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Existence and persistence of positive solution for a stochastic turbidostat model

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Abstract

A novel stochastic turbidostat model is investigated in this paper. The stochasticity in the model comes from the maximal growth rate influenced by white noise. Firstly, the existence and uniqueness of the positive solution for the system are demonstrated. Secondly, we analyze the persistence in mean and stochastic persistence of the system, respectively. Sufficient conditions about the extinction of the microorganism are obtained. Finally, numerical simulation results are given to support the theoretical conclusions.

Keywords: turbidostat model; white noise; persistence in mean; stochastic persistence; extinction

1 Introduction

The continuous culture of microorganism, such as the chemostat and the turbidostat, is a significant way in reality. According to the mechanism of turbidostat, a number of mathematical models were introduced to describe the change rate of microorganism and nutrient. Herbert *et al.* [1] constructed the basic model with a constant dilution rate, but they did not give a thorough analysis. Smith *et al.* [2] completely analyzed the model proposed by [1]. Li [3] introduced a competition turbidostat model with inconstant dilution rate $d + k_1x_1(t) + k_2x_2(t)$ and analyzed dynamic behaviors of the system. The above-mentioned systems are often described by ordinary differential equations.

Since the perturbation is inevitable in real conditions, an increasing number of researchers have realized that some phenomena, such as time delays [4] and stochastic factors [5], could also cause various dynamical behaviors that are different from the conclusions derived from ordinary differential equations. May [5] pointed out that parameters characterizing the natural biological systems have random influence. Mao [6] also emphasized the significant role of stochastic models in many ways of science and industry. Based on the above tendency and real conditions, the effect of stochastic disturbance on population dynamic has received and has been persistently receiving more attention. In microorganism cultivation, the sense of the stochasticity in the turbidostat may be arisen as a result of the destabilizations such as the uncertainty of the birth rate and the stochastic variations of environmental conditions. In particular, the survival rate is not always equivalent to constant due to the feedback phenomenon in the turbidostat. In order to explain those stochastic phenomena, scholars use white noise to represent destabilization from a biological point of view in generally. Standard Brownian motion $B(t)$ is a reasonable tool to describe the effect of white noise on a dynamic system.

Scholars have studied dynamical behaviors by constructing a stochastic model and obtained many important results. Imhof *et al.* [7] analyzed a deterministic single-substrate model that the dilution rate in the vessel is constant and derived the corresponding conclusion. They also set up a corresponding stochastic differential equation and investigated the extinction and the persistence of the system. Liu *et al.* [8, 9] put up a stochastic logistic model and established the sufficient condition of the stability of the positive solution. Campillo *et al.* [10] built the Fokker-Planck equation of stochastic chemostat and derived an adapted finite difference scheme to approximate the solution of the Fokker-Planck equation. Campillo *et al.* [11] constructed a set of stochastic chemostat models according to the different population scales and investigated the domain of validity for different scales. Zhang *et al.* [12] proposed a chemostat model with Holling type II functional response and stochastic perturbation and obtained sufficient conditions for the principle of competitive exclusion; they also provided numerical simulation to verify their results by using Milstein's higher order method. Zhao and Yuan [13] formulated a single-species stochastic chemostat model with periodic coefficients due to seasonal fluctuation; they obtained sufficient conditions for the existence of a random positive periodic solution and a globally attractive condition of the random periodic solution. Wang *et al.* [14] proposed a stochastic chemostat model with periodic wash-out rate and established sufficient conditions for the existence of a stochastic nontrivial positive periodic solution for the system. Lv *et al.* [15] proposed a stochastic competition chemostat model and derived the conditions of the threshold between persistence and extinction for the corresponding deterministic model and the stochastic model, respectively. Meng *et al.* [16] developed a stochastic chemostat model in a polluted environment and obtained the conditions of persistence and extinction for microorganism. They also pointed out that a small enough stochastic disturbance could cause the microorganism to die out even if the microorganism could be persistent in the deterministic model. More mathematical models about microorganism cultivation with constant dilution rate and perturbed phenomena could be found in [17–24]. The study of stochastic population models has been a focus of some scholars in recent years (see [25–41]).

From a biological point of view and real condition, if the concentration of the microorganism in the culture vessel is large but the wash-out rate is too small, it will affect the growth of the microorganism in the turbidostat. On the contrary, if the concentration of the microorganism is quite small, the fixed wash-out rate will cause the waste of the nutrient. Based on the above phenomenon, we construct a turbidostat model with linear wash-out rate $d + kx(t)$. Due to the stochastic destabilizations, the maximum growth rate, one of the essential parameters in microorganism cultivation, will undergo variations at different times in the turbidostat. That is, if the birth (death) rate increases (decreases) or the temperature and food are sufficient, the maximum growth rate occurs in advance. If the birth (death) rate decreases (increases) or the temperature and food are insufficient in the system, the maximum growth rate delays. Therefore, the maximum growth rate undergoes a random change.

In order to explain the above stochastic and feedback phenomena from mathematical aspect and provide researchers with more feasible suggestions about microbial cultivation, we consider a stochastic turbidostat model with white noise and Holling III functional response. We completely investigate the case that there exists stochastic destabilization for the maximum growth rate m. Therefore, we change the maximum growth rate m in the turbidostat model into a random variable $\tilde{m}$, in there $\tilde{m} = m + \alpha\dot{B}(t)$, where $\dot{B}(t)$ is white noise, *i.e.*, $B(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ (where $\mathcal{F}_t = \sigma\{(S(t), x(t)); 0 \leq t \leq \tau_e\}$ is σ-filed generated by $(S(t), x(t))$, $0 \leq t \leq \tau_e$). $\alpha \geq 0$ represents the intensity of white noise. On the basis of [3, 30] and the above analysis, we establish the following stochastic turbidostat model:

$$\begin{cases} dS = [(S^0 - S)(d + kx) - \frac{mS^2x}{a+S^2}]\,dt - \frac{\alpha S^2 x}{a+S^2}\,dB(t), \\ dx = [\frac{mS^2}{a+S^2} - (d + kx)]x\,dt + \frac{\alpha S^2 x}{a+S^2}\,dB(t), \end{cases} \tag{1.1}$$

where S and x represent the concentration of the nutrient and microorganism at the time t, respectively. S^0 expresses the input concentration of nutrition, $d + kx$ stands for the dilution rate of the turbidostat system. $\frac{mS^2}{a+S^2}$ is the Holling III functional response, m is the maximal growth rate and a is called the half-saturation constant. S^0, d, k, m and a are positive.

This paper is organized as follows. In Section 2, we determine the existence and uniqueness of a positive solution of system (1.1). In Section 3, we further investigate two kinds of persistence of system (1.1) and the extinction of microorganism in the turbidostat and obtain the corresponding break-even concentration. Finally, the effect of white noise on dynamical behaviors of system (1.1) is discussed in detail and specific examples are given to verify our theoretical conclusions.

2 Existence and uniqueness of positive solution

In this section, we demonstrate that system (1.1) has a unique global positive solution. The coefficients of (1.1) are not linear growth, but they are locally Lipschitz continuous. Thus for any initial value $(S_0, x_0) \in R^2_+$, there is a unique positive local solution $(S(t), x(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [6] (the time that the positive local solution $(S(t), x(t))$ does not satisfy). If we can show that $\tau_e = \infty$, then the positive solution $(S(t), x(t)) \in R^2_+$ for all $t \geq 0$.

Theorem 2.1 *If $d > kS^0$, for any initial value $(S_0, x_0) \in R^2_+$, there is a unique solution $(S(t), x(t))$ of system* (1.1) *such that $(S(t), x(t)) \in R^2_+$ for all $t \geq 0$ almost surely.*

Proof On the basis of the definition of $\mathcal{F}_t$, choose $\varepsilon_0 > 0$ such that $S_0 > \varepsilon_0$ and $x_0 > \varepsilon_0$ and define the stopping time t_ε as follows:

$$\tau_\varepsilon = \inf\bigl\{t \in [0, \tau_e) : S(t, \omega) \leq \varepsilon \text{ or } x(t, \omega) \leq \varepsilon\bigr\} \quad \text{for any } \varepsilon \geq \varepsilon_0 > 0,$$

where τ_ε is a random variable. For any $\omega \in \Omega$ and $\varepsilon > 0$, there exist $t_1, t_2, \ldots, t_n \in [0, \tau_e)$ such that $S(t_i, \omega) \leq \varepsilon$ $(i = 1, 2, \ldots, n)$ or $x(t_i, \omega) \leq \varepsilon$ $(i = 1, 2, \ldots, n)$. The stopping time $\tau_\varepsilon = \inf\{t_1, t_2, \ldots, t_n\}$, which means τ_ε is the first time such that $S(t, \omega) \leq \varepsilon$ or $x(t, \omega) \leq \varepsilon$.

Throughout this paper, we set $\inf \emptyset = \infty$ ($\emptyset$ represents the empty set). It is obvious that τ_ε is increasing as $\varepsilon \to 0$. Set $\tau_0 = \lim_{\varepsilon \to 0} \tau_\varepsilon$, whence $\tau_0 \leq \tau_e$ a.s. If we can demonstrate $\tau_0 = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(S(t), x(t)) \in R^2_+$ for all $t \geq 0$ a.s. Consequently, in order to prove Theorem 2.1, we only need to demonstrate that $\tau_0 = \infty$ a.s.

If the above statement is false, then there exist $\delta \in (0,1)$ and a constant $T > 0$ such that $P\{\tau_0 \leq T\} > \delta$. Therefore, we have $P\{\tau_\varepsilon \leq T\} > \delta$ for all $0 < \varepsilon \leq \varepsilon_0$.

According to system (1.1), the total biomass in the turbidostat satisfies $N(t) = S(t) + x(t) \geq 0$ because of the expression of the Brownian term. Besides, $N(t)$ satisfies the following equation:

$$\begin{aligned} \mathrm{d}N(t) &= \left[(d+kx)\left(S^0 - N(t)\right)\right]\mathrm{d}t \\ &\leq \left[dS^0 - \left(d - kS^0\right)N(t)\right]\mathrm{d}t. \end{aligned} \tag{2.1}$$

Define

$$\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = dS^0 - \left(d - kS^0\right)Z(t)$$

with the initial value $Z(0) = N(0) = S_0 + x_0$. After a simple calculation, it is easy to show that

$$Z(t) = \frac{dS^0}{d-kS^0} + \left[Z(0) - \frac{dS^0}{d-kS^0}\right]\mathrm{e}^{-(d-kS^0)t},$$

and for $t \in [0, \tau_e)$ we have

$$Z(t) \leq \max\left\{S_0 + x_0, \frac{dS^0}{d-kS^0}\right\}.$$

By the comparison theorem for differential equation, we have

$$N(t) \leq Z(t), \quad t \in [0, \tau_e) \text{ a.s.}$$

Therefore, we can get that, for $t \in [0, \tau_e)$,

$$N(t) \leq \max\left\{S_0 + x_0, \frac{dS^0}{d-kS^0}\right\} := C_1.$$

Define a function $V : R_+^2 \to \bar{R}_+$ as follows:

$$V\left(S(t), x(t)\right) = -\ln\frac{S(t)}{C_1} - \ln\frac{x(t)}{C_1}.$$

Obviously, V is nonnegative and definite. Using Itô's formula, we can obtain

$$\mathrm{d}V = LV\,\mathrm{d}t + \frac{\alpha S(x-S)}{a+S^2}\,\mathrm{d}B(t),$$

where

$$LV = -\frac{S^0(d+kx)}{S} + 2(d+kx) + \frac{mS(x-S)}{a+S^2} + \frac{1}{2}\frac{\alpha^2 S^2(x^2+S^2)}{(a+S^2)^2}.$$

Hence we can use the inequality $N(t) \leq C_1$ to conclude that

$$\begin{aligned} LV &\leq 2(d+kx) + \frac{mS(x-S)}{a+S^2} + \frac{1}{2}\frac{\alpha^2 S^2(x^2+S^2)}{(a+S^2)^2} \\ &\leq 2(d+kC_1) + \frac{mC_1^2}{a} + \frac{\alpha^2 C_1^4}{a^2} \\ &:= C_2, \end{aligned}$$

which yields the inequality

$$\mathrm{d}V \leq C_2\,\mathrm{d}t + \frac{\alpha S(x-S)}{a+S^2}\,\mathrm{d}B(t).$$

Integrating both sides for the above inequality from 0 to $\tau_\varepsilon \wedge T$ and taking expectation, we obtain

$$EV\bigl(S(\tau_\varepsilon \wedge T), x(\tau_\varepsilon \wedge T)\bigr) \leq V(S_0, x_0) + C_2 T.$$

Setting $\Omega_\varepsilon = \{\tau_\varepsilon \leq T\}$ for any nonnegative $\varepsilon \leq \varepsilon_0$, we can get $P(\Omega_\varepsilon) > \delta$. In view of the definition of the stoping time, we conclude, for every $\omega \in \Omega_\varepsilon$, that there exists at least one of $S(\tau_\varepsilon, \omega)$, $x(\tau_\varepsilon, \omega)$ is less than or equal to ε,

$$V\bigl(S(\tau_\varepsilon), x(\tau_\varepsilon)\bigr)1_{\{\omega\in\Omega_\varepsilon\}} \geq -\ln\frac{\varepsilon}{C_1}1_{\{\omega\in\Omega_\varepsilon\}}.$$

Consequently,

$$\begin{aligned} EV\bigl(S(\tau_\varepsilon), x(\tau_\varepsilon)\bigr)1_{\{\omega\in\Omega_\varepsilon\}} &\geq -P(\Omega_\varepsilon)1_{\{\omega\in\Omega_\varepsilon\}}\ln\frac{\varepsilon}{C_1} \\ &> -\delta\ln\frac{\varepsilon}{C_1}, \end{aligned}$$

which yields the inequality

$$V(S_0, x_0) + C_2 T \geq EV\bigl(S(\tau_\varepsilon), x(\tau_\varepsilon)\bigr)1_{\{\omega\in\Omega_\varepsilon\}} \geq -\delta\ln\frac{\varepsilon}{C_1}.$$

When $\varepsilon \to 0$, we have

$$V(S_0, x_0) + C_2 T \to \infty,$$

which leads to contradiction with

$$V(S_0, x_0) = -\ln\frac{S_0}{C_1} - \ln\frac{x_0}{C_1} < \infty.$$

Therefore we must have $\tau_0 = \infty$ a.s. □

3 Extinction and persistence of the model

Denote

$$\Gamma = \{(S,x) \in R_+^2 : S + x = S^0\}.$$

For the convenience of demonstration of the main results in this section, we give the following two remarks.

Remark 1 From (2.1), we know that Γ is a nonnegative invariant set for turbidostat stochastic model (1.1), which is essential characteristic for our theoretical analysis in the following section.

Remark 2 Yuan *et al.* [30] pointed out that if $m \leq D$ (the maximum growth rate is less than or equal to wash-out rate), microorganism in the system must be washed out. Moreover, this conclusion can also be found in [2] (Chapter 1 and Section 4 of Chapter 2) and [7]. If $m \leq d + kx$, the microorganism must be washed out in model (1.1). Thus we always assume $m > d + kx$ in this paper, which means $m > d$.

From a biological point of view and the mechanism of turbidostat, if the wash-out rate (the output constant of turbidostat) is larger than the maximum growth rate (the yield constant of turbidostat), there is no microorganism in the culture vessel. In other words, if the maximum growth rate is smaller than the wash-out rate, the population will be extinct. The proof in this section is based on $m > d + kx$.

On the basis of the positive invariant set $\Gamma = \{(S,x) \in R_+^2 : S + x = S^0\}$, we only need to investigate the following system:

$$\mathrm{d}x = \left[\frac{m(S^0 - x)^2}{a + (S^0 - x)^2} - (d + kx)\right]x\,\mathrm{d}t + \frac{\alpha(S^0 - x)^2 x}{a + (S^0 - x)^2}\,\mathrm{d}B(t), \tag{3.1}$$

with the initial value $x(0) = x_0 \in (0, S^0)$. We need the following definition and lemma in order to determine the main results.

Definition 3.1 ([42])

(I) The microorganism in system (1.1) is persistent if

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,\mathrm{d}s \geq \zeta$$

for some constant $\zeta > 0$;

(II) Microorganism in system (3.1) is stochastically persistent in the turbidostat if, for any $\epsilon \in (0,1)$, there are positive constants $B_1 = B_1(\epsilon)$ and $B_2 = B_2(\epsilon)$ such that, for any initial value $x_0 \in R_+$,

$$\lim_{t\to\infty} \inf P\bigl(x(t) \leq B_1\bigr) > 1 - \epsilon \quad \text{and} \quad \lim_{t\to\infty} \inf P\bigl(x(t) \geq B_2\bigr) > 1 - \epsilon.$$

Lemma 3.1 ([43]) *Let $f \in C[[0,\infty) \times \Omega, (0,\infty)]$. If there exist positive constants λ_0 and λ such that*

$$\log f(t) \geq \lambda t - \lambda_0 \int_0^t f(s)\,\mathrm{d}s + F(t), \quad a.s.$$

for all $t \geq 0$, *where* $F \in C[[0,\infty) \times \Omega, R]$ *and* $\lim_{t\to\infty} \frac{F(t)}{t} = 0$, *a.s. Then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \geq \frac{\lambda}{\lambda_0}, \quad a.s.$$

Theorem 3.1 *If the break-even concentration* $\lambda_1 < S^0$, *where*

$$\lambda_1 = \frac{d(a+(S^0)^2)}{mS^0} + \frac{\alpha^2(S^0)^3}{2m(a+(S^0)^2)}$$

for any given initial value $(S_0, x_0) \in R^2_+$, *the solution of turbidostat model* (1.1) *satisfies*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,\mathrm{d}s \geq \frac{maS^0}{(a+(S^0)^2(2mS^0+ka))}\bigl(S^0 - \lambda_1\bigr) > 0, \quad \textit{almost surely},$$

which means the microorganism in system (1.1) *is persistent.*

Proof Define a function $V(x(t)) = \ln x(t)$. Applying Itô's formula, we have

$$\mathrm{d}V = \mathcal{L}V\,\mathrm{d}t + \frac{\alpha S^2}{a+S^2}\,\mathrm{d}B(t), \tag{3.2}$$

where

$$\begin{aligned}
\mathcal{L}V &= \frac{mS^2}{a+S^2} - (d+kx) - \frac{1}{2}\frac{\alpha^2 S^4}{(a+S^2)^2} \\
&\geq \frac{m(S^0)^2}{a+(S^0)^2} - \frac{2mS^0}{a}x - d - kx - \frac{1}{2}\frac{\alpha^2(S^0)^4}{(a+(S^0)^2)^2} \\
&= \left[\frac{m(S^0)^2}{a+(S^0)^2} - d - \frac{1}{2}\frac{\alpha^2(S^0)^4}{(a+(S^0)^2)^2}\right] - \left(\frac{2mS^0}{a} + k\right)x.
\end{aligned}$$

Integrating (3.2) from 0 to t, we obtain

$$\begin{aligned}
\ln x(t) - \ln x(0) \geq{}& \left[\frac{m(S^0)^2}{a+(S^0)^2} - d - \frac{1}{2}\frac{\alpha^2(S^0)^4}{(a+(S^0)^2)^2}\right]t \\
&- \left(\frac{2mS^0}{a} + k\right)\int_0^t x(s)\,\mathrm{d}s + \int_0^t \frac{\alpha S^2(s)}{a+S^2(s)}\,\mathrm{d}B(s),
\end{aligned}$$

which means

$$\begin{aligned}
\frac{\ln x(t)}{t} \geq{}& \frac{mS^0}{a+(S^0)^2}\bigl(S^0 - \lambda_1\bigr) \\
&- \left(\frac{2mS^0}{a} + k\right)\frac{1}{t}\int_0^t x(s)\,\mathrm{d}s + \frac{1}{t}M(t) + \frac{1}{t}\ln x_0,
\end{aligned}$$

where $M(t) = \int_0^t \frac{\alpha S^2(s)}{a+S^2(s)}\,\mathrm{d}B(s)$ is a local continuous martingale with $M(0) = 0$. Define

$$Y_t = \langle M, M\rangle_t = \int_0^t \frac{\alpha^2 S^4}{(a+S^2)^2}\,\mathrm{d}t$$

is the quadratic variation process and $Y_t \leq (\frac{\alpha(S^0)^2}{a+(S^0)^2})^4 t$. Therefore,

$$\lim_{t\to\infty}\sup\frac{\langle M,M\rangle_t}{t} \leq \left(\frac{\alpha(S^0)^2}{a+(S^0)^2}\right)^4 < \infty.$$

Using the strong principle of large number, we obtain

$$\lim_{t\to\infty}\frac{M(t)}{t} = 0, \quad \text{almost surely,}$$

and

$$\lim_{t\to\infty}\frac{\ln x_0}{t} = 0, \quad \text{almost surely.}$$

If $\lambda_1 < S^0$, we can derive the following result by Lemma 3.1:

$$\lim_{t\to\infty}\inf\frac{1}{t}\int_0^t x(s)\,\mathrm{d}s \geq \frac{maS^0}{(a+(S^0)^2(2mS^0+ka))}\bigl(S^0-\lambda_1\bigr) > 0, \quad \text{almost surely.}$$

This completes the proof of Theorem 3.1. □

Consider the following time-homogeneous stochastic equation:

$$\mathrm{d}X(t) = b\bigl(X(t)\bigr)\,\mathrm{d}t + \alpha\bigl(X(t)\bigr)\,\mathrm{d}B(t) \quad \text{with } X(0) \in R_+.$$

Lemma 3.2 ([44]) *Let $X(t)$ be a time-homogeneous solution of the above one-dimensional time-homogeneous stochastic equation on E_1 (one-dimensional Euclidean space). Assume that:*

- (I) *In the domain $U \subset E_1$ and some neighborhood thereof, the diffusion $\alpha(X)$ is bounded away from zero;*
- (II) *If, for all $X \in E_1 \setminus U$, the mean time τ_X at which a path emerging from X reaches the set U is finite, and $\sup_{X\in K} E(\tau_X) < \infty$ for every compact subset $K \subset E_1$.*

Then the Markov process $X(t)$ has a stationary distribution $\pi(x)$.

Theorem 3.2 *If the break-even concentration $\lambda_2 < S^0$, where*

$$\lambda_2 = \frac{(a+(S^0)^2)d}{mS^0} + \frac{\alpha^2(S^0)^3}{m(a+(S^0)^2)},$$

for any given initial value $(S_0, x_0) \in R_+^2$, the microorganism $x(t)$ is stochastically persistent in the turbidostat and system (3.1) has a stationary distribution.

Proof Define a C^2-function $V: R_+ \to R_+$ for any $p \in (0,1)$ as follows:

$$V(x) = \frac{1}{x^p(t)}, \quad p \in (0,1).$$

Applying Itô's formula, one can obtain

$$\begin{aligned}
\mathrm{d}V &= \frac{-p}{x^{p+1}}\left\{\left[\frac{m(S^0-x)^2}{a+(S^0-x)^2}-(d+kx)\right]x\,\mathrm{d}t+\frac{\alpha(S^0-x)^2x}{a+(S^0-x)^2}\,\mathrm{d}B(t)\right\} \\
&\quad+\frac{p(p+1)\alpha^2(S^0-x)^4x^2}{2x^{p+2}(a+(S^0-x)^2)^2}\,\mathrm{d}t \\
&= -\frac{p}{x^p}\left\{\frac{m(S^0-x)^2}{a+(S^0-x)^2}-d-kx-\frac{(p+1)\alpha^2(S^0-x)^4}{2(a+(S^0-x)^2)^2}\right\}\mathrm{d}t \\
&\quad-\frac{p\alpha(S^0-x)^2}{x^p(a+(S^0-x)^2)}\,\mathrm{d}B(t),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathrm{d}V &= -\frac{p}{x^p}\left\{\frac{m(S^0)^2}{a+(S^0)^2}-d-\frac{(p+1)\alpha^2(S^0)^2}{2(a+(S^0)^2)^2}\right\}\mathrm{d}t+F(t)\,\mathrm{d}t \\
&\quad-\frac{p\alpha(S^0-x)^2}{x^p(a+(S^0-x)^2)}\,\mathrm{d}B(t),
\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}
F(t) &= \frac{p}{x^p}\left\{\frac{m(S^0)^2}{a+(S^0)^2}-\frac{m(S^0-x)^2}{a+(S^0-x)^2}-\frac{(p+1)\alpha^2(S^0)^4}{2(a+(S^0)^2)^2}+\frac{(p+1)\alpha^2(S^0-x)^4}{2(a+(S^0-x)^2)^2}+kx\right\} \\
&= \frac{p(2maS^0x-max^2)}{x^p(a+(S^0)^2)(a+(S^0-x)^2)}-\frac{p(p+1)\alpha^2}{2x^p}\left\{\left[\frac{(S^0)^2}{a+(S^0)^2}\right]^2-\left[\frac{(S^0-x)^2}{a+(S^0-x)^2}\right]^2\right\} \\
&\quad+\frac{pkx}{x^p} \\
&\le \frac{2pmaS^0x}{x^p(a+(S^0)^2)(a+(S^0-x)^2)}+\frac{pkx}{x^p} \\
&\le \left[\frac{2pmaS^0}{a(a+(S^0)^2)}+kp\right]\left(S^0\right)^{1-p}.
\end{aligned}$$

Let $\theta = p[\frac{m(S^0)^2}{a+(S^0)^2}-d-\frac{(p+1)\alpha^2(S^0)^4}{2(a+(S^0)^2)^2}]$, then we can choose p small enough such that $\theta > 0$. Multiplying (3.3) by $e^{\theta t}$ and taking an integration from 0 to t, we obtain

$$\begin{aligned}
\frac{1}{x^p(t)} &= e^{-\theta t}\frac{1}{x^p(0)}+\int_0^t F(s)e^{-\theta(t-s)}\,\mathrm{d}s-\int_0^t\frac{p\alpha(S^0-x(t))^2}{x^p(a+(S^0-x(t))^2)}\,\mathrm{d}B(s) \\
&\le \frac{1}{x^p(0)}+\frac{1}{\theta}\left[\frac{2pmaS^0}{a(a+(S^0)^2)}+kp\right]\left(S^0\right)^{1-p}-M(t),
\end{aligned}\tag{3.4}$$

where $M(t)=\int_0^t\frac{p\alpha(S^0-x)^2}{x^p(a+(S^0-x)^2)}\,\mathrm{d}B(s)$ is a continuous martingale with $M(0)=0$. Taking expectation on both sides of (3.4), we conclude that

$$\begin{aligned}
E\left[\frac{1}{x^p(t)}\right] &= \frac{1}{x^p(0)}+\int_0^t E\bigl(F(s)\bigr)e^{-\theta(t-s)}\,\mathrm{d}s-E\bigl(M(t)\bigr) \\
&\le \frac{1}{x^p(0)}+\frac{1}{\theta}\left[\frac{2pmaS^0}{a(a+(S^0)^2)}+kp\right]\left(S^0\right)^{1-p}.
\end{aligned}$$

Let $B_1 = S^0$, we have the following equality:

$$P\big(x(t) \le B_1\big) = P\big(x(t) \le S^0\big) = 1 \ge 1 - \epsilon,$$

on the positive invariant set Γ. Moreover, applying Chebyshev's inequality [6], we obtain

$$\begin{aligned} P\big(B_2 \le x(t)\big) &= P\left(\frac{1}{B_2^p} \ge \frac{1}{x^p(t)}\right) = 1 - P\left(\frac{1}{B_2^p} \le \frac{1}{x^p(t)}\right) \\ &\ge 1 - B_2^p E\left[\frac{1}{x^p(t)}\right] \\ &\ge 1 - B_2^p \left\{\frac{1}{x^p(0)} + \frac{1}{\theta}\left[\frac{2pmaS^0}{a(a+(S^0)^2)} + kp\right]\big(S^0\big)^{1-p}\right\}. \end{aligned}$$

We can choose B_2 such that $B_2^p\{\frac{1}{x^p(0)} + \frac{1}{\theta}[\frac{2pmaS^0}{a(a+(S^0)^2)} + kp](S^0)^{1-p}\} < \epsilon$, which implies that

$$P\big(B_2 \le x(t)\big) \ge 1 - \epsilon.$$

Therefore, the microorganism is stochastically persistent in the turbidostat. Next we prove that system (3.1) has a stationary distribution. Let $\varepsilon > 0$ be a small enough number and U be a bounded open subset with a regular boundary such that

$$\big\{x : \varepsilon \le x \le S^0 - \varepsilon\big\} \subset U \subset \bar{U} \subset \big(0, S^0\big),$$

where $\bar{U}$ represents the closure of U. Define a C^2-function $V : R_+ \to R_+$ as

$$V\big(x(t)\big) = \frac{1}{px^p(t)} + \frac{1}{S^0 - x(t)}$$

for any $p \in (0,1)$. Then apply Itô's formula to get

$$\begin{aligned} \mathrm{d}V = {}& -\frac{1}{x^p}\left\{\frac{a(S^0-x)^2}{a+(S^0-x)^2} - d - kx - \frac{(p+1)\alpha^2}{2}\frac{(S^0-x)^4}{(a+(S^0-x)^2)^2}\right\}\mathrm{d}t \\ &+ \left\{\frac{1}{(S^0-x)^2}\left[\frac{m(S^0-x)^2x}{a+(S^0-x)^2} - (d+kx)x\right] + \frac{1}{(S^0-x)^3}\frac{\alpha^2(S^0-x)^4x^2}{(a+(S^0-x)^2)^2}\right\}\mathrm{d}t \\ &+ \left\{\frac{-1}{x^{p+1}} + \frac{1}{(S^0-x)^2}\right\}\frac{\alpha(S^0-x)^2x}{a+(S^0-x)^2}\,\mathrm{d}B(t) \\ := {}& \mathcal{L}V\,\mathrm{d}t + \left\{\frac{-1}{x^{p+1}} + \frac{1}{(S^0-x)^2}\right\}\frac{\alpha(S^0-x)^2x}{a+(S^0-x)^2}\,\mathrm{d}B(t), \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}V \le {}& -\frac{1}{x^p}\left\{\frac{m(S^0)^2}{a+(S^0)^2} - d - \frac{(p+1)\alpha^2}{2}\frac{(S^0)^4}{(a+(S^0)^2)^2}\right\} + \left[\frac{2pmaS^0}{a(a+(S^0)^2)} + kp\right]\big(S^0\big)^{1-p} \\ &+ \frac{mS^0}{a} + \frac{\alpha^2(S^0)^3}{a^2} - \frac{dx}{(S^0-x)^2}. \end{aligned}$$

Use the inequality $\lambda_2 < S^0$ and $p \in (0,1)$ to check that, for sufficiently small $\varepsilon > 0$,

$$\mathcal{L}V(x) \le -1 \quad \text{for all } x \in \big(0, S^0\big) \setminus U,$$

which yields that (II) in Lemma 3.2 holds. It is easy to check that the diffusion $\sigma(x) = \frac{\alpha(S^0-x)^2x}{a+(S^0-x)^2}$ in system (3.1) is bounded away from zero for $x \in (0, S^0)$. Therefore, system (3.1) has a stationary distribution. This completes the proof of Theorem 3.2. □

Theorem 3.3 *If* $\frac{\alpha^2}{4} + kS^0 < d - \frac{m(S^0)^2}{a+(S^0)^2}$, *then for any initial condition* $(S_0, x_0) \in R^2_+$, *the microorganism* $x(t)$ *will be extinct with probability one in the turbidostat.*

Proof Defining a C^2-function $V(x(t)) = \ln x(t)$, we obtain the following equality by Itô's formula:

$$\mathrm{d}V = \left[\frac{mS^2}{a+S^2} - (d+kx) - \frac{\alpha^2S^4}{2(a+S^2)^2}\right]\mathrm{d}t + \frac{\alpha S^2}{a+S^2}\,\mathrm{d}B(t). \tag{3.5}$$

By equation (3.5), we define

$$h(S) = \frac{mS^2}{a+S^2} - d - kx - \frac{\alpha^2S^4}{2(a+S^2)^2}.$$

For $h(S)$, we can obtain

$$\begin{aligned} h(S) &\le \frac{\alpha^2aS^2}{(2\sqrt{a}S)^2} + \frac{m(S^0)^2}{a+(S^0)^2} - d + kS^0 \\ &= \frac{\alpha^2}{4} + \frac{m(S^0)^2}{a+(S^0)^2} - d + kS^0. \end{aligned} \tag{3.6}$$

By equations (3.5) and (3.6), we see that

$$\begin{aligned} \ln x(t) - \ln x_0 &= \int_0^t h(S)\,\mathrm{d}t + \int_0^t \frac{\alpha S^2}{a+S^2}\,\mathrm{d}B(t) \\ &\le \left[\frac{\alpha^2}{4} + \frac{m(S^0)^2}{a+(S^0)^2} - d + kS^0\right]t + \int_0^t \frac{\alpha S^2}{a+S^2}\,\mathrm{d}B(t), \end{aligned} \tag{3.7}$$

which yields the inequality

$$\frac{\ln x(t)}{t} \le \frac{\alpha^2}{4} + \frac{m(S^0)^2}{a+(S^0)^2} - d + kS^0 + \frac{\ln x_0}{t} + \frac{1}{t}M(t), \tag{3.8}$$

where $M(t) = \int_0^t \frac{\alpha S^2}{a+S^2}\,\mathrm{d}B(t)$ is a local continuous martingale with $M(0) = 0$. If $\frac{\alpha^2}{4} + \frac{m(S^0)^2}{a+(S^0)^2} - d + kS^0 < 0$, then taking the supremum and limit for (3.8), we get

$$\lim_{t\to\infty}\sup\frac{\ln x(t)}{t} \le \frac{\alpha^2}{4} + \frac{m(S^0)^2}{a+(S^0)^2} - d + kS^0 < 0, \quad \text{almost surely.} \tag{3.9}$$

That is, the microorganism $x(t)$ in the vessel will exponentially tend to zero. The proof of Theorem 3.3 is completed. □

4 Discussion and numerical simulation

In this paper, stochastic factors are introduced to the mathematical model of microorganism culture in the turbidostat. The maximum growth rate in system (1.1) is perturbed by stochastic phenomena such as the birth rate and the variation of environment. We show that oscillations may occur for system (1.1) when white noise exists. Under the condition of white noise and the feedback control of the turbidostat, system (1.1) may be persistent (stochastically persistent) as described by Theorem 3.1 (Theorem 3.2), and it has a stationary distribution. On the contrary, the microorganism $x(t)$ in the turbidostat might be extinct as displayed by Theorem 3.3 because of white noise and the feedback control of the turbidostat. We explicitly discuss those phenomena with the following three examples, respectively.

Computer simulation of the path of $(S(t), x(t))$ is provided with the initial value $(S(0), x(0)) = (0.4, 0.37)$. Let $S^0 = 0.77$, $d = 0.18$, $k = 0.2$, $m = 1.6$, $a = 1.7$, and choose $\alpha = 0$ or $\alpha = 0.3$.

$$\begin{cases} \mathrm{d}S(t) = [(0.77 - S(t))(0.18 + 0.2x(t)) - \frac{1.6S^2(t)x(t)}{1.7+S^2(t)}]\,\mathrm{d}t - \frac{0.3S^2(t)x(t)}{1.7+S^2(t)}\,\mathrm{d}B(t), \\ \mathrm{d}x(t) = [\frac{1.6S^2(t)}{1.7+S^2(t)} - (0.18 + 0.2x(t))]x(t)\,\mathrm{d}t + \frac{0.3S^2(t)x(t)}{1.7+S^2(t)}\,\mathrm{d}B(t). \end{cases} \tag{4.1}$$

When $\alpha = 0$, system (4.1) becomes a corresponding deterministic model. Figure 1 depicts the persistence of microorganism in the deterministic model. When $\alpha = 0.3$, which means system (4.1) has stochastic destabilization from internal or external factors, the break-even concentration

$$\lambda_1 = \frac{(a + (S^0)^2)d}{mS^0} + \frac{\alpha^2 S^0}{2m(a + (S^0)^2)} \approx 0.3406 < S^0 = 0.77.$$

Hence the microorganism in the turbidostat will be persistent according to Theorem 3.1 as is shown in Figure 2.

Computer simulation of the path of $(S(t), x(t))$ is provided with the initial value $(S(0), x(0)) = (0.3, 0.22)$. Let $S^0 = 0.52$, $d = 0.18$, $k = 0.2$, $m = 1.9$, $a = 1.2$, and choose $\alpha = 0$ or $\alpha = 0.3$.

$$\begin{cases} \mathrm{d}S(t) = [(0.52 - S(t))(0.18 + 0.2x(t)) - \frac{1.9S^2(t)x(t)}{1.2+S^2(t)}]\,\mathrm{d}t - \frac{0.3S^2(t)x(t)}{1.2+S^2(t)}\,\mathrm{d}B(t), \\ \mathrm{d}x(t) = [\frac{1.9S^2(t)}{1.2+S^2(t)} - (0.18 + 0.2x(t))]x(t)\,\mathrm{d}t + \frac{0.3S^2(t)x(t)}{1.2+S^2(t)}\,\mathrm{d}B(t). \end{cases} \tag{4.2}$$

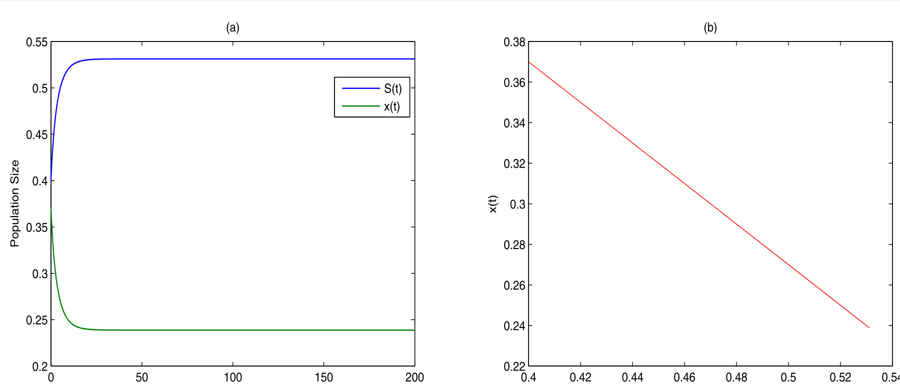

Figure 1 The time series and portrait phase of model (4.1) when $\alpha = 0$. (**a**) is the time series of $S(t)$ and $x(t)$; (**b**) is the portrait phase.

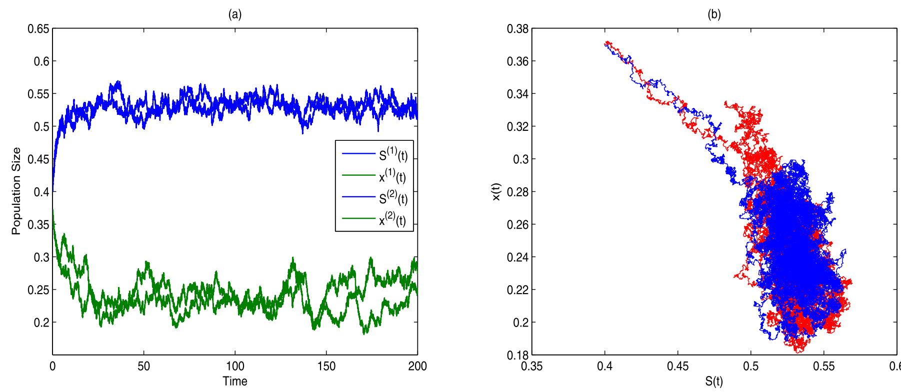

Figure 2 The time series and portrait phase of model (4.1) when $\alpha = 0.3$ ($S^{(1)}(t)$ and $S^{(2)}(t)$ are two sample paths about $S(t)$, $x^{(1)}(t)$ and $x^{(2)}(t)$ are two sample paths about $x(t)$.) (**a**) represents the time series of $S(t)$ and $x(t)$; (**b**) represents the portrait phase.

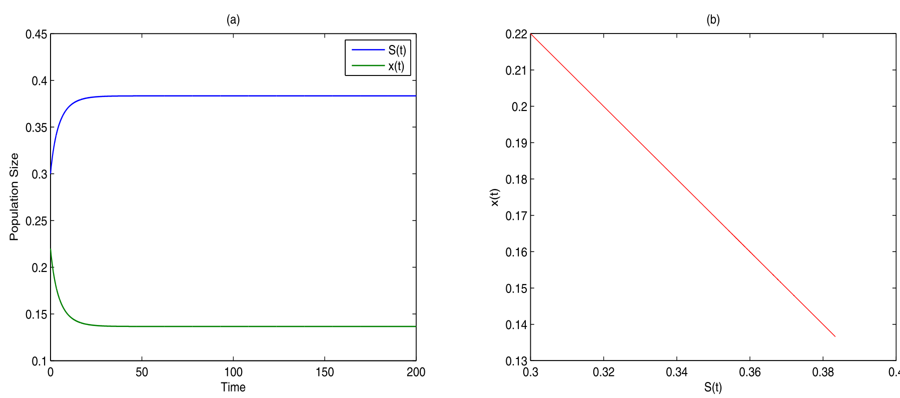

Figure 3 The time series and portrait phase of model (4.2) when $\alpha = 0$. (**a**) is the time series of $S(t)$ and $x(t)$; (**b**) is the portrait phase.

When $\alpha = 0$, system (4.2) is the corresponding deterministic model. The microorganism will be persistent as is depicted in Figure 3. When $\alpha = 0.3$, which means system (4.2) suffers stochastic destabilization from internal or external factors, the break-even concentration

$$\lambda_2 = \frac{(a + (S^0)^2)d}{mS^0} + \frac{\alpha^2 (S^0)^3}{m(a + (S^0)^2)} \approx 0.2724 < S^0 = 0.52.$$

Figure 4 provides the simulation of stochastic persistence for system (4.2).

Comparing Figure 1 (Figure 3) and Figure 2 (Figure 4), we find that the stochastic factors, such as the variation of environment, may cause sustained fluctuation for the microorganism, but the microorganism will also be persistent in the turbidostat because system (1.1) has a stationary distribution described in Theorem 3.2.

Computer simulation of the path of $(S(t), x(t))$ is provided with the initial value $(S(0), x(0)) = (0.4, 0.39)$. Let $S^0 = 0.79$, $d = 0.58$, $k = 0.2$, $m = 0.6$, $a = 1.1$, and choose $\alpha = 0$

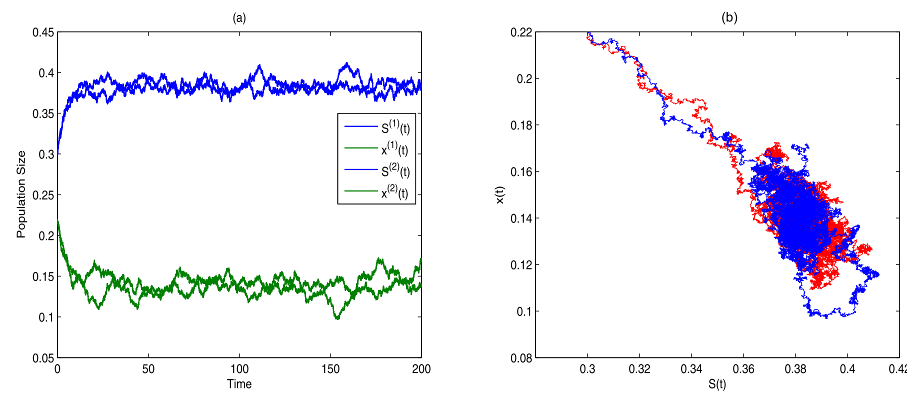

Figure 4 The time series and portrait phase of model (4.2) when $\alpha = 0.3$ ($S^{(1)}(t)$ and $S^{(2)}(t)$ are two sample paths about $S(t)$, $x^{(1)}(t)$ and $x^{(2)}(t)$ are two sample paths about $x(t)$.) (**a**) represents the time series of $S(t)$ and $x(t)$; (**b**) represents the portrait phase.

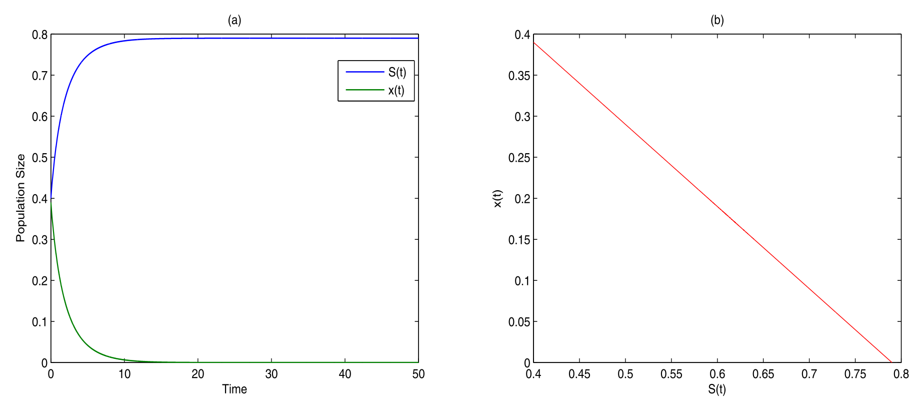

Figure 5 The time series and portrait phase of model (4.3) when $\alpha = 0$. (**a**) stands for the time series of $S(t)$ and $x(t)$; (**b**) stands for the portrait phase.

or $\alpha = 0.87$.

$$\begin{cases} dS(t) = [(0.79 - S(t))(0.58 + 0.2x(t)) - \frac{0.6S^2(t)x(t)}{1.1+S^2(t)}]\,dt - \frac{0.87S^2(t)x(t)}{1.1+S^2(t)}\,dB(t), \\ dx(t) = [\frac{0.6S^2(t)}{1.1+S^2(t)} - (0.58 + 0.2x(t))]x(t)\,dt + \frac{0.87S^2(t)x(t)}{1.1+S^2(t)}\,dB(t). \end{cases} \tag{4.3}$$

When $\alpha = 0$, system (4.3) is changed into the corresponding deterministic model. The microorganism $x(t)$ in the turbidostat will be extinct as the numerical simulation depicted in Figure 5. In addition, when $\alpha = 0.87$, system (4.3) has stochastic destabilization from internal or external factors and

$$\frac{\alpha^2}{4} + \frac{m(S^0)^2}{a + (S^0)^2} - d + kS^0 \approx -0.0156 < 0.$$

Therefore, in view of Theorem 3.3, the microorganism in the turbidostat will be extinct because of white noise and the feedback control of the turbidostat as is shown in Figure 6.

Comparing Figures 5 and 6, the microorganism in system (4.3) will be extinct both $\alpha = 0$ and $\alpha = 0.87$ because the turbidostat system has some negative feedback phenomenon

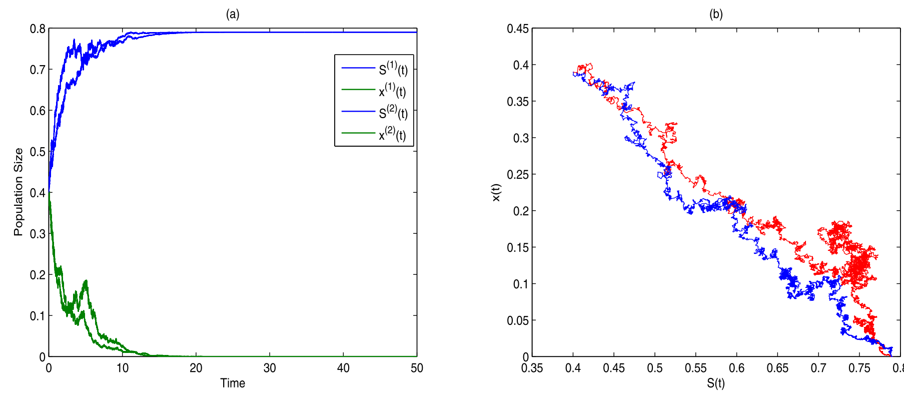

Figure 6 The time series and portrait phase of model (4.3) when $\alpha = 0.87$ ($S^{(1)}(t)$ and $S^{(2)}(t)$ are two sample paths about $S(t)$, $x^{(1)}(t)$ and $x^{(2)}(t)$ are two sample paths about $x(t)$.) (**a**) stands for the time series of $S(t)$ and $x(t)$; (**b**) stands for the portrait phase.

that might not help the persistence of microorganism. The stochastic system (when $\alpha = 0.87$) will also fluctuate with the deterministic system ($\alpha = 0$) because model (1.1) has a stationary distribution described in Theorem 3.2.

In view of Figure 2 (Figure 4) and Figure 6, the microorganism $x(t)$ in system (1.1) will be persistent or extinct under the condition of white noise ($\alpha \neq 0$), and both situations fluctuate with the deterministic model ($\alpha = 0$), which means the white noise has negative impact on the population dynamics. In addition, we can also find that the turbidostat system also has some negative effect on the population due to the feedback control phenomenon, which leads to the extinction of the population (see Figure 5).

For better explaining the white noise effects from a mathematical point of view, we rewrite the condition of Theorem 3.1 as

$$\alpha < \frac{1}{(S^0)^2}\sqrt{\left(a + \left(S^0\right)^2\right)\left(2m\left(S^0\right)^2 - d\left(a + \left(S^0\right)^2\right)\right)} := \alpha_0,$$

and change the condition of Theorem 3.2 into

$$\alpha < \sqrt{\frac{m(a + (S^0)^2)((S^0)^2 - d(a + (S^0)^2))}{(S^0)^3}} := \alpha_1.$$

If the intensity of white noise satisfies $\alpha < \alpha_0$ ($\alpha < \alpha_1$), then the destabilization will not cause the extinction but fluctuate with the deterministic model (see Figure 2(a) and Figure 4(a)).

For the condition of Theorem 3.3, if

$$\alpha < \sqrt{d - kS^0 - \frac{m(S^0)^2}{a + (S^0)^2}} := \alpha_2,$$

then the microorganism in both the deterministic model and the stochastic model will be extinct because of white noise and the feedback phenomenon of the turbidostat (see Figure 5(a) and Figure 6(a)).

To sum up all the analysis given above, we have investigated the dynamic behaviors of a turbidostat model with white noise. The importance of the conclusion in the realistic issue can be explained as follows. Since the stochastic perturbation is inevitable, it is reasonable to investigate the persistence of the stochastic system more than the stability of the deterministic model. Comparing the stochastic model (1.1) and the corresponding deterministic model ($\alpha = 0$ in model (1.1)) with numerical simulations, we find that stochastic phenomena, either the internal factors or the external phenomena, have negative effect on dynamical behaviors. To begin with, the break-even concentration of persistence for the stochastic model is larger than that for the deterministic model (when $\alpha = 0$). The condition of extinction is also larger than that in the deterministic model. Moreover, stochastic destabilization may cause the fluctuation centering on the value of deterministic model in the turbidostat as is depicted in Figure 1(a) and Figure 2(a), Figure 3(a) and Figure 4(a) and Figure 5(a) and Figure 6(a), which means the stochastic factors may affect the culture of microorganism in the turbidostat.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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