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Global analysis of a new nonlinear stochastic differential competition system with impulsive effect

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Abstract

We propose a new stochastic competition chemostat system with saturated growth rate and impulsive toxicant input. The main purpose of this paper is to study the stochastic dynamics of a high-dimensional impulsive stochastic chemostat model and find the threshold between persistence and extinction for the impulsive stochastic chemostat system. First, we investigate the stability of the periodic solution of a deterministic impulsive chemostat model and obtain the threshold between persistence and extinction for the system. Second, by using qualitative analysis method of impulsive stochastic differential equations, we obtain conditions for the extinction and persistence in mean of two microorganisms in the stochastic chemostat model. The results show that a stochastic disturbance or the impulsive effect can cause the microorganisms to go to extinction. Finally, we provide some examples together with numerical simulations to illustrate the analytical results and explain the biological implications.

MSC: 60H10; 65C30; 91B70

Keywords: stochastic chemostat system; impulsive toxicant input; Itô's formula; persistence in mean; nonlinear growth rate

1 Introduction

The chemostat is a kind of experimental device that can be used to cultivate microorganisms and plays an important role in many fields, such as microbiology, ecology, chemical engineering, and so on. Some analysis of a chemostat model and related results can be found in [1–6]. In addition, when microorganism individuals increase greatly, owing to the density-dependent population growth, the effect of saturation growth rate leads to a constant number of microorganism individuals. Comparing with bilinear growth rate, the saturated growth rate may be more suitable for many cases (see, e.g.,[7–9]).

Chemostat models have been applied to open natural environment [1, 2, 4, 7, 10–12]. Environmental pollution by industrial sewage or agricultural pesticides is one of the most serious social and ecological problems. The toxicant in the environment is a threat to the survival of the exposed microorganisms. Therefore, it is of great importance to investigate the effects of toxicant and obtain a theoretical threshold between the extinction and persistence of the microorganisms in a polluted environment [13–15]. In recent years, many works have been carried out to study the effects of toxicant on biological populations [16–18]. In the 1980s, some deterministic toxicant-population models were initially proposed by Hallam [16, 17]. From then on, many important and valuable deterministic models with toxicant effect were investigated by some scholars [14–17]. However, in the real world, a waste water with toxicant is always input impulsively, and the population system is inevitably affected by an impulsive toxicant input. Some authors have studied the effects of impulsive toxicant input on the persistence and extinction of microorganisms in a polluted environment [18–23].

Chemostat models are inevitably affected by the white noise stochastic disturbance; therefore the dynamics of a stochastic chemostat model may be different from that of a deterministic model. Some scholars have studied the dynamics behaviors of various kinds of stochastic systems and obtained many good results [24–42]. Recently, taking both impulsive toxicant input and white noises into account, persistence and extinction of a single-species population system in a polluted environment with random perturbations and impulsive toxicant input were explored [42, 43].

Recently, many scholars focus on the research of impulsive stochastic differential systems. Hence, the asymptotic stability of some impulsive stochastic differential systems were investigated, and many good results were obtained [44–49]. To capture essential features of the processes, the following several aspects should be considered in the formulation of chemostat models: (a) two-species competition for a limiting nutrient supplied at a constant rate; (b) impulsive toxicant input; (c) white noise stochastic disturbance; and (d) saturated growth rate. To our knowledge, there are only a few works that consider the qualitative analysis of high-dimensional impulsive stochastic chemostat competition models with saturated growth rate. Therefore, based on the four aspects, we propose a new competition model with white noise disturbance and impulsive toxicant input. For this new system, we explore the threshold between the extinction and persistence of two microorganisms and study the influences of impulsive toxicant input and stochastic disturbance on system dynamics. A deterministic chemostat competition model with saturated growth rate and pulsed toxicant input can be described by the following impulsive differential equation:

$$\begin{cases} \left.\begin{aligned} &\dot{S}(t) = Q(S_0 - S(t)) - \frac{\mu_1 S(t)x_1(t)}{\delta_1(a_1+x_1(t))} - \frac{\mu_2 S(t)x_2(t)}{\delta_2(a_2+x_2(t))}, \\ &\dot{x}_1(t) = \frac{\mu_1 S(t)x_1(t)}{a_1+x_1(t)} - Qx_1(t) - r_1c(t)x_1(t), \\ &\dot{x}_2(t) = \frac{\mu_2 S(t)x_2(t)}{a_2+x_2(t)} - Qx_2(t) - r_2c(t)x_2(t), \\ &\dot{c}(t) = -hc(t), \end{aligned}\right\} \quad t \neq n\tau, n \in Z^+, \\ S(n\tau^+) = S(n\tau), \qquad x_i(n\tau^+) = x_i(n\tau) \quad (i = 1,2), \\ c(n\tau^+) = c(n\tau) + u, \quad n \in Z^+, \end{cases} \tag{1}$$

where $S(t)$ denotes the concentration of the unconsumed nutrient at time t, $x_i(t)$ represents the concentration of the microorganism at time t ($i = 1,2$), $c(t)$ is the concentration of the toxicant in the chemostat at time t, S_0 and Q are the input concentration of the nutrient and the flow rate of the chemostat, respectively, μ_i is the maximal growth rate (or predation rate), a_i is the half-saturation constant ($i = 1,2$), r_i is the depletion rate coefficient of the microorganism population due to organismal pollutant concentration, δ_i is the yield of the microorganism $x_i(t)$ per unit mass of substrate ($i = 1,2$), h denotes the loss rate of toxicant in culture medium of the chemostat, u is the amount of toxicant pulsed each τ, where τ is the period of pulsing, and all the coefficients are positive constants. The function $\frac{\mu_i S(t)x_i(t)}{a_i+x_i(t)}$ represents saturated growth rate showing density effect of the microorganism population, which is different from $\frac{\mu_i S(t)x_i(t)}{a_i+S(t)}$ (see [6, 7, 9]).

Note that all parameters in system (1) can be affected by environmental noise, which always fluctuates around some average values. However, in this paper, we only consider the case that there is randomness involved in the maximal growth rate (or predation rate) μ_i, which is one of the crucial parameters, to the culture of microorganism. In this case, μ_i changes to a random variable $\mu_i + \sigma_i \dot{B}_i$, so that $\frac{\mu_i S(t)x_i(t)}{a_i+x_i(t)} \to \frac{\mu_i S(t)x_i(t)}{a_i+x_i(t)} + \frac{\sigma_i S(t)x_i(t)}{a_i+x_i(t)}\dot{B}_i(t)$, where $B_i(t)$ is a standard Brownian motion with intensity $\sigma_i^2 > 0$ ($i = 1,2$). Then a stochastic version can take the following form:

$$\begin{cases} \left.\begin{aligned} \mathrm{d}S(t) &= \Big(Q(S_0 - S(t)) - \tfrac{\mu_1 S(t)x_1(t)}{\delta_1(a_1+x_1(t))} - \tfrac{\mu_2 S(t)x_2(t)}{\delta_2(a_2+x_2(t))}\Big)\,\mathrm{d}t \\ &\quad - \tfrac{\sigma_1 S(t)x_1(t)}{\delta_1(a_1+x_1(t))}\,\mathrm{d}B_1(t) - \tfrac{\sigma_2 S(t)x_2(t)}{\delta_2(a_2+x_2(t))}\,\mathrm{d}B_2(t), \\ \mathrm{d}x_1(t) &= \Big(\tfrac{\mu_1 S(t)x_1(t)}{a_1+x_1(t)} - Qx_1(t) - r_1 c(t)x_1(t)\Big)\,\mathrm{d}t \\ &\quad + \tfrac{\sigma_1 S(t)x_1(t)}{a_1+x_1(t)}\,\mathrm{d}B_1(t), \\ \mathrm{d}x_2(t) &= \Big(\tfrac{\mu_2 S(t)x_2(t)}{a_2+x_2(t)} - Qx_2(t) - r_2 c(t)x_2(t)\Big)\,\mathrm{d}t \\ &\quad + \tfrac{\sigma_2 S(t)x_2(t)}{a_2+x_2(t)}\,\mathrm{d}B_2(t), \\ \mathrm{d}c(t) &= -hc(t)\,\mathrm{d}t, \end{aligned}\right\} \quad t \neq n\tau, n \in Z^+, \\ S(n\tau^+) = S(n\tau), \qquad x_i(n\tau^+) = x_i(n\tau) \quad (i = 1,2), \\ c(n\tau^+) = c(n\tau) + u, \quad n \in Z^+, \end{cases} \tag{2}$$

where σ_i is the environmental white noise disturbance coefficient ($i = 1,2$).

For convenience of description, we introduce the following definitions: $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous, whereas $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets); $B(t)$ is a scalar Brownian motion defined on this probability space; $S(t)$ and $x_i(t)$ are continuous at $t = n\tau$, $c(t)$ is left continuous at $t = n\tau$ and $c(n\tau^+) = \lim_{t\to n\tau^+} c(t)$; and $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(\theta)\,\mathrm{d}\theta$.

Next, we investigate the impulsive stochastic chemostat competition model with saturated growth response rates in a polluted environment. The main objective of this paper is to explore the extinction and persistence of a microorganism population and obtain the thresholds of the two chemostat models.

2 Deterministic system and auxiliary lemmas

For convenience of discussion, we introduce the following definition and some lemmas.

Definition 2.1 ([50, 51])

(i) The microorganisms $x_i(t)$ are said to be extinctive if $\lim_{t\to+\infty} x_i(t) = 0$ ($i = 1,2$) a.s.

(ii) The microorganisms $x_i(t)$ are said to be persistent if there exist positive constants λ_i such that $\liminf_{t\to+\infty} x_i(t) \geq \lambda_i$ ($i = 1,2$).

(iii) The microorganisms $x_i(t)$ are said to be persistent in the mean if there exist positive constants λ_i such that $\liminf_{t\to+\infty} \langle x_i(t)\rangle \geq \lambda_i$ ($i = 1,2$) a.s.

The subsystem of systems (1) and (2) is given by

$$\begin{cases} \mathrm{d}c(t) = -hc(t)\,\mathrm{d}t, & t \neq n\tau, n \in Z^+, \\ c(n\tau^+) = c(n\tau) + u, & n \in Z^+. \end{cases} \tag{3}$$

Lemma 2.1 [21, 22] *System* (3) *has a unique positive* τ*-periodic solution* $c^*(t)$ *and, for any solution* $c(t)$ *of* (3), $c(t) \to c^*(t)$ *as* $t \to +\infty$. *Moreover,* $c(t) > c^*(t)$ *for all* $t \geq 0$ *if* $c(0) > c^*(0)$, *where*

$$\begin{cases} c^*(t) = \frac{ue^{-h(t-n\tau)}}{1-e^{-h\tau}}, \\ c^*(0) = \frac{u}{1-e^{-h\tau}}, \end{cases} \tag{4}$$

for $t \in (n\tau, (n+1)\tau]$ *and* $n \in Z^+$.

Lemma 2.2 *For any positive solution* $(S(t), x_1(t), x_2(t), c(t))$ *of deterministic system* (1) *with initial value* $(S(0), x_1(0), x_2(0), c(0^+)) \in R^4_+$, *we have*

$$\limsup_{t\to+\infty} S(t) \leq S_0, \qquad \limsup_{t\to+\infty} x_1(t) \leq \delta_1 S_0,$$
$$\limsup_{t\to+\infty} x_2(t) \leq \delta_2 S_0, \qquad \lim_{t\to+\infty} \langle c(t) \rangle = \frac{u}{h\tau} \triangleq \bar{c}.$$

Proof From the first three equations of system (1) or (2), we have

$$\frac{\mathrm{d}(S(t) + \frac{1}{\delta_1}x_1(t) + \frac{1}{\delta_2}x_2(t))}{\mathrm{d}t} \leq QS_0 - Q\left(S(t) + \frac{1}{\delta}x_1(t) + \frac{1}{\delta_2}x_2(t)\right).$$

Thus we get

$$\lim_{t\to+\infty} \left(S(t) + \frac{1}{\delta_1}x_1(t) + \frac{1}{\delta_2}x_2(t)\right) \leq S_0.$$

Then

$$\limsup_{t\to+\infty} S(t) \leq S_0, \qquad \limsup_{t\to+\infty} x_i(t) \leq \delta_i S_0, \quad i = 1, 2.$$

By Lemma 2.1 we have

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t c(s)\,\mathrm{d}s = \lim_{t\to+\infty} \frac{1}{t}\int_0^t c^*(s)\,\mathrm{d}s = \frac{1}{\tau}\int_0^\tau c^*(t)\,\mathrm{d}t = \frac{u}{h\tau}.$$

The proof of Lemma 2.2 is completed. □

Similarly, we can obtain the same results for stochastic system (2), which is used in the following sections.

Define

$$\mathcal{R}_1 = \frac{\mu_1 S_0}{a_1(Q + \frac{r_1 u}{h\tau})}, \qquad \mathcal{R}_2 = \frac{\mu_2 S_0}{a_2(Q + \frac{r_2 u}{h\tau})}. \tag{5}$$

Lemma 2.3 *If $\mathcal{R}_1 < 1$ and $\mathcal{R}_2 < 1$, then system* (1) *has a unique stable 'microorganism-extinction' periodic solution $(S_0, 0, 0, c^*(t))$, which implies that the two microorganisms go extinct, whereas, if $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 > 1$, then the two microorganisms of system* (1) *are persistent.*

Lemma 2.3 is proved in the Appendix.

Remark 2.1 By Lemma 2.3, two thresholds $\mathcal{R}_1$ and $\mathcal{R}_2$ decide the persistence and extinction of the microorganisms that are related with the impulsive disturbance force, that is, the larger toxicant pulsed input u or the smaller period of pulsing τ can lead to the extinction of the microorganisms in the deterministic system (1) without white noise disturbance.

3 Dynamics of stochastic system

3.1 Extinction

In this section, we investigate the conditions for the extinction of the two microorganisms of system (2) under the stochastic white noise disturbance.

Lemma 3.1 *Let $(S(t), x_1(t), x_2(t), c(t))$ be a solution of system* (2) *with initial value $(S(0), x_1(0), x_2(0), c(0)) \in R^4_+$. Then*

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{\int_0^t \frac{\sigma_i S(\theta)}{a_i + x_i(\theta)}\,\mathrm{d}B_i(\theta)}{t} = 0,\\
&\lim_{t\to+\infty} \frac{\int_0^t \sigma_i S(\theta)\,\mathrm{d}B_i(\theta)}{t} = 0, \quad i = 1, 2, a.s.
\end{aligned}$$

Proof Let $Z(t) = \int_0^t \frac{\sigma_i S(\theta)}{a_i + x_i(\theta)}\,\mathrm{d}B_i(\theta)$ and $\xi > 2$. From Lemma 2.2 and Burkholder-Davis-Gundy inequality (see [52]) we have

$$\begin{aligned}
E\Big[\sup_{0\le\theta\le t} |Z(\theta)|^{\xi}\Big] &\le C_\xi E\bigg[\int_0^t \frac{\sigma_i^2 S^2(\theta)}{(a_i + x_i(\theta))^2}\,\mathrm{d}\theta\bigg]^{\frac{\xi}{2}}\\
&\le C_\xi t^{\frac{\xi}{2}} E\bigg[\sup_{0\le\theta\le t} \frac{\sigma_i^{\xi} S^{\xi}(\theta)}{(a_i + x_i(\theta))^{\xi}}\bigg]\\
&\le M_\xi C_\xi t^{\frac{\xi}{2}},
\end{aligned}$$

where $M_\xi = (\frac{S_0\sigma_i}{a_i})^{\xi}$. Let ε be an arbitrary positive constant. Then we can observe that

$$\begin{aligned}
\mathbb{P}\Big\{\omega : \sup_{k\delta\le t\le(k+1)\delta} |Z(t)|^{\xi} > (k\delta)^{1+\varepsilon+\frac{\xi}{2}}\Big\} &\le \frac{E(|Z((k+1)\delta)|^{\xi})}{(k\delta)^{1+\varepsilon+\frac{\xi}{2}}}\\
&\le \frac{M_\xi C_\xi [(k+1)\delta]^{\frac{\xi}{2}}}{(k\delta)^{1+\varepsilon+\frac{\xi}{2}}}\\
&\le \frac{2^{\frac{\xi}{2}} M_\xi C_\xi}{(k\delta)^{1+\varepsilon}}.
\end{aligned}$$

By the Borel-Cantelli lemma and Doob's martingale inequality (see [52]), for almost all $\omega \in \Omega$, we have that

$$\sup_{k\delta \le t \le (k+1)\delta} \left|Z(t)\right|^{\xi} \le (k\delta)^{1+\varepsilon+\frac{\xi}{2}} \tag{6}$$

for all but finitely many k. Thus, there exists a positive $k_0(\omega)$ such that, for almost all $\omega \in \Omega$, (6) holds when $k \ge k_0(\omega)$. Hence, if $k \ge k_0(\omega)$ and $k\delta \le t \le (k+1)\delta$, then, for almost all $\omega \in \Omega$,

$$\frac{\ln |Z(t)|^{\xi}}{\ln t} \le \frac{(1+\varepsilon+\frac{\xi}{2})\ln(k\delta)}{\ln(k\delta)} = 1+\varepsilon+\frac{\xi}{2}.$$

Thus we have

$$\limsup_{t\to+\infty} \frac{\ln |Z(t)|}{\ln t} \le \frac{1+\varepsilon+\frac{\xi}{2}}{\xi}.$$

Letting $\varepsilon \to 0$, we obtain

$$\limsup_{t\to+\infty} \frac{\ln |Z(t)|}{\ln t} \le \frac{1}{2}+\frac{1}{\xi}.$$

Then, for an arbitrary small positive constant ϵ ($\epsilon < \frac{1}{2}-\frac{1}{\xi}$), there exist a constant $T(\omega)$ and a set Ω_ϵ such that $\mathbb{P}(\Omega_\epsilon) \ge 1-\epsilon$ and, for $t \ge T(\omega)$, $\omega \in \Omega_\epsilon$,

$$\ln\left|Z(t)\right| \le \left(\frac{1}{2}+\frac{1}{\xi}+\epsilon\right)\ln t.$$

Therefore,

$$\limsup_{t\to+\infty} \frac{Z(t)}{t} \le \limsup_{t\to+\infty} \frac{t^{\frac{1}{2}+\frac{1}{\xi}+\epsilon}}{t} = 0.$$

Note that

$$\liminf_{t\to+\infty} \frac{|Z(t)|}{t} \ge 0.$$

Then we have

$$\lim_{t\to+\infty} \frac{|Z(t)|}{t} = 0 \quad \text{a.s.}$$

that is,

$$\lim_{t\to+\infty} \frac{Z(t)}{t} = \lim_{t\to+\infty} \frac{\int_0^t \frac{\sigma_i S(\theta)}{a_i+x_i(\theta)}\,\mathrm{d}B_i(\theta)}{t} = 0 \quad \text{a.s.}$$

Similarly, we can obtain

$$\lim_{t\to+\infty} \frac{\int_0^t \sigma_i S(\theta)\,\mathrm{d}B_i(\theta)}{t} = 0, \quad i = 1, 2, \text{a.s.}$$

This completes the proof of Lemma 3.1. □

Define

$$\mathcal{R}_1^* = \frac{\mu_1 S_0}{a_1(Q+\frac{r_1 u}{h\tau})} - \frac{\sigma_1^2 S_0^2}{2a_1^2(Q+\frac{r_1 u}{h\tau})} = \mathcal{R}_1 - \frac{\sigma_1^2 S_0^2}{2a_1^2(Q+\frac{r_1 u}{h\tau})},$$
$$\mathcal{R}_2^* = \frac{\mu_2 S_0}{a_2(Q+\frac{r_2 u}{h\tau})} - \frac{\sigma_2^2 S_0^2}{2a_2^2(Q+\frac{r_2 u}{h\tau})} = \mathcal{R}_2 - \frac{\sigma_2^2 S_0^2}{2a_2^2(Q+\frac{r_2 u}{h\tau})},$$

where $\mathcal{R}_1$, $\mathcal{R}_2$ are the thresholds of the deterministic system (1) given in (5).

Theorem 3.1 *Let* $(S(t),x_1(t),x_2(t),c(t))$ *be the solution of system* (2) *with initial value* $(S(0),x_1(0),x_2(0),c(0)) \in R_+^4$. *If* (i) $\sigma_i > \frac{\mu_i}{\sqrt{2(Q+\frac{r_i u}{h\tau})}}$ *for* $i=1,2$ *or* (ii) $\mathcal{R}_i^* < 1$ *and* $\sigma_i \le \sqrt{\frac{a_i\mu_i}{S_0}}$ *for* $i=1,2$, *then the two microorganisms of system* (2) *go to extinction almost surely, that is,* $\lim_{t\to+\infty} x_i(t) = 0$ $(i=1,2)$ *a.s.; moreover,* $\lim_{t\to+\infty} S(t) = S_0$ *a.s. and* $\lim_{t\to+\infty} c(t) = c^*(t)$ *for* $t\in(n\tau,(n+1)\tau]$ *and* $n\in Z^+$.

Proof Applying Itô's formula to system (2) leads to

$$\begin{aligned} \mathrm{d}\ln x_i(t) &= \left(\frac{\mu_i S(t)}{a_i+x_i(t)} - Q - r_i c(t) - \frac{\sigma_i^2 S^2(t)}{2(a_i+x_i(t))^2}\right)\mathrm{d}t \\ &\quad + \frac{\sigma_i S(t)}{a_i+x_i(t)}\,\mathrm{d}B_i(t), \quad i=1,2. \end{aligned} \tag{7}$$

Case (i). Integrating both sides of (7) from 0 to t results in

$$\begin{aligned} \ln x_i(t) &= -\frac{\sigma_i^2}{2}\int_0^t \left(\frac{\mu_i}{\sigma_i^2} - \frac{S(t)}{a_i+x_i(t)}\right)^2 \mathrm{d}t - Qt - r_i\int_0^t c(\theta)\,\mathrm{d}\theta + \frac{\mu_i^2}{2\sigma_i^2}t + M_i(t) + \ln x_i(0) \\ &\le -Qt - r_i\int_0^t c(\theta)\,\mathrm{d}\theta + \frac{\mu_i^2}{2\sigma_i^2}t + M_i(t) + \ln x_i(0), \end{aligned} \tag{8}$$

where $M_i(t) = \int_0^t \frac{\sigma_i S(\theta)}{a_i+x_i(\theta)}\,\mathrm{d}B_i(\theta)$, $i=1,2$. Dividing both sides of (8) by t, we observe that

$$\frac{\ln x_i(t)}{t} \le -\left(Q + r_i\langle c(t)\rangle - \frac{\mu_i^2}{2\sigma_i^2}\right) + \frac{M_i(t)}{t} + \frac{\ln x_i(0)}{t}. \tag{9}$$

The process $M_i(t)$ $(i=1,2)$ is a local continuous martingale with $M_i(0)=0$, and from Lemma 3.1 we have

$$\lim_{t\to+\infty}\frac{M_i(t)}{t} = 0, \quad i=1,2, \text{a.s.}$$

Since $\sigma_i > \frac{\mu_i}{\sqrt{2(Q+\frac{r_i u}{h\tau})}}$ for $i=1,2$, we have $-(Q + r_i\langle c(t)\rangle - \frac{\mu_i^2}{2\sigma_i^2}) < 0$.

Taking the limit superior of both sides of (9), we can observe that

$$\limsup_{t\to+\infty}\frac{\ln x_i(t)}{t} \le -\left(Q + r_i\langle c(t)\rangle - \frac{\mu_i^2}{2\sigma_i^2}\right) < 0 \quad \text{a.s.},$$

which implies $\lim_{t\to+\infty} x_i(t) = 0$, $i=1,2$, a.s.

Case (ii). Integrating both sides of (7) from 0 to t and dividing by t lead to

$$\begin{aligned}\frac{\ln x_i(t)}{t} &= \frac{1}{t}\int_0^t \left(\frac{\mu_i S(\theta)}{a_i + x_i(\theta)} - Q - r_i c(\theta) - \frac{\sigma_i^2 S^2(\theta)}{2(a_i + x_i(\theta))^2}\right) \mathrm{d}\theta + \frac{M_i(t)}{t} + \frac{\ln x_i(0)}{t} \\ &\le \left(\frac{\mu_i S_0}{a_i} - \bigl(Q + r_i\langle c(t)\rangle\bigr) - \frac{\sigma_i^2 S_0^2}{2a_i^2}\right) + \frac{M_i(t)}{t} + \frac{\ln x_i(0)}{t} \\ &= \bigl(Q + r_i\langle c(t)\rangle\bigr)\left(\frac{\mu_i S_0}{a_i(Q + r_i\langle c(t)\rangle)} - \frac{\sigma_i^2 S_0^2}{2a_i^2(Q + r_i\langle c(t)\rangle)} - 1\right) \\ &\quad + \frac{M_i(t)}{t} + \frac{\ln x_i(0)}{t}.\end{aligned} \tag{10}$$

Taking the limit superior of both sides of (10) yields

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{t} \le (Q + r_i\bar{c})\bigl(\mathcal{R}_i^* - 1\bigr) < 0 \quad \text{a.s.,}$$

which means $\lim_{t\to+\infty} x_i(t) = 0$ a.s.

Without loss of generality, we may assume that $0 < x_i(t) < \varepsilon_i$ $(i = 1, 2)$ for an arbitrarily small positive quantity ε_i and all $t \ge 0$. By the first equation of system (2) we have

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} \ge Q\bigl(S_0 - S(t)\bigr) - \left(\frac{u_1\varepsilon_1}{\delta_1 a_1} + \frac{u_2\varepsilon_2}{\delta_2 a_2} + \frac{\sigma_1\varepsilon_1}{\delta_1 a_1}\bigl|\dot{B}_1(t)\bigr| + \frac{\sigma_2\varepsilon_2}{\delta_2 a_2}\bigl|\dot{B}_2(t)\bigr|\right)S(t). \tag{11}$$

As $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, taking the limit inferior of both sides of (11) gives

$$\liminf_{t\to+\infty} S(t) \ge S_0 \quad \text{a.s.} \tag{12}$$

By the proof of Lemma 2.2 we have

$$\lim_{t\to+\infty} S(t) \le S_0 + \varepsilon_1 + \varepsilon_2 \quad \text{a.s.}$$

Then, letting $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, we have

$$\limsup_{t\to+\infty} S(t) \le S_0 \quad \text{a.s.} \tag{13}$$

From (12) and (13) we have

$$\lim_{t\to+\infty} S(t) = S_0 \quad \text{a.s.}$$

From (3) and Lemma 2.1 we can observe that

$$\lim_{t\to+\infty} c(t) = c^*(t)$$

for $t \in (n\tau, (n+1)\tau]$ and $n \in Z^+$. □

Remark 3.1 Theorem 3.1 shows that the two microorganisms will die out if the white noise disturbance is large or $\mathcal{R}_i^* < 1$ and the white noise disturbance is not too large. Note that the expression of $\mathcal{R}_i^*$ is a difference compared with two thresholds of system (1), $\mathcal{R}_i$. This implies that the conditions for $x_i(t)$ to go to extinction in the deterministic system (1) are stronger than in the corresponding stochastic model (2).

3.2 Persistence in mean

Theorem 3.2 *Let* $(S(t),x_1(t),x_2(t),c(t))$ *be a solution of system* (2) *with initial value* $(S(0),x_1(0),x_2(0),c(0))\in R_+^4$.

(i) *If* $\mathcal{R}_1^*>1$, $\mathcal{R}_2^*<1$, *and* $\sigma_2\le\sqrt{\frac{a_2\mu_2}{S_0}}$, *then the microorganism* x_2 *dies out, and the microorganism* x_1 *is persistent in mean; moreover,* x_1 *satisfies*

$$\liminf_{t\to+\infty}\langle x_1(t)\rangle\ge\frac{a_1\delta_1Q(Q+r_1\overline{c})}{(\mu_1+\delta_1Q)(Q+r_1c^*(0))}\bigl(\mathcal{R}_1^*-1\bigr)\quad a.s.$$

(ii) *If* $\mathcal{R}_2^*>1$, $\mathcal{R}_1^*<1$, *and* $\sigma_1\le\sqrt{\frac{a_1\mu_1}{S_0}}$, *then the microorganism* x_1 *dies out, and the microorganism* x_2 *is persistent in mean; moreover,* x_2 *satisfies*

$$\liminf_{t\to+\infty}\langle x_2(t)\rangle\ge\frac{a_2\delta_2Q(Q+r_2\overline{c})}{(\mu_2+\delta_2Q)(Q+r_2c^*(0))}\bigl(\mathcal{R}_2^*-1\bigr)\quad a.s.$$

(iii) *If* $\mathcal{R}_1^*>1$ *and* $\mathcal{R}_2^*>1$, *then the two microorganisms* x_1 *and* x_2 *are persistent in mean; moreover,* x_1 *and* x_2 *satisfy*

$$\liminf_{t\to+\infty}\langle x_1(t)+x_2(t)\rangle\ge\frac{1}{\Delta_{\max}}\sum_{i=1}^{2}a_i(Q+r_i\overline{c})\bigl(\mathcal{R}_i^*-1\bigr)\quad a.s.,$$

where

$$\Delta_{\max}=\max\left\{\bigl(Q+r_1c^*(0)\bigr)\left(\frac{\mu_1+\mu_2}{\delta_1Q}+1\right),\bigl(Q+r_2c^*(0)\bigr)\left(\frac{\mu_1+\mu_2}{\delta_2Q}+1\right)\right\}.$$

Proof Case (i). By Theorem 3.1, since $\mathcal{R}_2^*<1$ and $\sigma_2\le\sqrt{\frac{a_2\mu_2}{S_0}}$, we have $\lim_{t\to+\infty}x_2(t)=0$ a.s. Since $\mathcal{R}_1^*>1$, we have that, for ε small enough such that $0<x_2(t)<\varepsilon$ for all t large enough,

$$\frac{\mu_1(S_0-(\frac{Q+r_2c^*(0)}{\delta_3Q}\varepsilon))}{a_1(Q+r\overline{c})}-\frac{\sigma_1^2S_0^2}{2a_1^2(Q+r_1\overline{c})}>1\quad\text{a.s.}$$

Integrating both sides of system (2) from 0 to t and dividing by t yield

$$\begin{aligned}\Theta(t)&\triangleq\frac{S(t)-S(0)}{t}+\frac{1}{\delta_1}\frac{x_1(t)-x_1(0)}{t}+\frac{1}{\delta_2}\frac{x_2(t)-x_2(0)}{t}\\&\ge QS_0-Q\langle S(t)\rangle-\left(\frac{Q+r_1c^*(0)}{\delta_1}\right)\langle x_1(t)\rangle-\left(\frac{Q+r_2c^*(0)}{\delta_2}\right)\langle x_2(t)\rangle\\&\ge QS_0-Q\langle S(t)\rangle-\left(\frac{Q+r_1c^*(0)}{\delta_1}\right)\langle x_1(t)\rangle-\left(\frac{Q+r_2c^*(0)}{\delta_2}\right)\varepsilon.\end{aligned}$$

Then we get

$$\langle S(t)\rangle\ge\left(S_0-\left(\frac{Q+r_2c^*(0)}{\delta_2Q}\right)\varepsilon\right)-\left(\frac{Q+r_1c^*(0)}{\delta_1Q}\right)\langle x_1(t)\rangle-\frac{\Theta(t)}{Q}.\tag{14}$$

Applying Itô's formula to system (2) leads to

$$\begin{aligned}
&\mathrm{d}\big(a_1 \ln x_1(t) + x_1(t)\big) \\
&\quad = \left(\mu_1 S(t) - a_1\big(Q + r_1 c(t)\big) - \big(Q + r_1 c(t)\big)x_1(t) - \frac{a_1 \sigma_1^2 S^2(t)}{2(a_1 + x_1(t))^2}\right)\mathrm{d}t + \sigma_1 S(t)\,\mathrm{d}B_1(t) \\
&\quad \geq \left(\mu_1 S(t) - a_1\big(Q + r_1 c(t)\big) - \big(Q + r_1 c^*(0)\big)x_1(t) - \frac{\sigma_1^2 S_0^2}{2a_1}\right)\mathrm{d}t + \sigma_1 S(t)\,\mathrm{d}B_1(t). \qquad (15)
\end{aligned}$$

Integrating on both sides of (15) from 0 to t and dividing by t yield

$$\begin{aligned}
&\frac{a_1(\ln x_1(t) - \ln x_1(0))}{t} + \frac{x_1(t) - x_1(0)}{t} \\
&\quad \geq \mu_1\langle S(t)\rangle - a_1\big(Q + r_1\langle c(t)\rangle\big) - \big(Q + r_1 c^*(0)\big)\langle x_1(t)\rangle - \frac{\sigma_1^2 S_0^2}{2a_1} + \frac{M_1(t)}{t} \\
&\quad \geq \mu_1\left(S_0 - \left(\frac{Q + r_2 c^*(0)}{\delta_2 Q}\right)\varepsilon\right) - a_1\big(Q + r_1\langle c(t)\rangle\big) - \frac{\sigma_1^2 S_0^2}{2a_1} \\
&\qquad - \left(\frac{\mu_1(Q + r_1 c^*(0))}{\delta_1 Q} + \big(Q + r_1 c^*(0)\big)\right)\langle x_1(t)\rangle - \frac{\mu_1 \Theta(t)}{Q} + \frac{M_1(t)}{t} \\
&\quad = a_1\big(Q + r_1\langle c(t)\rangle\big)\left(\frac{\mu_1(S_0 - \frac{Q + r_2 c^*(0)}{\delta_2 Q}\varepsilon)}{a_1(Q + r_1\langle c(t)\rangle)} - \frac{\sigma_1^2 S_0^2}{2a_1(Q + r_1\langle c(t)\rangle)} - 1\right) \\
&\qquad - \left(\frac{\mu_1(Q + r_1 c^*(0))}{\delta_1 Q} + \big(Q + r_1 c^*(0)\big)\right)\langle x_1(t)\rangle - \frac{\mu_1 \Theta(t)}{Q} + \frac{M_1(t)}{t}, \qquad (16)
\end{aligned}$$

where $M_1(t) = \int_0^t \sigma_1 S(\theta)\,\mathrm{d}B_1(\theta)$. Inequality (16) can be rewritten as

$$\begin{aligned}
\langle x_1(t)\rangle &\geq \frac{1}{\Delta}\Bigg[a_1\big(Q + r_1\langle c(t)\rangle\big)\left(\frac{\mu_1(S_0 - \frac{Q + r_2 c^*(0)}{\delta_2 Q}\varepsilon)}{a_1(Q + r_1\langle c(t)\rangle)} - \frac{\sigma_1^2 S_0^2}{2a_1^2(Q + r_1\langle c(t)\rangle)} - 1\right) \\
&\qquad - \frac{\mu_1 \Theta(t)}{Q} + \frac{M_1(t)}{t} - \left(\frac{a_1(\ln x_1(t) - \ln x_1(0))}{t} + \frac{x_1(t) - x_1(0)}{t}\right)\Bigg] \\
&\geq \begin{cases}
\frac{1}{\Delta}[a_1(Q + r_1\langle c(t)\rangle)(\frac{\mu_1(S_0 - \frac{Q + r_2 c^*(0)}{\delta_2 Q}\varepsilon)}{a_1(Q + r_1\langle c(t)\rangle)} - \frac{\sigma_1^2 S_0^2}{2a_1^2(Q + r_1\langle c(t)\rangle)} - 1) \\
\quad - \frac{\mu_1 \Theta(t)}{Q} + \frac{M_1(t)}{t} + \frac{a_1 \ln x_1(0)}{t} - \frac{x_1(t) - x_1(0)}{t}], \quad 0 < x_1(t) < 1; \\
\frac{1}{\Delta}[a_1(Q + r_1\langle c(t)\rangle)(\frac{\mu_1(S_0 - \frac{Q + r_2 c^*(0)}{\delta_2 Q}\varepsilon)}{a_1(Q + r_1\langle c(t)\rangle)} - \frac{\sigma_1^2 S_0^2}{2a_1^2(Q + r_1\langle c(t)\rangle)} - 1) \\
\quad - \frac{\mu_1 \Theta(t)}{Q} + \frac{M_1(t)}{t} - \frac{a_1(\ln x_1(t) - \ln x_1(0))}{t} - \frac{x_1(t) - x_1(0)}{t}], \quad 1 \leq x_1(t),
\end{cases} \qquad (17)
\end{aligned}$$

where $\Delta = \frac{(Q + r_1 c^*(0))(\mu_1 + \delta_1 Q)}{\delta_1 Q}$.

By Lemma 3.1 we have $\lim_{t\to+\infty} \frac{M_1(t)}{t} = 0$ a.s. According to Lemma 2.2, we can find that $x_1(t) \leq \delta_1 S_0$. Thus we have $\lim_{t\to+\infty} \frac{x_1(t)}{t} = 0$ and $\lim_{t\to+\infty} \frac{\ln x_1(t)}{t} = 0$ a.s. as $x_1(t) \geq 1$ and $\lim_{t\to+\infty} \Theta(t) = 0$ a.s. Taking the limit inferior of both sides of (17) results in

$$\begin{aligned}
\liminf_{t\to+\infty}\langle x_1(t)\rangle &\geq \frac{a_1(Q + r_1\overline{c})}{\Delta}\left[\frac{\mu_1 S_0}{a_1(Q + r_1\overline{c})} - \frac{\sigma_1^2 S_0^2}{2a_1^2(Q + r_1\overline{c})} - 1\right] \\
&= \frac{a_1 \delta_1 Q(Q + r_1\overline{c})}{(\mu_1 + \delta_1 Q)(Q + r_1 c^*(0))}\big(\mathcal{R}_1^* - 1\big) > 0.
\end{aligned}$$

Similarly, we can prove Case (ii), and we omit it here.

Case (iii). Note that

$$\langle S(t)\rangle = S_0 - \frac{Q + r_2\langle c(t)\rangle}{\delta_2 Q}\langle x_2(t)\rangle - \frac{Q + r_1\langle c(t)\rangle}{\delta_1 Q}\langle x_1(t)\rangle - \frac{\Theta(t)}{Q}. \tag{18}$$

Define

$$V(t) = \ln\bigl[x_1^{a_1}(t)x_2^{a_2}(t)\bigr] + \bigl[x_1(t) + x_2(t)\bigr].$$

Note that $V(t)$ is bounded. Then we have

$$\begin{aligned} D^+V(t) &= \left[(\mu_1 + \mu_2)S(t) - \sum_{i=1}^{2}\bigl(Q + r_i c(t)\bigr)\bigl(a_i + x_i(t)\bigr) - \sum_{i=1}^{2}\frac{a_i\sigma_i^2 S^2(t)}{2(a_i + x_i(t))^2}\right]\mathrm{d}t \\ &\quad + \sum_{i=1}^{2}\sigma_i S(t)\,\mathrm{d}B_i(t) \\ &\geq \left[(\mu_1 + \mu_2)S(t) - \sum_{i=1}^{2}a_i\bigl(Q + r_i c(t)\bigr) - \sum_{i=1}^{2}x_i(t)\bigl(Q + r_i c^*(0)\bigr) - \sum_{i=1}^{2}\frac{\sigma_i^2 S_0^2}{2a_i}\right]\mathrm{d}t \\ &\quad + \sum_{i=1}^{2}\sigma_i S(t)\,\mathrm{d}B_i(t). \end{aligned} \tag{19}$$

Integrating both sides of (19) from 0 to t and dividing by t yield

$$\begin{aligned} &\frac{V(t)}{t} - \frac{V(0)}{t} \\ &\quad\geq (\mu_1 + \mu_2)\langle S(t)\rangle - \sum_{i=1}^{2}a_i\bigl(Q + r_i\langle c(t)\rangle\bigr) - \sum_{i=1}^{2}\langle x_i(t)\rangle\bigl(Q + r_i c^*(0)\bigr) \\ &\qquad - \sum_{i=1}^{2}\frac{\sigma_i^2 S_0^2(t)}{2a_i} + \sum_{i=1}^{2}\frac{M_i}{t} \\ &\quad= (\mu_1 + \mu_2)S_0 - \sum_{i=1}^{2}\bigl(Q + r_i\langle c(t)\rangle\bigr)a_i - \sum_{i=1}^{2}\frac{\sigma_i^2 S_0}{2a_i} \\ &\qquad - \sum_{i=1}^{2}\left[\frac{(\mu_1 + \mu_2)(Q + r_i\langle c(t)\rangle)}{\delta_i Q} + \bigl(Q + r_i c^*(0)\bigr)\right]\langle x_i(t)\rangle \\ &\qquad - \frac{\mu_1 + \mu_2}{Q}\Theta(t) + \sum_{i=1}^{2}\frac{M_i}{t} \\ &\quad\geq (\mu_1 + \mu_2)S_0 - \sum_{i=1}^{2}\bigl(Q + r_i\langle c(t)\rangle\bigr)a_i - \sum_{i=1}^{2}\frac{\sigma_i^2 S_0}{2a_i} - \Delta_{\max}\bigl[\langle x_1(t)\rangle + \langle x_2(t)\rangle\bigr] \\ &\qquad - \frac{\mu_1 + \mu_2}{Q}\Theta(t) + \sum_{i=1}^{2}\frac{M_i}{t}, \end{aligned} \tag{20}$$

where $M_i(t) = \int_0^t \sigma_i S(\theta)\,\mathrm{d}B_i(\theta)$. Inequality (20) can be rewritten as

$$\begin{aligned}\langle x_1(t)\rangle + \langle x_2(t)\rangle \geq \frac{1}{\Delta_{\max}}\Bigg[&(\mu_1+\mu_2)S_0 - \sum_{i=1}^{2}\big(Q + r_i\langle c(t)\rangle\big)a_i - \sum_{i=1}^{2}\frac{\sigma_i^2 S_0}{2a_i}\\ &-\frac{\mu_1+\mu_2}{Q}\Theta(t) - \frac{V(t)}{t} + \frac{V(0)}{t} + \sum_{i=1}^{2}\frac{M_i}{t}\Bigg].\end{aligned} \tag{21}$$

Since $0 < S \leq S_0$, we have

$$\limsup_{t\to+\infty}\frac{\langle M_i(t), M_i(t)\rangle_t}{t} \leq \sigma^2 S_0^2 < \infty \quad \text{a.s.}$$

By Lemma 3.1 we observe that $\lim_{t\to+\infty}\frac{M_i(t)}{t} = 0$ a.s. for $i = 1, 2$. According to Lemma 2.2, we have $\lim_{t\to+\infty}\Theta(t) = 0$ and $\lim_{t\to+\infty}\frac{V(t)}{t} = 0$.

Taking the limit inferior of both sides of (21) yields

$$\liminf_{t\to+\infty}\langle x_1(t) + x_2(t)\rangle \geq \frac{1}{\Delta_{\max}}\sum_{i=1}^{2}a_i(Q + r_i\bar{c})\big(\mathcal{R}_i^* - 1\big) > 0 \quad \text{a.s.}$$

This completes the proof of Theorem 3.2. □

Remark 3.2 Theorem 3.2 shows that the two microorganisms will be persistent if the white noise disturbances are small enough such that $\mathcal{R}_i^* > 1$; conversely, if the white noise disturbances are large enough, then the two microorganisms will go to extinction. This implies that the stochastic disturbance may cause the populations to die out.

4 Conclusion and simulations

In this paper, we investigate the dynamics of an impulsive stochastic competition chemostat model with saturated growth rate in a polluted environment. We obtain sufficient conditions for extinction and persistence of both deterministic and stochastic systems. From the expressions of the thresholds of the stochastic system (2) we can observe that $\mathcal{R}_i^* < \mathcal{R}_i$, $i = 1, 2$, which means that the conditions for those two microorganisms to die out in the deterministic model (1) are stronger than those in the corresponding stochastic system (2). This implies that a persistent deterministic system may become extinct in the case of white noise stochastic disturbance.

On one hand, [44–49] investigated the asymptotic stability of some impulsive stochastic differential systems and obtained many good results. On the other hand, [53, 54] investigated qualitative properties for persistence and extinction of one-dimensional impulsive stochastic single-species population models. Based on the works [53, 54], we consider the qualitative analysis of the high-dimensional impulsive stochastic multi-species population model, which leads to a more complex and difficult stochastic analysis. Moreover, we use impulsive stochastic inequality technique to discuss the question according to three different cases. The main aim of the paper is to study the stochastic dynamics of the high-dimensional impulsive stochastic chemostat model and find the threshold between persistence and extinction of the microorganisms. In a sense, we improve and develop the theoretical method in [53, 54].

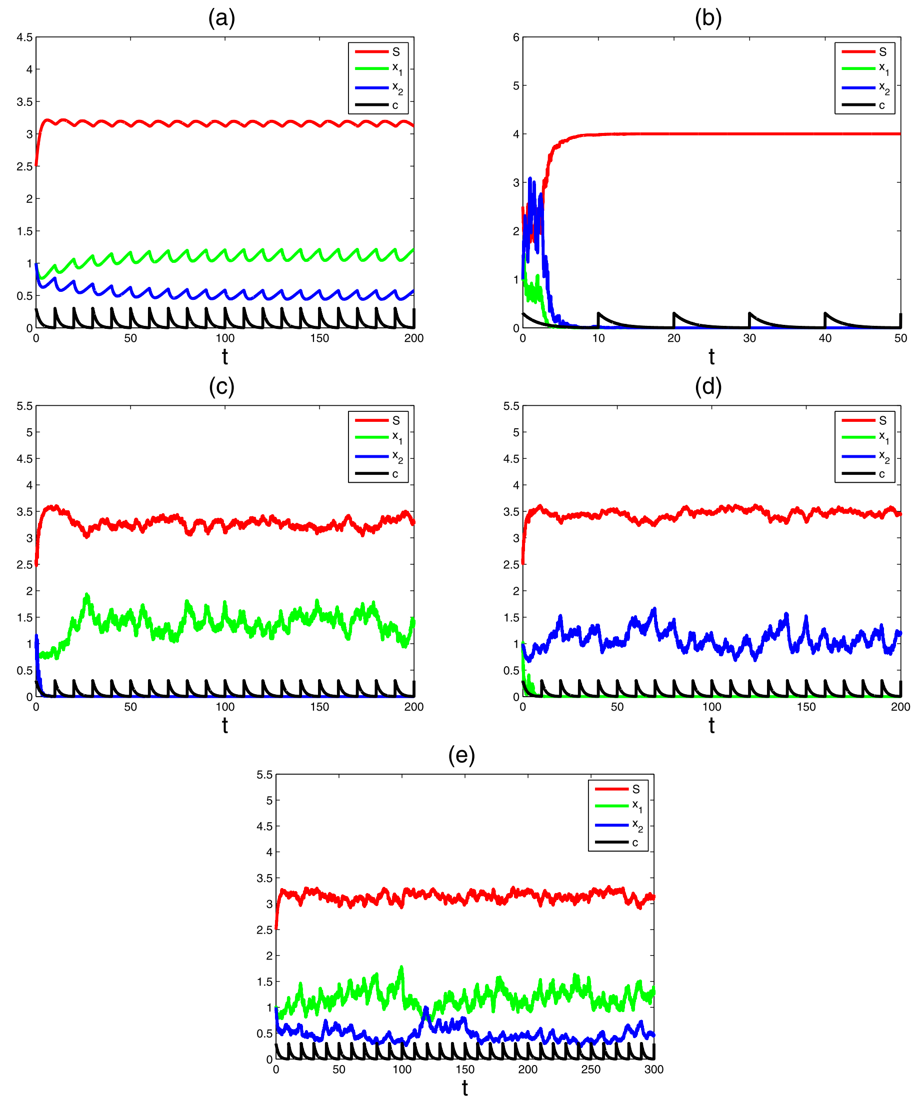

Figure 1 Computer simulation of the paths $S(t)$, $x_1(t)$, $x_2(t)$, $c(t)$ for the deterministic chemostat model (1) and the stochastic chemostat model (2) with parameters $S_0 = 4$, $Q = 0.5$, $r_1 = 0.5$, $r_2 = 0.9$, $\delta_1 = 2$, $\delta_2 = 2.2$, $a_1 = 15$, $a_2 = 7.5$, $\mu_1 = 2.7$, $\mu_2 = 1.4$, $h = 0.5$, $u = 0.3$, $\tau = 10$ and the initial values $S(0) = 2.5$, $x_1(0) = 1$, $x_2(0) = 1$, $c(0) = 0.3$. (a) Time series for $S(t)$, $X_1(t)$, $X_2(t)$, $c(t)$ with parameters $\sigma_1 = 0$, $\sigma_2 = 0$. **(b)** Time series for $S(t)$, $X_1(t)$, $X_2(t)$, $c(t)$ with parameters $\sigma_1 = 2.4$, $\sigma_2 = 1.2$. **(c)** Time series for $S(t)$, $X_1(t)$, $X_2(t)$, $c(t)$ with parameters $\sigma_1 = 0.2$, $\sigma_2 = 1.2$. **(d)** Time series for $S(t)$, $X_1(t)$, $X_2(t)$, $c(t)$ with parameters $\sigma_1 = 2.4$, $\sigma_2 = 0.1$. **(e)** Time series for $S(t)$, $X_1(t)$, $X_2(t)$, $c(t)$ with parameters $\sigma_1 = 0.2$, $\sigma_2 = 0.1$.

Next, we employ the Euler method to simulate the dynamics of the deterministic and stochastic systems to support our theoretical results. We choose some parameters in systems (1) and (2) as follows: $S_0 = 4$, $Q = 0.5$, $r_1 = 0.5$, $r_2 = 0.9$, $\delta_1 = 2$, $\delta_2 = 2.2$, $a_1 = 15$, $a_2 = 7.5$, $\mu_1 = 2.7$, $\mu_2 = 1.4$, $h = 0.5$, $u = 0.3$, $\tau = 10$, and the initial values are $S(0) = 2.5$, $x_1(0) = 1$, $x_2(0) = 1$, $c(0) = 0.3$.

In Figure 1, we can see that

(a) $\sigma_1 = 0$, $\sigma_2 = 0$, $\mathcal{R}_1 = 1.3585 > 1$, $\mathcal{R}_2 = 1.3478 > 1$;

(b) $\sigma_1 = 2.4$, $\sigma_2 = 1.2$, $\mathcal{R}_1^* = 0.9721 < 1$, $\mathcal{R}_2^* = 0.9781 < 1$.

This shows that the persistent two microorganisms of a deterministic system (see Figure 1(a)) can become extinct under the white noise stochastic disturbance (see Figure 1(b)), and thus the simulation is consistent with the theoretical results of Lemma 2.3 and Theorem 3.1. When $\mathcal{R}_i^* = \mathcal{R}_i - \frac{\sigma_i^2 S_0^2}{2a_i^2(Q+r_i\bar{c})} < 1 < \mathcal{R}_i$, a persistent deterministic system goes to extinction due to the white noise disturbance.

Next, we choose σ_1 and σ_2 with different values. When σ_1 is small and σ_2 is large ($\sigma_1 = 0.2$, $\sigma_2 = 1.2$), here $\mathcal{R}_1^* = 1.3558 > 1$ and $\mathcal{R}_2^* = 0.9781 < 1$. Thus, the microorganism x_2 goes to extinction, and the microorganism x_1 is persistent (see Figure 1(c)). Conversely, when σ_1 is large and σ_2 is small ($\sigma_1 = 2.4$, $\sigma_2 = 0.1$), here $\mathcal{R}_1^* = 0.9721 < 1$, $\mathcal{R}_2^* = 1.3452 > 1$. Figure 1(d) shows that the microorganism x_1 goes to extinction and the microorganism x_2 is persistent. Moreover, for small noise intensities, $\sigma_1 = 0.2$ and $\sigma_2 = 0.1$, both microorganisms are persistent (see Figure 1(e)). This supports our theoretical results in Theorem 3.2, and we observe that the white noise has unfavorable effects on the persistence of microorganisms.

Figure 2(a) shows that a greater impulsive toxicant input can lead to the extinction of the two microorganisms, whereas the microorganism populations can be persistent in the smaller impulsive toxicant input environment (see Figure 2(b) and Figure 2(c)). This supports our theoretical results in Theorems 3.1 and 3.2, and we observe that the impulsive toxicant input has unfavorable effects on the persistence of microorganisms.

From the theoretical analysis and simulations we can find that if the intensity of the white noise or impulsive input is small, then the microorganisms can still be persistent just as in the deterministic system, whereas for the large intensity of the white noise or impulsive input, microorganisms may become extinct. Therefore, noises and impulsive

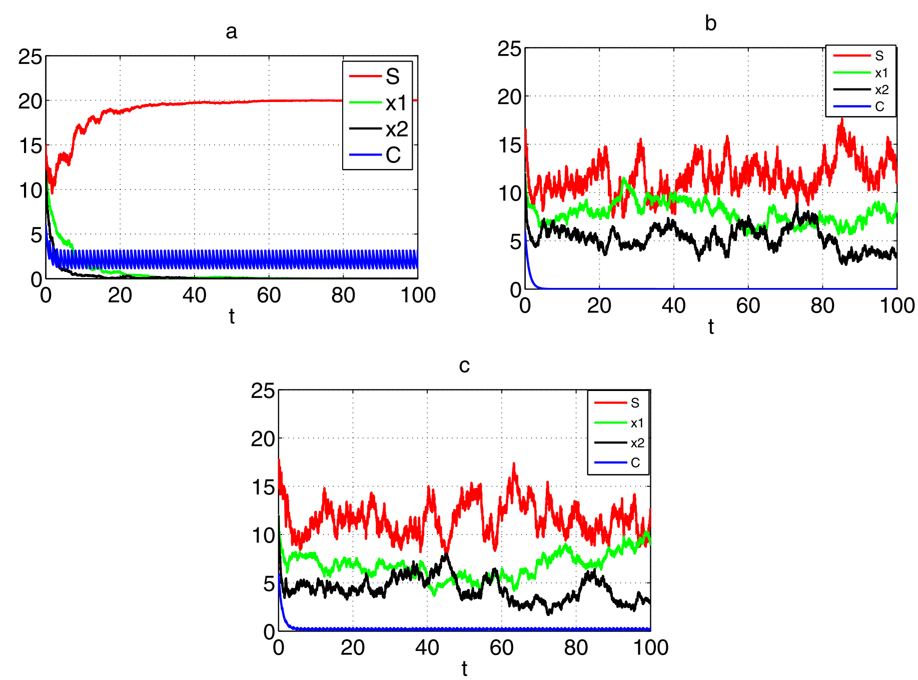

Figure 2 Computer simulation of the paths $S(t)$, $x_1(t)$, $x_2(t)$, $c(t)$ of the stochastic chemostat model (2) for impulsive effects. (a) Time series for $S(t)$, $X_1(t)$, $X_2(t)$, $c(t)$ with parameters $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $u = 2$. **(b)** Time series for $S(t)$, $X_1(t)$, $X_2(t)$, $c(t)$ with parameters $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $u = 0$. **(c)** Time series for $S(t)$, $X_1(t)$, $X_2(t)$, $c(t)$ with parameters $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $u = 0.2$.

effects go against the survival of microorganisms. The first three equations of system (2) can also be considered as a nonautonomous and nonimpulsive periodic system with periodic coefficient $c(t)$ (with period τ). Moreover, the extinction and persistence of two microorganisms are discussed in three cases. The theoretical method can also be used to explore the thresholds of some high-dimensional impulsive stochastic differential systems and some nonautonomous periodic systems.

Some problems in this direction deserve further investigation. It is interesting to study other kinds of high-dimensional impulsive stochastic Lotka-Volterra systems, such as predator-prey system and cooperation system, or introduce a Markov process or Lévy jumps into the impulsive stochastic environment. This our future research work should continue to be concerned about.

Appendix

The proof of Lemma 2.3.

Proof By Lemma 2.1 there is a unique 'microorganism-extinction' periodic solution $(S_0, 0, 0, c^*(t))$ in system (1). The local stability of the periodic solution $(S_0, 0, 0, c^*(t))$ may be determined by considering the behavior of small amplitude perturbations of the solution. Let $S(t) = S_0 + \phi(t)$, $x_1(t) = v_1(t)$, $x_2(t) = v_2(t)$, $c(t) = c^*(t) + w(t)$. Then we have

$$\begin{pmatrix} \phi(t) \\ v_1(t) \\ v_2(t) \\ w(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} \phi(0) \\ v_1(0) \\ v_2(0) \\ w(0) \end{pmatrix}, \quad 0 \le t \le \tau,$$

where $\Phi(t)$ satisfies

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \begin{pmatrix} -Q & * & * & 0 \\ 0 & \frac{\mu_1 S_0}{a_1} - Q - r_1 c^*(t) & 0 & 0 \\ 0 & 0 & \frac{\mu_2 S_0}{a_2} - Q - r_2 c^*(t) & 0 \\ 0 & 0 & 0 & -h \end{pmatrix} \Phi(t),$$

$\Phi(0) = E_{4\times 4}$ is the identity matrix, and the fundamental solution matrix is

$$\Phi(t) = \begin{pmatrix} \exp(-Q\tau) & * & * & 0 \\ 0 & \exp(\int_0^\tau \frac{\mu_1 S_0}{a_1} - Q - r_1 c^*(t)\,\mathrm{d}t) & 0 & 0 \\ 0 & 0 & \exp(\int_0^\tau \frac{\mu_2 S_0}{a_2} - Q - r_2 c^*(t)\,\mathrm{d}t) & 0 \\ 0 & 0 & 0 & \exp(-h\tau) \end{pmatrix}.$$

There is no need to calculate the exact form of $(*)$ because it is not required in the analysis that follows. The linearization of the impulsive equations of (1) becomes

$$\begin{pmatrix} \phi(n\tau^+) \\ v_1(n\tau^+) \\ v_2(n\tau^+) \\ w(n\tau^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(n\tau) \\ v_1(n\tau) \\ v_2(n\tau) \\ w(n\tau) \end{pmatrix} = E_{4\times 4} \begin{pmatrix} \phi(n\tau) \\ v_1(n\tau) \\ v_2(n\tau) \\ w(n\tau) \end{pmatrix}.$$

The stability of the periodic solution $(S_0, 0, 0, c^*(t))$ is determined by the eigenvalues of $M = E_{4\times 4}\Phi(t)$, that is,

$$M = \begin{pmatrix} \exp(-Q\tau) & * & * & 0 \\ 0 & \exp(\int_0^\tau \frac{\mu_1 S_0}{a_1} - Q - r_1 c^*(t)\,\mathrm{d}t) & 0 & 0 \\ 0 & 0 & \exp(\int_0^\tau \frac{\mu_2 S_0}{a_2} - Q - r_2 c^*(t)\,\mathrm{d}t) & 0 \\ 0 & 0 & 0 & \exp(-h\tau) \end{pmatrix},$$

and there is no need to calculate the exact form of $(*)$. The eigenvalues of the upper triangular matrix M are

$$\lambda_1 = \mathrm{e}^{-Q\tau} < 1, \qquad \lambda_2 = \mathrm{e}^{\int_0^\tau \frac{\mu_1 S_0}{a_1} - Q - r_1 c^*(t)\,\mathrm{d}t},$$
$$\lambda_3 = \mathrm{e}^{\int_0^\tau \frac{\mu_2 S_0}{a_2} - Q - r_2 c^*(t)\,\mathrm{d}t}, \qquad \lambda_4 = \mathrm{e}^{-h\tau} < 1.$$

According to Floquet theory of impulsive equations, $(S_0, 0, 0, c^*(t))$ is stable if $\lambda_2 < 1$ and $\lambda_3 < 1$, that is, $\mathcal{R}_1 < 1$ and $\mathcal{R}_2 < 1$.

Now we prove that system (1) is persistent if $\mathcal{R}_1 > 1$ and $\mathcal{R}_2 > 1$.

For the equation $S(t) + \frac{1}{\delta_1}x_1(t) + \frac{1}{\delta_2}x_2(t)$, integrating it from 0 to t and dividing by t, for t large enough, we have

$$\begin{aligned} \epsilon(t) &\triangleq \frac{S(t) - S(0)}{t} + \frac{1}{\delta_1}\frac{x_1(t) - x_1(0)}{t} + \frac{1}{\delta_2}\frac{x_2(t) - x_2(0)}{t} \\ &\geq QS_0 - Q\langle S(t)\rangle - \frac{Q + r_1 c^*(0)}{\delta_1}\langle x_1(t)\rangle - \frac{Q + r_2 c^*(0)}{\delta_2}\langle x_2(t)\rangle. \end{aligned}$$

Then we have

$$\langle S(t)\rangle \geq S_0 - \frac{Q + r_1 c^*(0)}{\delta_1 Q}\langle x_1(t)\rangle - \frac{Q + r_2 c^*(0)}{\delta_2 Q}\langle x_2(t)\rangle - \frac{\epsilon(t)}{Q}. \tag{22}$$

Define $V(t) = a_1 \ln x_1(t) + a_2 \ln x_2(t) + x_1(t) + x_2(t)$, which is a bounded function. Then we get

$$\begin{aligned} D^+V(t) &= \mu_1 S(t) - a_1\big(Q + r_1 c(t)\big) - Qx_1(t) - r_1 c(t)x_1(t) + \mu_2 S(t) \\ &\quad - a_2\big(Q + r_2 c(t)\big) - Qx_2(t) - r_2 c(t)x_2(t) \\ &\geq (\mu_1 + \mu_2)S(t) - \sum_{i=1}^{2} a_i\big(Q + r_i c(t)\big) - \sum_{i=1}^{2}\big(Q + r_i c^*(0)\big)x_i(t). \end{aligned} \tag{23}$$

Integrating both sides of (23) from 0 to t and dividing by t yield

$$\begin{aligned} \frac{V(t)}{t} - \frac{V(0)}{t} &\geq (\mu_1 + \mu_2)\langle S(t)\rangle - \sum_{i=1}^{2} a_i\big(Q + r_i\langle c(t)\rangle\big) - \sum_{i=1}^{2}\big(Q + r_i c^*(0)\big)\langle x_i(t)\rangle \\ &\geq (\mu_1 + \mu_2)S_0 - \sum_{i=1}^{2} a_i\big(Q + r_i\langle c(t)\rangle\big) \\ &\quad - \sum_{i=1}^{2}\left[\big(Q + r_i c^*(0)\big)\left(\frac{\mu_1 + \mu_2}{\delta_i Q} + 1\right)\right]\langle x_i(t)\rangle - \frac{(\mu_1 + \mu_2)\epsilon(t)}{Q} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{2} a_i\bigl(Q + r_i\langle c(t)\rangle\bigr)\left[\frac{\mu_i S_0}{a_i(Q + r_i\langle c(t)\rangle)} - 1\right] \\
&\quad - \sum_{i=1}^{2}\left[\bigl(Q + r_i c^*(0)\bigr)\left(\frac{\mu_1 + \mu_2}{\delta_i Q} + 1\right)\right]\bigl\langle x_i(t)\bigr\rangle - \frac{(\mu_1 + \mu_2)\epsilon(t)}{Q}. \qquad (24)
\end{aligned}$$

Notice that $0 < S \leq S_0$ and $0 < x_i(t) \leq \delta_i S_0$. Then $\lim_{t\to+\infty} \frac{V(t)}{t} = 0$ and $\lim_{t\to+\infty} \epsilon(t) = 0$. Taking the limit inferior of both sides of (24) leads to

$$\liminf_{t\to+\infty}\bigl\langle x_1(t) + x_2(t)\bigr\rangle \geq \frac{1}{\Delta_{\max}} \sum_{i=1}^{2} a_i(Q_i + r_i\bar{c})(\mathcal{R}_i - 1) > 0,$$

where

$$\Delta_{\max} = \max\left\{\bigl(Q + r_1 c^*(0)\bigr)\left(\frac{\mu_1 + \mu_2}{\delta_1 Q} + 1\right), \bigl(Q + r_2 c^*(0)\bigr)\left(\frac{\mu_1 + \mu_2}{\delta_2 Q} + 1\right)\right\}.$$

This completes the proof. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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