (s+\alpha+\gamma\delta)^{(-\beta_1)} \big((s+\alpha)^{(-\gamma)}\big)^{\delta} \\
&\le \frac{2L_1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \big(t-\sigma(s)\big)^{(\alpha-1)} (s+\alpha+\gamma\delta)^{(-\beta_1)} (s+\alpha)^{(-\gamma\delta)} \\
&\le \frac{2L_1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \big(t-\sigma(s)\big)^{(\alpha-1)} (s+\alpha)^{(-\beta_1-\gamma\delta)} \\
&= \frac{2L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} (t+\alpha)^{(\alpha-\beta_1-\gamma\delta)} \\
&\le \frac{2L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} t^{(\alpha-\beta_1-\gamma\delta)} \\
&\le \varepsilon.
\end{aligned}
$$

Hence we can get that

$$
\|Tu_n - Tu\| = \sup_{t\in\mathbb{N}_\alpha} \big|Tu_n(t) - Tu(t)\big| \to 0 \quad \text{as } n \to \infty.
$$

Thus T is continuous in S_1.

In the following, we prove that TS_1 is relatively compact. For $t_1, t_2 \in \mathbb{N}_{\alpha+K_1}$, we have

$$
\begin{aligned}
\big|Tu(t_2) - Tu(t_1)\big| &\le \left| \frac{u_0}{\Gamma(\alpha)} t_2^{(\alpha-1)} - \frac{u_0}{\Gamma(\alpha)} t_1^{(\alpha-1)} \right| \\
&\quad + \Bigg| \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t_2-\alpha} \big(t_2-\sigma(s)\big)^{(\alpha-1)} f\big(s+\alpha, u(s+\alpha)\big) \\
&\quad - \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t_1-\alpha} \big(t_1-\sigma(s)\big)^{(\alpha-1)} f\big(s+\alpha, u(s+\alpha)\big) \Bigg| \\
&\le \frac{u_0}{\Gamma(\alpha)} t_2^{(\alpha-1)} + \frac{u_0}{\Gamma(\alpha)} t_1^{(\alpha-1)} \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t_2-\alpha} \big(t_2-\sigma(s)\big)^{(\alpha-1)} f\big(s+\alpha, u(s+\alpha)\big) \right| \\
&\quad + \left| \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t_1-\alpha} \big(t_1-\sigma(s)\big)^{(\alpha-1)} f\big(s+\alpha, u(s+\alpha)\big) \right| \\
&\le \frac{u_0}{\Gamma(\alpha)} t_2^{(\alpha-1)} + \frac{u_0}{\Gamma(\alpha)} t_1^{(\alpha-1)} + \frac{L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} t_2^{(\alpha-\beta_1-\gamma\delta)} \\
&\quad + \frac{L_1\Gamma(1-\beta_1-\gamma\delta)}{\Gamma(1+\alpha-\beta_1-\gamma\delta)} t_1^{(\alpha-\beta_1-\gamma\delta)} \\
&\le 2\varepsilon.
\end{aligned}
$$

Therefore $\{Tu : u \in S_1\}$ is a bounded and uniformly Cauchy subset by Definition 2.5. Hence TS_1 is relatively compact by Theorem 2.1. Due to Schauder's fixed point theorem, T has a fixed point. All functions in S_1 tend to 0 as $t \to \infty$, and hence the solution of (1.1) is attractive. The proof is completed. □

3.3 Results in a weighted space

Let ℓ_α^∞ be the set of all real sequences $u = \{u(t)\}_{t=\alpha}^\infty$ with the norm $\|u\| = \sup_{t\in\mathbb{N}_\alpha} |u(t)|$. Then $(\ell_\alpha^\infty, \|\cdot\|)$ is a Banach space. For any $u \in \ell_\alpha^\infty$, we introduce a new norm as follows:

$$\|u\|_{\beta_2} = \sup_{t\in\mathbb{N}_\alpha} \left\{ \frac{|u(t)|}{t^{(\beta_2)}} \right\},$$

where $\beta_2 \in (\alpha, 1)$. It is easy to see that $(\ell_\alpha^\infty, \|\cdot\|_{\beta_2})$ is also a Banach space.

Define the closed subset

$$S_{\beta_2} = \left\{ u \in \ell_\alpha^\infty : \|u\|_{\beta_2} \le \frac{(\alpha|u_0| + M)\Gamma(\alpha + 1 - \beta_2)}{\Gamma(\alpha + 1)} \right\}.$$

Obviously, S_{β_2} is a bounded, closed, and convex subset of the Banach space $(\ell_\alpha^\infty, \|\cdot\|_{\beta_2})$.

Definition 3.1 A function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is β_2-continuous if for any $\varepsilon > 0$, there exists $\delta = \delta_\varepsilon > 0$ such that for all $(t, u_1), (t, u_2) \in [0,\infty) \times \mathbb{R}$, we have

$$\bigl|f(t,u_1) - f(t,u_2)\bigr| < \varepsilon,$$

provided that $\frac{|u_1-u_2|}{t^{(\beta_2)}} < \delta$.

Obviously, u is a solution of (1.1) if and only if the operator T has a fixed point.

Theorem 3.3 *Assume that* (H_2) *holds and* f *is* β_2*-continuous. Then the fractional difference equation* (1.1) *has at least one solution in* S_{β_2}. *Further, the solutions of* (1.1) *are attractive.*

Proof We first show that T is continuous in S_{β_2} and maps S_{β_2} into S_{β_2}. Applying (3.4), for any $u \in S_{\beta_2}$, we have

$$\begin{aligned}
\frac{|Tu(t)|}{t^{(\beta_2)}} &\le \frac{|u_0|t^{(\alpha-1)}}{\Gamma(\alpha)t^{(\beta_2)}} + \frac{1}{\Gamma(\alpha)t^{(\beta_2)}} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} \bigl|f\bigl(s+\alpha, u(s+\alpha)\bigr)\bigr| \\
&\le \frac{|u_0|t^{(\alpha-1)}}{\Gamma(\alpha)t^{(\beta_2)}} + \frac{M}{\Gamma(\alpha)t^{(\beta_2)}} \sum_{s=0}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{(\alpha-1)} \\
&= \frac{|u_0|t^{(\alpha-1)}}{\Gamma(\alpha)t^{(\beta_2)}} + \frac{Mt^{(\alpha)}}{\Gamma(\alpha+1)t^{(\beta_2)}} \\
&= \frac{|u_0|\Gamma(t-\beta_2+1)}{\Gamma(\alpha)\Gamma(t-\alpha+2)} + \frac{M\Gamma(t-\beta_2+1)}{\Gamma(\alpha+1)\Gamma(t-\alpha+1)} \\
&\le \frac{(\alpha|u_0| + M)\Gamma(\alpha+1-\beta_2)}{\Gamma(\alpha+1)},
\end{aligned}$$

which implies that T maps S_{β_2} into S_{β_2}.

Let $u_n, u \in S_{\beta_2}$, $n = 1, 2, \dots$, and $\lim_{n\to\infty} u_n = u$, which means that $\|u_n - u\|_{\beta_2} \to 0$ as $n \to \infty$, that is,

$$\sup_{t\in\mathbb{N}_\alpha} \frac{|u_n(t) - u(t)|}{t^{(\beta_2)}} \to 0 \quad \text{as } n \to \infty.$$

Combining this with (2.9), we have

$$\begin{aligned}
&\frac{|Tu_n(t)-Tu(t)|}{t^{(\beta_2)}}\\
&\quad\le \frac{1}{\Gamma(\alpha)t^{(\beta_2)}}\sum_{s=0}^{t-\alpha}\big(t-\sigma(s)\big)^{(\alpha-1)}\big|f\big(s+\alpha,u_n(s+\alpha)\big)-f\big(s+\alpha,u(s+\alpha)\big)\big|\\
&\quad\le \frac{\varepsilon}{\Gamma(\alpha)t^{(\beta_2)}}\sum_{s=0}^{t-\alpha}\big(t-\sigma(s)\big)^{(\alpha-1)}\\
&\quad=\frac{\varepsilon t^{(\alpha)}}{\Gamma(\alpha+1)t^{(\beta_2)}}\\
&\quad\le \varepsilon.
\end{aligned}$$

Hence we get that

$$\|Tu_n-Tu\|_{\beta_2}=\sup_{t\in\mathbb{N}_\alpha}\left\{\frac{|Tu_n(t)-Tu(t)|}{t^{(\beta_2)}}\right\}\to 0\quad\text{as } n\to\infty,$$

and thus T is continuous in S_{β_2}.

In the following, we prove that TS_{β_2} is relatively compact. To this end, we define

$$Hu(t)=\frac{Tu(t)}{t^{(\beta_2)}}\quad\text{for } t\in\mathbb{N}_\alpha, u\in S_{\beta_2}.$$

According to this process, it is easy to see that $\{Hu(\cdot):u\in S_{\beta_2}\}$ is uniformly bounded. Next, we prove that $\{Hu(\cdot):u\in S_{\beta_2}\}$ is uniformly Cauchy. For $t_1,t_2\in\mathbb{N}_\alpha$ such that $t_1<t_2$, we have

$$\begin{aligned}
&\big|Hu(t_2)-Hu(t_1)\big|\\
&\quad=\left|\frac{Tu(t_2)}{t_2^{(\beta_2)}}-\frac{Tu(t_1)}{t_1^{(\beta_2)}}\right|\\
&\quad\le\left|\frac{u_0t_2^{(\alpha-1)}}{\Gamma(\alpha)t_2^{(\beta_2)}}-\frac{u_0t_1^{(\alpha-1)}}{\Gamma(\alpha)t_1^{(\beta_2)}}\right|+\left|\frac{1}{\Gamma(\alpha)t_2^{(\beta_2)}}\sum_{s=0}^{t_2-\alpha}\big(t_2-\sigma(s)\big)^{(\alpha-1)}f\big(s+\alpha,u(s+\alpha)\big)\right.\\
&\qquad\left.-\frac{1}{\Gamma(\alpha)t_1^{(\beta_2)}}\sum_{s=0}^{t_1-\alpha}\big(t_1-\sigma(s)\big)^{(\alpha-1)}f\big(s+\alpha,u(s+\alpha)\big)\right|\\
&\quad\le\frac{|u_0|}{\Gamma(\alpha)}\left(\frac{\Gamma(t_2-\beta_2+1)}{\Gamma(t_2-\alpha+2)}+\frac{\Gamma(t_1-\beta_2+1)}{\Gamma(t_1-\alpha+2)}\right)\\
&\qquad+\left|\frac{1}{\Gamma(\alpha)t_2^{(\beta_2)}}\sum_{s=0}^{t_2-\alpha}\big(t_2-\sigma(s)\big)^{(\alpha-1)}f\big(s+\alpha,u(s+\alpha)\big)\right|\\
&\qquad+\left|\frac{1}{\Gamma(\alpha)t_1^{(\beta_2)}}\sum_{s=0}^{t_1-\alpha}\big(t_1-\sigma(s)\big)^{(\alpha-1)}f\big(s+\alpha,u(s+\alpha)\big)\right|\\
&\quad\le\frac{|u_0|}{\Gamma(\alpha)}\left(\frac{\Gamma(t_2-\beta_2+1)}{\Gamma(t_2-\alpha+2)}+\frac{\Gamma(t_1-\beta_2+1)}{\Gamma(t_1-\alpha+2)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{M}{\Gamma(\alpha) t_2^{(\beta_2)}} \sum_{s=0}^{t_2-\alpha} \bigl(t_2 - \sigma(s)\bigr)^{(\alpha-1)} + \frac{M}{\Gamma(\alpha) t_1^{(\beta_2)}} \sum_{s=0}^{t_1-\alpha} \bigl(t_1 - \sigma(s)\bigr)^{(\alpha-1)} \\
&\le \frac{|u_0|}{\Gamma(\alpha)} \left(\frac{\Gamma(t_2 - \beta_2 + 1)}{\Gamma(t_2 - \alpha + 2)} + \frac{\Gamma(t_1 - \beta_2 + 1)}{\Gamma(t_1 - \alpha + 2)} \right) + \frac{M t_2^{(\alpha)}}{\Gamma(\alpha+1) t_2^{(\beta_2)}} + \frac{M t_1^{(\alpha)}}{\Gamma(\alpha+1) t_1^{(\beta_2)}} \\
&\le \frac{|u_0|}{\Gamma(\alpha)} \left(\frac{\Gamma(t_2 - \beta_2 + 1)}{\Gamma(t_2 - \alpha + 2)} + \frac{\Gamma(t_1 - \beta_2 + 1)}{\Gamma(t_1 - \alpha + 2)} \right) \\
&\quad + \frac{M}{\Gamma(\alpha+1)} \left(\frac{\Gamma(t_2 - \beta_2 + 1)}{\Gamma(t_2 - \alpha + 1)} + \frac{\Gamma(t_1 - \beta_2 + 1)}{\Gamma(t_1 - \alpha + 1)} \right) \\
&= \frac{|u_0|}{\Gamma(\alpha)} \left[t_2^{\alpha-\beta_2-1} \left(1 + O\left(\frac{1}{t_2}\right) \right) + t_1^{\alpha-\beta_2-1} \left(1 + O\left(\frac{1}{t_1}\right) \right) \right] \\
&\quad + \frac{M}{\Gamma(\alpha+1)} \left[t_2^{\alpha-\beta_2} \left(1 + O\left(\frac{1}{t_2}\right) \right) + t_1^{\alpha-\beta_2} \left(1 + O\left(\frac{1}{t_1}\right) \right) \right] \\
&\to 0 \quad \text{as } t_1 \to \infty, t_2 \to \infty.
\end{aligned}
$$

Therefore, $\{Hu : u \in S_{\beta_2}\}$ is a bounded and uniformly Cauchy subset by Definition 2.5. Thus $\{\frac{Tu(t)}{t^{(\beta_2)}} : u \in S_{\beta_2}\}$ is relatively compact by Theorem 2.1. Hence $T(S_{\beta_2})$ is relatively compact in S_{β_2}. Due to Schauder's fixed point theorem, T has a fixed point. All functions in S_{β_2} tend to 0 as $t \to \infty$, and hence the solution of (1.1) is attractive. The proof is completed. □

4 Example

As applications of our main results, we consider the following examples.

Example 4.1 Consider the equation

$$
\begin{cases}
\Delta_{-0.5}^{0.5} u(t) = (t + 0.7)^{(-0.7)} \sin u(t + 0.5), & t \in \mathbb{N}_0, \\
\Delta_{-0.5}^{-0.5} u(t)|_{t=0} = 0,
\end{cases}
\tag{4.1}
$$

where $f(t, u(t)) = (t + 0.2)^{(-0.7)} \sin u(t)$, $t \in \mathbb{N}_{0.5}$. Since

$$
\bigl| f(t, u_1) - f(t, u_2) \bigr| = \bigl| (t + 0.2)^{(-0.7)} (\sin u_1 - \sin u_2) \bigr| \tag{4.2}
$$

$$
\le \frac{\Gamma(1.7)}{\Gamma(2.4)} |u_1 - u_2| \tag{4.3}
$$

and

$$
\bigl| f(t, u) \bigr| = \bigl| (t + 0.2)^{(-0.7)} \sin u \bigr| \le \frac{\Gamma(1.7)}{\Gamma(2.4)} \quad \text{for } t \in \mathbb{N}_{0.5} \text{ and } u \in \ell_{0.5}^{\infty}, \tag{4.4}
$$

(H_1) and (H_2) hold. By Theorem 3.1 we get that (4.1) has at least one attractive solution.

Example 4.2 Consider

$$
\begin{cases}
\Delta_{-0.5}^{0.5} u(t) = (t + 0.7)^{(-0.7)} u^{1.2}(t + 0.5), & t \in \mathbb{N}_0, \\
\Delta_{-0.5}^{-0.5} u(t)|_{t=0} = 0,
\end{cases}
\tag{4.5}
$$

where $f(t,u(t)) = (t+0.2)^{(-0.7)}u^{1.2}(t)$, $t \in \mathbb{N}_{0.5}$. Since $\alpha = 0.5$, $\beta_1 = 0.7$, $\delta = 1.2$, $\gamma = 0.2$, we have that

$$\alpha + \gamma - 1 < 0, \qquad 1 - \beta_1 - \gamma\delta > 0, \qquad \alpha + \gamma - \beta_1 - \gamma\delta < 0, \tag{4.6}$$

and

$$\left|f(t,u)\right| \le L_1(t+\gamma\delta)^{(-\beta_1)}|u|^{\delta} \quad \text{for } t \in \mathbb{N}_{0.5} \text{ and } u \in \ell_{0.5}^{\infty}. \tag{4.7}$$

Then (H_3) holds. By Theorem 3.2 we get that (4.5) has at least one attractive solution.

Example 4.3 Consider

$$\begin{cases} \Delta_{-0.5}^{0.5}u(t) = (t+0.7)^{(-0.7)}\sin((t+0.5)^{(-0.6)}u(t+0.5)), \quad t \in \mathbb{N}_0, \\ \Delta_{-0.5}^{-0.5}u(t)|_{t=0} = 0, \end{cases} \tag{4.8}$$

where $f(t,u(t)) = (t+0.2)^{(-0.7)}\sin(t^{(-0.6)}u(t))$, $t \in \mathbb{N}_{0.5}$. Let $\beta_2 = 0.6 \in (0.5,1)$. Since

$$\begin{aligned} \left|f(t,u_1) - f(t,u_2)\right| &= \left|(t+0.2)^{(-0.7)}\left(\sin\left(t^{(-0.6)}u_1\right) - \sin\left(t^{(-0.6)}u_2\right)\right)\right| \\ &\le \frac{\Gamma(1.7)}{\Gamma(2.4)}t^{(-0.6)}t^{(0.6)}\frac{|u_1-u_2|}{t^{(0.6)}} \\ &\le \frac{\Gamma(1.7)}{\Gamma(2.4)}\frac{|u_1-u_2|}{t^{(0.6)}} \end{aligned} \tag{4.9}$$

and

$$\left|f(t,u)\right| = \left|(t+0.2)^{(-0.7)}\sin\left(t^{(-0.6)}u(t)\right)\right| \le \frac{\Gamma(1.7)}{\Gamma(2.4)} \quad \text{for } t \in \mathbb{N}_{0.5} \text{ and } u \in \ell_{0.5}^{\infty}. \tag{4.10}$$

Then f is β_2-continuous, and (H_2) holds. By Theorem 3.3 we get that (4.8) has at least one attractive solution.

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The authors declare that they have no competing interests.

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