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Integral spectral Tchebyshev approach for solving space Riemann-Liouville and Riesz fractional advection-dispersion problems

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Abstract

The principal aim of this paper is to analyze and implement two numerical algorithms for solving two kinds of space fractional linear advection-dispersion problems. The proposed numerical solutions are spectral and they are built on assuming the approximate solutions to be certain double shifted Tchebyshev basis. The two typical collocation and Petrov-Galerkin spectral methods are applied to obtain the desired numerical solutions. The special feature of the two proposed methods is that their applications enable one to reduce, through integration, the fractional problem under investigation into linear systems of algebraic equations, which can be efficiently solved via any suitable solver. The convergence and error analysis of the double shifted Tchebyshev basis are carefully investigated, aiming to illustrate the correctness and feasibility of the proposed double expansion. Finally, the efficiency, applicability, and high accuracy of the suggested algorithms are demonstrated by presenting some numerical examples accompanied with comparisons with some other existing techniques discussed in the literature.

1 Introduction

Fractional calculus is a very important branch of mathematical analysis. This branch is basically interested in investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals). Fractional ordinary differential equations (FODEs) and fractional partial differential equations (FPDEs) have attracted considerable interest from a large number of researchers due to their ability to model a lot of phenomena in engineering, control theory, chemical physics, stochastic processes, anomalous diffusion, rheology, biology, and other sciences, such as medicine and neuronal dynamics. Due to the growing interest in these kinds of differential equations, obtaining approximate solutions of them is of fundamental importance and hence it is very useful to develop numerical techniques for solving these types of equations.

The approach of employing spectral methods is very effective in handling ordinary differential equations as well as fractional differential equations. This approach is basically built on assuming the approximate solutions to be linear combinations of certain basis functions. These basis functions may be orthogonal or nonorthogonal. Approximations by orthogonal basis functions occupy a considerable part in the literature. In fact, there are three commonly used spectral methods, namely the collocation, the tau, and the Galerkin method and its variants. The collocation method is very effective in a wide range of practical problems, particularly in the nonlinear ones (see for example [1]). The tau method is useful for treating boundary value problems of complicated boundary conditions (see for example [2]). In the Galerkin and Petrov-Galerkin methods, one chooses two sets of basis functions, which are called 'trial functions' and 'test functions'. These two sets are identical in the Galerkin method, however, in the Petrov-Galerkin method, they are not identical, so the main advantage of employing the Petrov-Galerkin method is its flexibility in choosing test functions. The Galerkin and Petrov-Galerkin methods have been applied successfully in various situations. For example, the authors in [3, 4] have constructed efficient spectral Galerkin algorithms and Petrov-Galerkin algorithms for handling various even- and odd-order boundary value problems. One of the advantages of the Galerkin and Petrov-Galerkin methods is that they enable one to investigate carefully the algebraic systems resulting from their applications and their structures and complexities.

There are three types of FPDEs, namely space FPDEs, time FPDEs and space-time FPDEs. One of the most important fractional differential equation is the fractional advection-diffusion equation (FADE). Due to the importance of this equation (see, for example, [5]), a variety of papers with different numerical techniques have been proposed to handle it. For example, Shen *et al.* in [6] derived the fundamental solution for the space-time Riesz-Caputo FADE with an initial condition. The authors in [7] have derived some analytical solutions for the multi-term time-space Caputo-Riesz FADEs on a finite domain. The Adomian decomposition method is employed in [8] for solving an intermediate fractional advection-dispersion equation. The finite element method is used to handle the fractional advection-dispersion equation.

In this study, we are concerned with introducing numerical integral spectral solutions for two kinds of the FADEs. We apply the collocation and Petrov-Galerkin methods for this purpose. The main idea behind the proposed approach is to solve the integrated form of the equation. The advantage of using this approach is that its use enables one to reduce the solution of the equation with its boundary and initial conditions into a system of linear or nonlinear algebraic equations. The linear system can be efficiently solved using the Gaussian elimination solver or by any other suitable solver, while the nonlinear system can be solved with the aid of Newton's iterative method.

The paper is organized as follows. In the next section, some necessary definitions and mathematical preliminaries of fractional calculus are presented. Besides, some properties of shifted Tchebyshev polynomials are given. Sections 3 and 4 are devoted to solving FADEs by implementing and presenting two numerical algorithms based on the application of the collocation and Petrov-Galerkin methods. Section 5 focuses on investigating the convergence and error analysis of the suggested Tchebyshev double expansion. The numerical results and comparisons are displayed in Section 6. Finally, some conclusions are reported in Section 7.

2 Some fundamental properties of fractional calculus

This section is devoted to presenting some fundamental definitions and preliminary facts of the fractional calculus theory. For fundamentals of this branch, the reader is referred to [9].

Definition 1 The Riemann-Liouville fractional integral operator I^{α} of order α on the usual Lebesgue space $L_1[0,1]$ is defined as

$$I^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau, & \alpha>0,\\ f(t), & \alpha=0.\end{cases} \tag{2.1}$$

The following properties are satisfied by the operator I^{α}:

(i) $I^{\alpha}I^{\beta}=I^{\alpha+\beta}$,

(ii) $I^{\alpha}I^{\beta}=I^{\beta}I^{\alpha}$,

(iii) $I^{\alpha}t^{\nu}=\dfrac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}t^{\nu+\alpha}$,

where $\alpha,\beta\geq 0$, and $\nu>-1$.

Definition 2 The left and right handed Riemann-Liouville fractional-order operators are defined, respectively, as follows:

$$\left({}_{a}^{R}D_{x}^{\gamma}f\right)(x)=\frac{1}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_a^x (x-\tau)^{m-\gamma-1}f(\tau)\,d\tau, \tag{2.2}$$

$$\left({}_{x}^{R}D_{b}^{\gamma}f\right)(x)=\frac{(-1)^m}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_x^b (\tau-x)^{m-\gamma-1}f(\tau)\,d\tau, \tag{2.3}$$

where $m-1\leq\gamma<m$, $m\in\mathbb{N}$.

Definition 3 For a function f defined on the interval $I=[a,b]$, the left and right handed Caputo fractional-order derivatives are defined as

$$\left({}_{a}^{C}D_{x}^{\gamma}f\right)(x)=\frac{1}{\Gamma(m-\gamma)}\int_a^x (x-\tau)^{m-\gamma-1}f^{(m)}(\tau)\,d\tau,\quad \gamma>0,t>0, \tag{2.4}$$

$$\left({}_{x}^{C}D_{b}^{\gamma}f\right)(x)=\frac{(-1)^m}{\Gamma(m-\gamma)}\int_x^b (\tau-x)^{m-\gamma-1}f^{(m)}(\tau)\,d\tau,\quad \gamma>0,t>0, \tag{2.5}$$

where $m-1\leq\gamma<m$, $m\in\mathbb{N}$.

It is worthwhile to mention here that the operator ${}_{0}^{C}D_{t}^{\alpha}$ satisfies the following fundamental properties, for $n-1\leq\alpha<n$:

$$\begin{aligned}&\left({}_{0}^{C}D_{t}^{\alpha}I^{\alpha}f\right)(t)=f(t),\\ &{}_{0}^{C}D_{t}^{\alpha}t^k=\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}t^{k-\alpha},\quad k\text{ is a positive integer}, k\geq\lceil\alpha\rceil,\end{aligned} \tag{2.6}$$

where $\lceil\alpha\rceil$ is the well-known ceiling function.

Property 1 (See [10]) The relation between the Riemann-Liouville fractional derivative and the Caputo fractional derivative is given by

$$\left({}_0^R D_x^\gamma f\right)(x) = \left({}_0^C D_x^\gamma f\right)(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{\Gamma(j-\gamma+1)} x^{j-\gamma}. \tag{2.7}$$

Definition 4 The Riesz fractional-order derivative of order γ is defined (see [11]) as

$$\frac{\partial^\gamma f}{\partial |x|^\gamma} = -c\left[{}_{-\infty}^{R} D_x^\gamma f(x) + {}_x^R D_\infty^\gamma f(x)\right], \tag{2.8}$$

where $c = \frac{1}{2} \sec \frac{\gamma\pi}{2}$.

For a function f defined over the interval $I = [a,b]$, where $f(a) = f(b) = 0$, we extend the function to be $f(x) = 0$, for all $x > b$ and $x < a$. Thus we have [6]

$$\frac{\partial^\gamma f(x)}{\partial |x|^\gamma} = -c\left[{}_a^R D_x^\gamma f(x) + {}_x^R D_b^\gamma f(x)\right]. \tag{2.9}$$

For several other properties of fractional derivatives and integrals, see [9, 12].

2.1 Some properties of shifted Tchebyshev polynomials

Let $T_n(t)$; $t \in [-1,1]$, denote the standard first kind Tchebyshev polynomial of degree n. It is well known that the set $\{T_n(t); t \in [-1,1]\}_{n\ge 0}$ form a complete orthogonal system for $L_w^2(-1,1)$, where $w = \frac{1}{\sqrt{1-t^2}}$. Moreover, Tchebyshev polynomials satisfy the following orthogonality relation:

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}\,dt = \begin{cases} \frac{\pi}{\kappa_i}, & i = j, \\ 0, & i \neq j, \end{cases} \quad \text{where } \kappa_i = \begin{cases} 1, & i = 0, \\ 2, & \text{otherwise.} \end{cases}$$

We denote by $T_j^\tau(t)$ the shifted Tchebyshev polynomials defined on $(0,\tau)$ as

$$T_j^\tau(t) = T_j\left(\frac{2t}{\tau} - 1\right).$$

The shifted Tchebyshev polynomials form a complete orthogonal system for $L_w^2(0,\tau)$, where $w = \frac{1}{\sqrt{\tau t - t^2}}$. The orthogonality relation satisfied by $T_i^\tau(t)$ is

$$\int_0^\tau \frac{T_i^\tau(t) T_j^\tau(t)}{\sqrt{\tau t - t^2}}\,dt = \begin{cases} \frac{\pi}{\kappa_i}, & i = j, \\ 0, & i \neq j. \end{cases}$$

The linearization of the product of two Tchebyshev polynomials is given by (see [13])

$$T_\ell^\tau(s) T_j^\tau(s) = \frac{1}{2}\left[T_{|\ell-j|}^\tau(s) + T_{\ell+j}^\tau(s)\right]. \tag{2.10}$$

These polynomials (see [13]) have the following two analytic forms:

$$T_j^\tau(t) = j \sum_{s=0}^{j} \frac{(-1)^{j-s} 2^{2s} (j+s-1)!}{(j-s)!(2s)!\tau^s} t^s, \tag{2.11}$$

and

$$T_j^{\tau}(t) = j\sum_{s=0}^{j} \frac{(-1)^s (j+s-1)!}{(s)!(\frac{1}{2})_s (j-s)!\tau^s}(\tau - t)^s. \tag{2.12}$$

Now if we let $f(t) \in L^2[0,\tau]$, then $f(t)$ may be expanded in terms of a shifted Tchebyshev basis as

$$f(t) = \sum_{i=0}^{\infty} c_i T_i^{\tau}(t),$$

where

$$c_i = \frac{\kappa_i}{\pi}\int_0^{\tau} \frac{f(t)T_i^{\tau}(t)}{\sqrt{\tau t - t^2}}\,dt. \tag{2.13}$$

For more properties of Tchebyshev polynomials and their shifted ones, see for example [13].

3 Integral transforms for two kinds of the space fractional advection-dispersion equations

This section is devoted to transforming two kinds of space FADEs into their integral forms in order to treat them numerically by our proposed techniques in the upcoming section.

3.1 The first kind of space fractional advection-dispersion problem

Consider the following space right-handed Riemann-Liouville fractional advection-dispersion problem (see [14]):

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t} + v(x,t){}_0^R D_x^{\beta} u(x,t) - k(x,t){}_0^R D_x^{\gamma} u(x,t) = f(x,t,u),\\ &0 < \beta < 1, \qquad 1 < \gamma < 2, \qquad (x,t) \in \Omega := (0,\ell) \times (0,\tau),\end{aligned} \tag{3.1}$$

governed by the nonhomogeneous boundary conditions

$$u(0,t) = u_0(t), \qquad u(\ell,t) = u_\ell(t), \quad 0 < t < \tau, \tag{3.2}$$

and the initial condition

$$u(x,0) = f_0(x), \quad 0 < x < \ell, \tag{3.3}$$

where $u(x,t)$ represents the concentration, $k(x,t)$ and $v(x,t)$ are the dispersion coefficients, and $f(x,t,u)$ is the source term which may be linear or nonlinear.

In order to proceed in developing our two spectral algorithms for treating (3.1)-(3.3), we make use of the following transformation:

$$u(x,t) = g(x,t) + \left(1 - \frac{x}{\ell}\right)u_0(t) + \frac{x}{\ell}u_\ell(t), \tag{3.4}$$

and we take into consideration the following relations:

$$
\begin{aligned}
{}_0^RD_x^\beta 1 &= \frac{x^{-\beta}}{\Gamma(1-\beta)}, \qquad {}_0^RD_x^\beta \frac{x}{\ell} = \frac{x^{1-\beta}}{\ell\Gamma(2-\beta)}, \\
{}_0^RD_x^\gamma 1 &= \frac{(1-\gamma)x^{-\gamma}}{\Gamma(2-\gamma)}, \qquad {}_0^RD_x^\gamma \frac{x}{\ell} = \frac{x^{1-\gamma}}{\ell\Gamma(2-\gamma)},
\end{aligned}
$$

to convert equation (3.1), subject to the nonhomogeneous boundary conditions (3.2) and the initial condition (3.3), into the following modified one:

$$
\begin{aligned}
&\frac{\partial g(x,t)}{\partial t} + \nu(x,t){}_0^RD_x^\beta g(x,t) - k(x,t){}_0^RD_x^\gamma g(x,t) = f_1(x,t,g), \\
&0<\beta<1, \qquad 1<\gamma<2, \qquad (x,t)\in\Omega := (0,\ell)\times(0,\tau),
\end{aligned} \tag{3.5}
$$

subject to the homogeneous Dirichlet boundary conditions

$$
g(0,t) = g(\ell,t) = 0, \quad 0<t<\tau, \tag{3.6}
$$

and the initial condition

$$
g(x,0) = \tilde{f}_0(x), \quad 0<x<\ell, \tag{3.7}
$$

where

$$
\begin{aligned}
f_1(x,t,g) = f(x,t,u) &- \frac{x^{-\beta}\nu(x,t)}{\Gamma(1-\beta)}\left[u_0(t) + \frac{x}{\ell(1-\beta)}\big(u_\ell(t)-u_0(t)\big)\right] \\
&+ \frac{x^{-\gamma}k(x,t)}{\Gamma(1-\gamma)}\left[u_0(t) + \frac{x}{\ell(1-\gamma)}\big(u_\ell(t)-u_0(t)\big)\right] \\
&- \left(1-\frac{x}{\ell}\right)u_0'(t) - \frac{x}{\ell}u_\ell'(t),
\end{aligned} \tag{3.8}
$$

and

$$
\tilde{f}_0(x) = f_0(x) - \left(1-\frac{x}{\ell}\right)u_0(0) - \frac{x}{\ell}u_\ell(0). \tag{3.9}
$$

Note 1 If $\beta = 1$, $\gamma = 2$, equation (3.8) has the following form:

$$
f_1(x,t,g) = f(x,t,u) + \frac{\nu(x,t)}{\ell}\big(u_0(t)-u_\ell(t)\big) - \left(1-\frac{x}{\ell}\right)u_0'(t) - \frac{x}{\ell}u_\ell'(t). \tag{3.10}
$$

Now, the integral form of equation (3.5) governed by the conditions (3.6) and (3.7) is

$$
\left.
\begin{array}{l}
g(x,t) = -\int_0^t \nu(x,s){}_0^RD_x^\beta g(x,s)\,ds + \int_0^t k(x,s){}_0^RD_x^\gamma g(x,t)\,ds + F(x,t,g), \\
0<\beta<1, \qquad 1<\gamma<2, \qquad (x,t)\in\Omega := (0,\ell)\times(0,\tau), \\
g(0,t) = g(\ell,t) = 0, \quad 0<t<\tau,
\end{array}
\right\} \tag{3.11}
$$

and

$$
F(x,t,g) = \tilde{f}_0(x) + \int_0^t f_1(x,s,g)\,ds. \tag{3.12}
$$

3.2 The second kind of space fractional advection-dispersion problem

Consider the space Riemann-Liouville fractional advection-dispersion problem with Riesz space fractional derivatives (see [15]):

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t}+\kappa_\beta\frac{\partial^\beta u(x,t)}{\partial|x|^\beta}-\kappa_\gamma\frac{\partial^\gamma u(x,t)}{\partial|x|^\gamma}=f(x,t,u),\\ &0<\beta<1,\qquad 1<\gamma<2,\qquad (x,t)\in\Omega:=(0,\ell)\times(0,\tau), \end{aligned}\tag{3.13}$$

governed by the nonhomogeneous boundary conditions

$$u(0,t)=u_0(t),\qquad u(\ell,t)=u_\ell(t),\quad 0<t<\tau,\tag{3.14}$$

and the initial condition

$$u(x,0)=f_0(x),\quad 0<x<\ell,\tag{3.15}$$

where $u(x,t)$ represents the concentration, κ_β and κ_γ represent the dispersion coefficient and the average fluid velocity, respectively, and $f(x,t,u)$ is the source term which may be linear or nonlinear.

Now, with the aid of the transformation formula

$$u(x,t)=g(x,t)+\left(1-\frac{x}{\ell}\right)u_0(t)+\frac{x}{\ell}u_\ell(t),\tag{3.16}$$

equation (3.13), subject to the nonhomogeneous boundary conditions (3.14) and the initial condition (3.15), is turned into the following modified equation:

$$\begin{aligned} &\frac{\partial g(x,t)}{\partial t}+\kappa_\beta\frac{\partial^\beta g(x,t)}{\partial|x|^\beta}-\kappa_\gamma\frac{\partial^\gamma g(x,t)}{\partial|x|^\gamma}=f_1(x,t,g),\\ &0<\beta<1,\qquad 1<\gamma<2,\qquad (x,t)\in\Omega:=(0,\ell)\times(0,\tau), \end{aligned}\tag{3.17}$$

subject to the homogeneous Dirichlet boundary conditions

$$g(0,t)=g(\ell,t)=0,\quad 0<t<\tau,\tag{3.18}$$

and the initial condition

$$g(x,0)=\tilde{f}_0(x),\quad 0<x<\ell,\tag{3.19}$$

where

$$\begin{aligned} f_1(x,t,u)=f(x,t,u)&+\varsigma_\beta u_0\left(x^{-\beta}-\frac{x^{1-\beta}}{\ell(1-\beta)}+\frac{(\ell-x)^{1-\beta}}{\ell(1-\beta)}\right)\\ &+\varsigma_\beta u_\ell\left(\frac{x^{1-\beta}}{\ell(1-\beta)}+(\ell-x)^{-\beta}-\frac{(\ell-x)^{1-\beta}}{\ell(1-\beta)}\right)\\ &-\varsigma_\gamma u_0\left(x^{-\gamma}-\frac{x^{1-\gamma}}{\ell(1-\gamma)}+\frac{(\ell-x)^{1-\gamma}}{\ell(1-\gamma)}\right)\\ &-\varsigma_\gamma u_\ell\left((\ell-x)^{-\gamma}+\frac{x^{1-\gamma}}{\ell(1-\gamma)}-\frac{(\ell-x)^{1-\gamma}}{\ell(1-\gamma)}\right)-\frac{x}{\ell}u_\ell'-\left(1-\frac{x}{\ell}\right)u_0', \end{aligned}\tag{3.20}$$

where

$$\varsigma_\mu = \frac{\kappa_\mu \sec(\frac{\mu\pi}{2})}{2\Gamma(1-\mu)}$$

and

$$\tilde{f}_0(x) = f_0(x) - \left(1 - \frac{x}{\ell}\right) u_0(0) - \frac{x}{\ell} u_\ell(0). \tag{3.21}$$

Note 2 If $\beta = 1$, $\gamma = 2$, equation (3.20) has the following form:

$$f_1(x,t,g) = f(x,t,u) - \left(1 - \frac{x}{\ell}\right) u_0'(t) - \frac{x}{\ell} u_\ell'(t). \tag{3.22}$$

Now, the integral form of equation (3.17) governed by the conditions (3.18) and (3.19) is

$$\left.\begin{array}{l} g(x,t) = -\kappa_\beta \int_0^t \frac{\partial^\beta g(x,s)}{\partial |x|^\beta}\, ds + \kappa_\gamma \int_0^t \frac{\partial^\gamma g(x,s)}{\partial |x|^\gamma}\, ds + F(x,t,g), \\ 0 < \beta < 1, \qquad 1 < \gamma < 2, \qquad (x,t) \in \Omega := (0,\ell) \times (0,\tau), \\ g(0,t) = g(\ell,t) = 0, \quad 0 < t < \tau, \end{array}\right\} \tag{3.23}$$

and

$$F(x,t,g) = \tilde{f}_0(x) + \int_0^t f_1(x,s,g)\, ds. \tag{3.24}$$

4 Numerical spectral treatment for two kinds of space fractional linear advection-dispersion problems

This section is concerned with explaining in detail two spectral algorithms for numerically solving two kinds of space fractional linear advection-dispersion problems. First, we select a unified double Tchebyshev expansion as basis functions, and then apply the two well-known spectral methods, namely the collocation and Petrov-Galerkin methods.

4.1 Choice of the basis functions

We choose the following two families of orthogonal polynomials:

$$\phi_i(x) = x(\ell - x) T_i^\ell(x), \quad i = 0,1,2,\ldots, \tag{4.1}$$

$$\psi_j(t) = T_j^\tau(t), \quad j = 0,1,2,\ldots. \tag{4.2}$$

It is not difficult to show that the polynomials $\{\phi_i(x)\}_{i\geq 0}$ are linearly independent and orthogonal with respect to the weight function $w(x) = \frac{1}{x^{5/2}(\ell-x)^{5/2}}$ on $[0,\ell]$. Moreover, it is also clear that each member of them fulfills the boundary conditions (3.6). The orthogonality relation for $\{\phi_i(x)\}_{i\geq 0}$ is

$$\int_0^\ell \frac{\phi_i(x)\phi_j(x)\, dx}{x^{5/2}(\ell - x)^{5/2}} = h_i = \begin{cases} \frac{\pi}{\kappa_i}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.3}$$

Now, we define the following two spaces:

$$V = \{y \in H^2_{w(x,t)}(\Omega) : y(0,t) = y(\ell,t) = 0, t \in [0,\tau]\}, \tag{4.4}$$
$$V_M = \operatorname{span}\{\phi_i(x)\psi_j(t) : i,j = 0,1,\ldots,M\},$$

where $H^2_{w(x,t)}(\Omega)$, $\Omega = (0,\ell) \times (0,\tau]$ is the Sobolev space defined in [16], and

$$w(x,t) = \frac{1}{x^{5/2}(\ell - x)^{5/2}t^{1/2}(\tau - t)^{1/2}}.$$

Now, let $g(x,t) \in V$. Then this function can be expanded in the following double expansion:

$$g(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\phi_i(x)\psi_j(t), \tag{4.5}$$

and the coefficients c_{ij} are given by the formula

$$c_{ij} = \frac{1}{h_i h_j}\int_0^{\tau}\int_0^{\ell} \frac{g(x,t)\phi_i(x)\psi_j(t)}{x^{5/2}(\ell - x)^{5/2}t^{1/2}(\tau - t)^{1/2}}\,dx\,dt. \tag{4.6}$$

Numerically, $g(x,t)$ can be approximated by the truncated double series

$$g(x,t) \approx g_M(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\phi_i(x)\psi_j(t). \tag{4.7}$$

In the following, we are going to state and prove three important theorems concerned with the basis functions $\phi_i(x)$ and $\psi_j(t)$. In the first, we give a new formula for the fractional derivatives of the polynomials $\phi_i(x)$ in the sense of Riemann-Liouville, while the second gives the Riesz fractional derivatives for the same basis. In the third theorem, an integral formula for $\psi_j(t)$ is given.

Theorem 1 *Let* $\alpha \in (1,2)$. *Then the following fractional derivative* (*in the Riemann-Liouville sense*) *relation is valid*:

$$\begin{aligned} {}_0^R D_x^{\alpha}\phi_i(x) = {}& i\sum_{k=0}^{i} \frac{(-1)^{i+k+1}(k+1)(i+k-1)!\Gamma(\alpha-k-1)\sin(\pi(\alpha-k))}{\pi(\frac{1}{2})_k(i-k)!\ell^{k-1}} \\ & \times \left(x^{-\alpha+k+1} - \frac{(k+2)}{(-\alpha+k+2)\ell}x^{-\alpha+k+2}\right). \end{aligned} \tag{4.8}$$

Proof The power form representation for $T_i^{\ell}(x)$ given in (2.11) enables one to write $\phi_i(x)$ as

$$\phi_i(x) = i\sum_{k=0}^{i} \frac{(-1)^{i-k}(i+k-1)!2^{2k}}{(2k)!(i-k)!\ell^{k-1}}x^{k+1} - i\sum_{k=0}^{i} \frac{(-1)^{i-k}(i+k-1)!2^{2k}}{(2k)!(i-k)!\ell^{k}}x^{k+2}. \tag{4.9}$$

If the operator ${}_0^RD_x^\alpha$ is applied to both sides of (4.9), and making use of the formula

$$ {}_0^RD_x^\alpha x^k = \frac{k!}{\Gamma(1+k-\alpha)} x^{k-\alpha}, $$

then we get

$$ \begin{aligned} {}_0^RD_x^\alpha \phi_i(x) = {} & i \sum_{k=0}^{i} \frac{(-1)^{i-k}(i+k-1)!2^{2k}(k+1)!}{(2k)!(i-k)!\Gamma(k+2-\alpha)\ell^{k-1}} x^{k+1-\alpha} \\ & - i \sum_{k=0}^{i} \frac{(-1)^{i-k}(i+k-1)!2^{2k}(k+2)!}{(2k)!(i-k)!\Gamma(k+3-\alpha)\ell^{k}} x^{k+2-\alpha}. \end{aligned} \tag{4.10} $$

If we make use of the relation

$$ \Gamma(\xi)\Gamma(1-\xi) = \frac{\pi}{\sin(\xi\pi)}, $$

then after performing some rather lengthy manipulations, we get the desired equation (4.8). □

Note 3 By writing $a_k \lessapprox b_k$, we mean that there exists a generic constant C such that $a_k < C\,b_k$ for large k.

Lemma 1 *For* $x \in [0,1]$ *and* $0 < \alpha < 1$, *we have*

$$ \left| {}_0^RD_x^\alpha \phi_i(x) \right| \lessapprox i. \tag{4.11} $$

Proof From relation 2.7 and knowing that $\phi_i(0) = 0$, we have

$$ \left({}_0^RD_x^\alpha \phi_i \right)(x) = \left({}_0^CD_x^\alpha \phi_i \right)(x), $$

so it suffices to prove the lemma for Caputo's definition:

$$ \left({}_0^CD_x^\alpha \phi_i \right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\tau)^{-\alpha} \phi_i'(\tau)\,d\tau. \tag{4.12} $$

Noting that $\phi_i(x) = (1-2x)\cos(i\theta) - i\sqrt{x-x^2}\sin(i\theta)$, where $\theta = \arccos(2x-1)$, and since $\sqrt{x-x^2} \le \frac{1}{2}$, we get

$$ \left| {}_0^CD_x^\alpha \phi_i(x) \right| \le \frac{1+\frac{i}{2}}{\Gamma(1-\alpha)} \int_0^x (x-\tau)^{-\alpha}\,d\tau = \frac{x^{1-\alpha}(1+\frac{i}{2})}{\Gamma(2-\alpha)}, $$

and since $x^{1-\alpha} \le 1$, then the lemma is proved. □

Lemma 2 *For* $x \in [0,1]$ *and* $1 < \gamma < 2$, *we have*

$$ \left| {}_0^RD_x^\gamma \phi_i(x) \right| \lessapprox i^2. \tag{4.13} $$

Proof The proof of this lemma is similar to the proof of Lemma 1. □

Theorem 2 *For $\beta \in (0,2) - \{1\}$, the following Riesz fractional derivatives relation is valid*:

$$\frac{\partial^{\beta}\phi_i(x)}{\partial|x|^{\beta}} = i\sum_{k=0}^{i}\frac{(i+k-1)!(k+1)!\Gamma(\beta-k-2)\sec(\frac{\pi\beta}{2})\sin(\pi(\beta-k))}{2\pi(2k)!(i-k)!\ell^{k}} \times\Big[\big((-4)^{k}\big((\ell-x)^{1+k-\beta}\big(-(2+k)x+\ell\beta\big)+(-1)^{i}x^{1+k-\beta}\big((2+k)(-\ell+x)+\ell\beta\big)\big)\big)\Big]. \tag{4.14}$$

Proof Based on the two analytic forms of $T_i^{\ell}(x)$ given in (2.11) and (2.12) and if we perform similar manipulations as in the proof of Theorem 1, then we get the desired result. □

Similarly we can prove the following estimates for the Riesz fractional derivatives of $\phi_i(x)$.

Lemma 3 *For $x \in [0,1]$ and $0 < \beta < 1$, we have*

$$\left|\frac{\partial^{\beta}\phi_i(x)}{\partial|x|^{\beta}}\right| \lessapprox i. \tag{4.15}$$

Lemma 4 *For $x \in [0,1]$ and $1 < \gamma < 2$, we have*

$$\left|\frac{\partial^{\gamma}\phi_i(x)}{\partial|x|^{\gamma}}\right| \lessapprox i^2. \tag{4.16}$$

Theorem 3 *If $\psi_j(t)$ is defined as in* (4.2), *then the following integral formula holds*:

$$\int_0^t \psi_j(s)\,ds = j\sum_{k=0}^{j}\frac{(-1)^{j-k}(j+k-1)!2^{2k}}{(k+1)(j-k)!(2k)!\tau^{k}}t^{k+1}. \tag{4.17}$$

Proof The proof is easily obtained by integrating relation (2.11) over the interval $[0,t]$. □

4.2 Numerical algorithms for handling equation (3.11)

This section is devoted to describing in detail two numerical algorithms for handling equation (3.11). The first algorithm depends on the application of the typical collocation method, while the second depends on the application of the Petrov-Galerkin method. The main idea behind the two proposed algorithms is based on making use of Theorems 1 and 3 along with the application of the collocation and Petrov-Galerkin methods in order to transform equation (3.11) into a system of linear or nonlinear algebraic equations in the unknown expansion coefficients c_{ij}. The linear system is solved using Gauss elimination, and the nonlinear one is solved via Newton's iterative method.

4.2.1 The collocation approach

Consider the following approximate solution of equation (3.11):

$$g_M(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{M}c_{ij}\phi_i(x)\psi_j(t). \tag{4.18}$$

Now, let $v(x,t)$ and $k(x,t) \in V$. Then these two functions can be expanded in the following double expansions:

$$v(x,t) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} v_{pq} T_p^{\ell}(x) T_q^{\tau}(t), \tag{4.19}$$

$$k(x,t) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} k_{pq} T_p^{\ell}(x) T_q^{\tau}(t), \tag{4.20}$$

and the coefficients v_{pq} and k_{pq} are given by the formulas

$$v_{pq} = \frac{1}{h_p h_q} \int_0^{\tau} \int_0^{\ell} \frac{v(x,t) T_p^{\ell}(x) T_q^{\tau}(t),}{x^{1/2}(\ell - x)^{1/2} t^{1/2}(\tau - t)^{1/2}}\, dx\, dt, \tag{4.21}$$

$$k_{pq} = \frac{1}{h_p h_q} \int_0^{\tau} \int_0^{\ell} \frac{k(x,t) T_p^{\ell}(x) T_q^{\tau}(t),}{x^{1/2}(\ell - x)^{1/2} t^{1/2}(\tau - t)^{1/2}}\, dx\, dt. \tag{4.22}$$

Now consider the following approximations:

$$v(x,s) \approx v_M(x,s) = \sum_{p=0}^{M}\sum_{q=0}^{M} v_{pq} T_p^{\ell}(x) T_q^{\tau}(s), \tag{4.23}$$

and

$$k(x,s) \approx k_M(x,s) = \sum_{p=0}^{M}\sum_{q=0}^{M} k_{pq} T_p^{\ell}(x) T_q^{\tau}(s). \tag{4.24}$$

In order to apply the collocation method, and due to (4.18), we note that the residual of (3.11) takes the form

$$\begin{aligned} R(x,t) = {} & \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\phi_i(x)\psi_j(t) + \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\, {}_0^R D_x^{\beta} \phi_i(x) \int_0^t v(x,s)\psi_j(s)\, ds \\ & - \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\, {}_0^R D_x^{\gamma} \phi(x) \int_0^t k(x,s)\, \psi_j(s)\, ds - F(x,t,g), \end{aligned} \tag{4.25}$$

and therefore

$$\begin{aligned} R(x,t) = {} & \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\phi_i(x)\psi_j(t) + \sum_{i,j=0}^{M}\sum_{p,q=0}^{M} c_{ij} v_{pq} T_p^{\ell}(x)\, {}_0^R D_x^{\beta} \phi_i(x) \int_0^t T_q^{\tau}(s)\psi_j(s)\, ds \\ & - \sum_{i,j=0}^{M}\sum_{p,q=0}^{M} c_{ij} k_{pq} T_p^{\ell}(x)\, {}_0^R D_x^{\gamma} \phi_i(x) \int_0^t T_q^{\tau}(s)\psi_j(s)\, ds - F(x,t,g). \end{aligned} \tag{4.26}$$

Now, making use of the two power form representations of the polynomials $\phi_i(x)$ and $\psi_j(t)$, and the two relations in (2.10), (4.8) and (4.17) enable us to write the residual $R(x,t)$ in the following form:

$$
\begin{aligned}
R(x,t) = {} & \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{j} c_{ij}\frac{(-1)^{i+j-k-s}2^{2k}2^{2s}(i+k-1)!(j+s-1)!}{(2k)!(2s)!(i-k)!(j-s)!\tau^{s}\ell^{k}}(\ell-x)x^{k+1}t^{s} \\
& + \sum_{i,j=0}^{M}\sum_{p,q=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{p} ipc_{ij}v_{pq} \\
& \times \frac{(-1)^{i+k+p-s+1}(k+1)2^{2s}(p+s-1)!(i+k-1)!\Gamma(\beta-k-1)\sin(\pi(\beta-k))}{2\sqrt{\pi}(p-s)!(2s)!(i-k)!\Gamma(k+\frac{1}{2})\ell^{s+k-1}} \\
& \times \left(x^{-\beta+s+k+1} - \frac{(k+2)}{(-\beta+k+2)\ell}x^{-\beta+s+k+2}\right) \\
& \times \left(\zeta\sum_{s=0}^{\zeta}\frac{(-1)^{\zeta-s}(\zeta+s-1)!2^{2s}}{(s+1)(\zeta-s)!(2s)!\tau^{s}}t^{s+1} + \eta\sum_{s=0}^{\eta}\frac{(-1)^{\eta-s}(\eta+s-1)!2^{2s}}{(s+1)(\eta-s)!(2s)!\tau^{s}}t^{s+1}\right) \\
& - \sum_{i,j=0}^{M}\sum_{p,q=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{p} ipc_{ij}k_{pq} \\
& \times \frac{(-1)^{i+k+p-s+1}(k+1)2^{2s}(p+s-1)!(i+k-1)!\Gamma(\gamma-k-1)\sin(\pi(\gamma-k))}{2\sqrt{\pi}(p-s)!(2s)!(i-k)!\ell^{r+k-1}\Gamma(k+\frac{1}{2})} \\
& \times \left(x^{-\gamma+s+k+1} - \frac{(k+2)}{(-\gamma+k+2)\ell}x^{-\gamma+s+k+2}\right) \\
& \times \left(\zeta\sum_{s=0}^{\zeta}\frac{(-1)^{\zeta-s}(\zeta+s-1)!2^{2s}}{(s+1)(\zeta-s)!(2s)!\tau^{s}}t^{s+1} + \eta\sum_{s=0}^{\eta}\frac{(-1)^{\eta-s}(\eta+s-1)!2^{2s}}{(s+1)(\eta-s)!(2s)!\tau^{s}}t^{s+1}\right) \\
& - F(x,t,g), \qquad (4.27)
\end{aligned}
$$

where $\zeta = |q-j|$ and $\eta = q+j$. Now, we apply the typical collocation method to equation (4.27). In fact, if this equation is collocated at the following set of points: $\{(\frac{i\ell}{M+2}, \frac{j\tau}{M+2}) : 1 \le i,j \le M+1\}$, then we obtain a system of algebraic equations of dimension $(M+1)^2$ in the unknown expansion coefficients $\{c_{ij} : 0 \le i,j \le M\}$. This linear algebraic system is solved by the Gaussian elimination procedure or by any suitable solver, while the nonlinear system is solved with the aid of Newton's iterative method. Hence, the desired spectral solution can be obtained.

4.2.2 Petrov-Galerkin approach

This section is devoted to introducing an alternative algorithm for finding a spectral solution to equation (3.11). The main advantage of employing the Petrov-Galerkin method is its flexibility in choosing test functions, since we can choose a suitable set of functions that is not identical to the set of trial functions. Now, we choose the test functions to be

$$\rho_{mn}(x,t) = x^{m}t^{n}.$$

Then the application of the Petrov-Galerkin method leads to

$$\int_0^{\tau}\int_0^{\ell} R(x,t)\rho_{mn}(x,t)\,dx\,dt = 0, \quad 1 \le m,n \le M+1, \qquad (4.28)$$

where $R(x,t)$ is defined as in (4.27).

Now, equation (4.28) can be written alternatively as

$$\begin{aligned}
&\sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{j} c_{ij}\frac{(-1)^{i+j-k-s}2^{2k+2s}(i+k-1)!(j+s-1)!\ell^{m+3}\tau^{n+1}}{(2k)!(2s)!(i-k)!(j-s)!(k+m+2)(k+m+3)(n+s+1)} \\
&\quad+\sum_{i,j=0}^{M}\sum_{p,q=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{j} ijpc_{ij}v_{pq} \\
&\quad\times\frac{(-1)^{i+j-s+k+1}(k+1)(i+k-1)!\Gamma(\beta-k-1)\sin(\pi(\beta-k))(j+s-1)!2^{2s}}{(s+1)(j-s)!(2s)!\tau^{s}\sqrt{\pi}(i-k)!\ell^{k-1}\Gamma(k+\frac{1}{2})} \\
&\quad\times\sum_{s=0}^{p}\frac{(-1)^{p-s}2^{2s}(p+s-1)!}{(p-s)!(2s)!} \\
&\quad\times\left(\zeta\sum_{s=0}^{\zeta}\frac{(-1)^{\zeta-s}(\zeta+s-1)!2^{2s}\tau^{n+2}}{(s+1)(\zeta-s)!(2s)!(2+n+s)}+\eta\sum_{s=0}^{\eta}\frac{(-1)^{\eta-s}(\eta+s-1)!2^{2s}\tau^{n+2}}{(s+1)(\eta-s)!(2s)!(2+n+s)}\right) \\
&\quad\times\left(\frac{\ell^{-\beta+k+m+s+2}}{-\beta+k+m+s+2}-\frac{(k+2)\ell^{-\beta+k+m+s+2}}{(-\beta+k+2)(-\beta+k+m+s+3)}\right) \\
&\quad-\sum_{i,j=0}^{M}\sum_{p,q=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{j} ijpc_{ij}k_{pq} \\
&\quad\times\frac{(-1)^{i+k-s+1}(k+1)(i+k-1)!(j+s-1)!2^{2s}\Gamma(\gamma-k-1)\sin(\pi(\gamma-k))}{(s+1)(j-s)!(2s)!\tau^{s}\sqrt{\pi}(i-k)!\ell^{s+k-1}\Gamma(k+\frac{1}{2})} \\
&\quad\times\sum_{s=0}^{p}\frac{(-1)^{p-s}2^{2r}(p+s-1)!}{(p-s)!(2s)!} \\
&\quad\times\left(\zeta\sum_{s=0}^{\zeta}\frac{(-1)^{\zeta-s}(\zeta+s-1)!2^{2s}\tau^{n+2}}{(s+1)(\zeta-s)!(2s)!(n+s+2)}+\eta\sum_{s=0}^{\eta}\frac{(-1)^{\eta-s}(\eta+s-1)!2^{2s}\tau^{n+2}}{(s+1)(\eta-s)!(2s)!(n+s+2)}t^{n+s+1}\right) \\
&\quad\times\left(\frac{\ell^{-\gamma+k+m+s+2}}{-\beta+k+m+s+2}-\frac{(k+2)\ell^{-\gamma+k+m+s+2}}{(-\gamma+k+2)(-\gamma+k+m+s+3)}\right)-F_{m,n}=0, \qquad (4.29)
\end{aligned}$$

where $F_{m,n}=\int_0^\tau\int_0^\ell F(x,t,g)\rho_{mn}(x,t)\,dx\,dt$. Hence, a system of equations of dimension $(M+1)^2$ in the unknown expansion coefficients c_{ij} is generated. This system can be efficiently solved by the Gauss elimination technique, and hence an approximate spectral solution can be obtained.

4.3 Numerical algorithms for handling equation (3.23)

In this section, we describe how the collocation and Petrov-Galerkin procedures can be employed to handle equation (3.23). As we have done in Section 4.2.1, collocation and Petrov-Galerkin are employed along with Theorems 2 and 3 to convert equation (3.23) into a system of linear algebraic equations in the unknown expansion coefficients c_{ij}.

4.3.1 The collocation approach

Now, consider the following approximate spectral solution of equation (3.23):

$$g_M(x,t)=\sum_{i=0}^{M}\sum_{j=0}^{M}c_{ij}\phi_i(x)\psi_j(t). \qquad (4.30)$$

In order to apply the collocation method, and due to (4.30), we note that the residual of (3.23) takes the form

$$\begin{aligned} R(x,t) = {}& \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\phi_i(x)\psi_j(t) + \kappa_\beta \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\frac{\partial^\beta \phi_i(x)}{\partial |x|^\beta}\int_0^t \psi_j(s)\,ds \\ & - \kappa_\gamma \sum_{i=0}^{M}\sum_{j=0}^{M} c_{ij}\frac{\partial^\gamma \phi_i(x)}{\partial |x|^\gamma}\int_0^t \psi_j(s)\,ds - F(x,t,g). \end{aligned} \tag{4.31}$$

The latter formula, along with the aid of the two power form representations of the polynomials $\phi_i(x)$ and $\psi_j(t)$, and the two relations in (4.14) and (4.17), enable one to write the residual $R(x,t)$ in the following form:

$$\begin{aligned} R(x,t) = {}& \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{j} ij c_{ij}\frac{(-1)^{j-s+k}(i+k-1)!(j+s-1)!2^{2s}\sqrt{\pi}}{\Gamma(k+\frac{1}{2})(i-k)!(j-s)!k!(2s)!\ell^k\tau^s} x(\ell-x)^{k+1}t^s \\ & + \kappa_\beta \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{i} c_{ij} i \sum_{k=0}^{i} \frac{(i+k-1)!\Gamma(2+k)\Gamma(\beta-k-2)\sec(\frac{\pi\beta}{2})\sin(\pi(\beta-k))}{2\pi(2k)!(i-k)!\ell^k} \\ & \times \Big(\Big((-4)^k\big((-1)^{2k}(\ell-x)^{1+k-\beta}\big(-(2+k)x+\ell\beta\big) + (-1)^i x^{1+k-\beta}\big((2+k)(-\ell+x)+\ell\beta\big)\big)\Big) \\ & \times j\sum_{s=0}^{j}\frac{(-1)^{j-s}(j+s-1)!2^{2s}}{(s+1)(j-s)!(2s)!\tau^s}t^{s+1} \\ & - \kappa_\gamma \sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{i} c_{ij} i \sum_{k=0}^{i} \frac{(i+k-1)!\Gamma(2+k)\Gamma(\gamma-k-2)\sec(\frac{\pi\gamma}{2})\sin(\pi(\gamma-k))}{2\pi(2k)!(i-k)!\ell^k} \\ & \times \Big(\Big((-4)^k\big((-1)^{2k}(\ell-x)^{1+k-\gamma}\big(-(2+k)x+\ell\gamma\big) + (-1)^i x^{1+k-\gamma}\big((2+k)(-\ell+x)+\ell\gamma\big)\big)\Big) \\ & \times j\sum_{s=0}^{j}\frac{(-1)^{j-s}(j+s-1)!2^{2s}}{(s+1)(j-s)!(2s)!\tau^s}t^{s+1} - F(x,t,g). \end{aligned} \tag{4.32}$$

Now, we apply the typical collocation method. In fact, the residual $R(x,t)$ is enforced to vanish at the following set of points: $\{(\frac{i\ell}{M+2},\frac{j\tau}{M+2}) : 1 \le i,j \le M+1\}$. Then we obtain a linear algebraic system of equations of dimension $(M+1)^2$ in the unknown expansion coefficients $\{c_{ij} : 0 \le i,j \le M\}$ which is solved by Gauss elimination solver.

4.3.2 Petrov-Galerkin approach

To apply the Petrov-Galerkin method to equation (3.23), we choose the test functions to be

$$\rho_{mn}(x,t) = x^m t^n,$$

and therefore the application of the Petrov-Galerkin method leads to

$$\int_0^\tau \int_0^\ell R(x,t)\rho_{mn}(x,t)\,dx\,dt = 0, \quad 1 \le m,n \le M+1, \tag{4.33}$$

where $R(x,t)$ is defined as in (4.32). Now, equation (4.33) can be written alternatively as

$$\begin{aligned}
&\sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{j} ijc_{ij}\frac{(-1)^{j-s+k}(i+k-1)!(j+s-1)!2^{2s}\sqrt{\pi}}{\Gamma(k+\frac{1}{2})(i-k)!(j-s)!k!(2s)!\ell^k\tau^s}\int_0^{\ell}\int_0^{\tau}x^{m+1}(\ell-x)^{k+1}t^{n+s}\,dt\,dx\\
&\quad+(\kappa_\beta-\kappa_\gamma)\sum_{i=0}^{M}\sum_{j=0}^{M}\sum_{k=0}^{i}\sum_{s=0}^{j} ijc_{ij}\\
&\quad\times\frac{(i+k-1)!\Gamma(2+k)\Gamma(\beta-k-2)\sec(\frac{\pi\beta}{2})\sin(\pi(\beta-k))(-1)^{j-s}(j+s-1)!2^{2s}}{2\pi(2k)!(i-k)!\ell^k(s+1)(j-s)!(2s)!\tau^s}\\
&\quad\times\int_0^{\ell}\int_0^{\tau}\Bigl(\Bigl(\Bigl((-4)^k\bigl((-1)^{2k}(\ell-x)^{1+k-\beta}\\
&\quad\times\bigl(-(2+k)x+\ell\beta\bigr)+(-1)^i x^{1+k-\beta}\bigl((2+k)(-\ell+x)+\ell\beta\bigr)\bigr)\Bigr)\Bigr)\\
&\quad\times x^m t^{n+s+1}\,dt\,dx-F_{m,n}=0,
\end{aligned}\tag{4.34}$$

where

$$F_{m,n}=\int_0^{\tau}\int_0^{\ell}F(x,t,g)\rho_{mn}(x,t)\,dx\,dt.$$

Equation (4.34) generates a linear system of dimension $(M+1)^2$ in the unknown expansion coefficients c_{ij}, which is efficiently solved using Gauss elimination solver.

5 Convergence and error analysis of the suggested double expansion

In this section we concentrate on investigating the convergence and error analysis of the suggested double Tchebyshev expansion. Three important theorems are stated and proved for this purpose. The first theorem shows that the double Tchebyshev expansion of a function $u(x,t)$ converges uniformly to $u(x,t)$. The second and third theorems discuss the error analysis of the full discretization scheme for the two problems (3.5) and (3.17).

First, the following lemma is useful.

Lemma 5 (See [17], p.742) *Let $u(x)$ be a function such that $u(k)=a_k$. Suppose that the following assumptions are satisfied*:

1. *$u(x)$ is a continuous, positive, decreasing function for $x\ge n$.*
2. *$\sum a_n$ is convergent, and $R_n=\sum_{k=n+1}^{\infty}a_k$.*

Then

$$R_n\le\int_n^{\infty}u(x)\,dx.$$

Theorem 4 *A function*

$$g(x,t)=x(\ell-x)f(x)h(t)\in L^2_{w(x,t)}(\Omega), w(x,t)=\frac{1}{x^{5/2}(\ell-x)^{5/2}t^{1/2}(\tau-t)^{1/2}}$$

with $|f^{(3)}(x)|\le M_1$, $|h^{(3)}(x)|\le M_2$ can be expanded as an infinite sum of the basis $\{\phi_i(x)\psi_j(t)\}_{0\le i,j\le M}$, and the series converges uniformly to $g(x,t)$. Moreover, the following *inequality is satisfied by the expansion coefficients in* (4.18):

$$|c_{ij}| \lessapprox \frac{1}{i^4 j^4} \quad \forall i,j > 3. \tag{5.1}$$

Proof Relation (4.6) implies that

$$c_{ij} = \frac{1}{h_i h_j} \int_0^\tau \int_0^\ell \frac{g(x,t)\phi_i(x)\psi_j(t)}{x^{5/2}(\ell - x)^{5/2} t^{1/2}(\tau - t)^{1/2}}\, dx\, dt. \tag{5.2}$$

We make use of the two substitutions:

$$\frac{2t}{\tau} - 1 = \cos\xi, \qquad \frac{2x}{\ell} - 1 = \cos\theta,$$

to convert the coefficients c_{ij} in (5.2) into the form

$$\begin{aligned} c_{ij} = {} & \frac{1}{h_i h_j} \int_0^\pi f\left(\frac{\ell}{2}(1+\cos\theta)\right) \cos(i\theta)\, d\theta \\ & \times \int_0^\pi h\left(\frac{\tau}{2}(1+\cos\xi)\right) \cos(j\xi)\, d\xi. \end{aligned} \tag{5.3}$$

Application of the integration by parts three times to the right hand side of equation (5.3) enables one to write c_{ij} in the form

$$\begin{aligned} c_{ij} = {} & \frac{\ell^3 \tau^3}{64 h_i h_j} \int_0^\pi f^{(3)}\left(\frac{\ell}{2}(1+\cos\theta)\right) \Omega_i(\theta) \sin(\theta)\, d\theta \\ & \times \int_0^\pi h^{(3)}\left(\frac{\tau}{2}(1+\cos\xi)\right) \Omega_j(\xi) \sin(\xi)\, d\xi, \end{aligned} \tag{5.4}$$

where

$$\Omega_i(\theta) = \frac{i\sin(\theta)((i^2+11)\cos(2\theta) - i^2 + 19)\cos(\theta i) - 6\cos(\theta)\sin(\theta i)((i^2+1)\cos(2\theta) - i^2 + 4)}{2(i^2-9)(i^2-4)(i-i^3)}.$$

In virtue of the two assumptions $|f^{(3)}(x)| \le M_1$, $|h^{(3)}(t)| \le M_2$ and the well-known inequality $|T_i^\ell(x)| \le 1$, and since

$$\left|\Omega_i(\theta)\right| \le \frac{i^3 + 6i^2 + 15i + 15}{(i-3)_7} \lessapprox i^{-4},$$

we get the theorem. □

Theorem 5 *Let $\mathcal{E}_M$ be the residual of equation* (3.5), i.e.,

$$\mathcal{E}_M = \left| D_t g_M(x,t) + \nu(x,t)\, {}_0^R D_x^\beta g_M(x,t) - k(x,t)\, {}_0^R D_x^\gamma g_M(x,t) - f_1(x,t,g_M) \right|.$$

If $f_1(x,t,z)$ satisfies the Lipschitz condition in the variable z, and $\nu(x,t)$ and $k(x,t)$ are bounded functions, then we have the following estimate:

$$\mathcal{E}_M \lessapprox M^{-1}.$$

Proof We have

$$\begin{aligned}\mathcal{E}_M &= \Big|D_t g_M(x,t) + \nu(x,t)\,{}_0^R D_x^\beta g_M(x,t) - k(x,t)\,{}_0^R D_x^\gamma g_M(x,t) \\ &\quad - f_1(x,t,g) - f_1(x,t,g_M) + f_1(x,t,g)\Big| \\ &\le \Big|D_t g_M(x,t) + \nu(x,t)\,{}_0^R D_x^\beta g_M(x,t) - k(x,t)\,{}_0^R D_x^\gamma g_M(x,t) - f_1(x,t,g)\Big| \\ &\quad + \big|f_1(x,t,g_M) - f_1(x,t,g)\big| \\ &\le \Big|D_t g_M(x,t) + \nu(x,t)\,{}_0^R D_x^\beta g_M(x,t) - k(x,t)\,{}_0^R D_x^\gamma g_M(x,t) - f_1(x,t,g)\Big| + L|g_M - g| \\ &\le \Big|D_t(g_M - g) + \nu(x,t)\,{}_0^R D_x^\beta (g_M - g) - k(x,t)\,{}_0^R D_x^\gamma (g_M - g)\Big| + L|g_M - g|.\end{aligned}$$

Now, noting that

$$g - g_M = \sum_{j=M+1}^{\infty}\sum_{i=0}^{M} c_{i,j}\phi_i^\ell(x)T_j^\tau(t) + \sum_{j=0}^{\infty}\sum_{i=M+1}^{\infty} c_{i,j}\phi_i^\ell(x)T_j^\tau(t),$$

and based on Theorem 4, Lemmas 1 and 2 and the hypothesis of the theorem, we get the desired result. □

Theorem 6 *Let $\mathcal{F}_M$ be the residual of equation* (3.17), i.e.,

$$\mathcal{F}_M = \left|\frac{\partial g_M}{\partial t} + \kappa_\beta \frac{\partial^\beta g_M}{\partial |x|^\beta} - \kappa_\gamma \frac{\partial^\gamma g_M}{\partial |x|^\gamma} - f_1(x,t,g_M)\right|.$$

If $f_1(x,t,z)$ satisfies the Lipschitz condition in the variable z, then we have the following estimate:

$$\mathcal{F}_M \lessapprox M^{-1}.$$

Proof The steps of the proof are similar to those followed in the proof of Theorem 5, but based on Lemmas 3 and 4 instead of Lemmas 1 and 2. □

Now, the following theorem investigates the stability of the suggested double Tchebyshev expansion.

Theorem 7 *Under the assumptions of Theorem* 4, *we have* $\|g_{M+1}(x,t) - g_M(x,t)\|_\omega \lessapprox M^{-4}$.

Proof We have

$$\begin{aligned}&\big\|g_{M+1}(x,t) - g_M(x,t)\big\|_\omega^2 \\ &\quad = \left\|\sum_{i=0}^{M} c_{i,M+1}\phi_{M+1}^\ell(x)T_j^\tau(t) + \sum_{j=0}^{M} c_{M+1,j}\phi_{M+1}^\ell(x)T_j^\tau(t) + c_{M+1,M+1}\phi_{M+1}^\ell(x)T_{M+1}^\tau(t)\right\|_\omega^2.\end{aligned}$$

With the aid of the identity

$$\big\|\phi_i(x)T_j^\ell(t)\big\|_\omega^2 = h_i h_j \le \pi^2,$$

we get

$$\left\| g_{M+1}(x,t) - g_M(x,t) \right\|_{\omega}^2 < \pi^2 \left(\sum_{i=0}^{M} c_{i,M+1}^2 + \sum_{i=0}^{M} c_{M+1,j}^2 + c_{M+1,M+1}^2 \right).$$

Now, based on the result of Theorem 4, and following [18], and after some manipulations, we get the desired result. □

6 Numerical results and comparisons

This section is devoted to presenting some numerical results obtained by the application of the two proposed numerical methods. All the results are obtained using the software Mathematica 9. The results are accompanied by a comparison with results from the literature, obtained by applying some other numerical techniques.

Example 1 Consider the following equation (see [14]):

$$\frac{\partial u}{\partial t} = \Gamma(1.2)x^{1.8}\left({}_0D_x^{\gamma} u\right)(x,t) - u^3 + u^2 + \mu(x,t), \quad (x,t) \in (0,1) \times (0,\tau), \tag{6.1}$$

where

$$\mu(x,t) = e^{-t}\left(-x^9 + 3x^8 - 3x^7 + 2x^5 - x^4 + 6x^3 - 3x^2\right), \tag{6.2}$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \quad t \in [0,\tau],$$

and the initial condition

$$u(x,0) = 5\left(x^2 - x^3\right), \quad x \in [0,1],$$

where $\gamma = 1.8$, such that the exact solution of (6.1) is $u(x,t) = 5e^{-t}(x^2 - x^3)$. In [14], the authors assumed that the solution of (6.1) is given by $u_M(x,t) = \sum_{i=0}^{M} a_i(t)P_i^{(\alpha\cdot\beta)}(x)$. Then they obtained a linear system of differential equations in the unknowns $a_i(t)$, and they solved it using the finite difference method with step size τ. In Table 1 we list the maximum pointwise error of Example 1, while in Table 2, we compare the best errors of Example 1. We mention here that the computational complexity required to solve a linear system of differential equations of dimension 7 with step size $\frac{1}{300}$ is much smaller than the computational cost to solve a linear system of algebraic equations in the unknown expansion coefficients c_{ij} of dimension 49. Our computational cost is of order $\frac{2}{3}(M+1)^3 \approx 78{,}433$, while their computational cost is of order $\tau 4^{M+1} \ln 4 \approx 9{,}564{,}537$. Let E_C and E_{PG} denote, respectively, the maximum pointwise errors resulting from the application of the two methods, namely, the shifted Tchebyshev collocation method (SCCM) and the shifted Tchebyshev Petrov-Galerkin method (SCPGM).

Example 2 Consider the following equation (see [19]):

$$\frac{\partial u}{\partial t} + \nu \frac{\partial u}{\partial x} = \sigma\, {}_0D_x^{\gamma} u + e^{-t}x^3\left(4\nu - x - x^{1-\gamma}\right), \quad (x,t) \in (0,1) \times (0,\tau), \tag{6.3}$$

Table 1 Maximum pointwise error of Example 1

M	γ	E_C	E_{PG}	γ	E_C	E_{PG}	γ	E_C	E_{PG}
4	1.5	$2.1 \cdot 10^{-4}$	$8.5 \cdot 10^{-5}$	1.8	$6.2 \cdot 10^{-4}$	$3.4 \cdot 10^{-5}$	1.9	$2.7 \cdot 10^{-4}$	$8.6 \cdot 10^{-5}$
5		$3.8 \cdot 10^{-6}$	$5.2 \cdot 10^{-7}$		$5.4 \cdot 10^{-6}$	$3.6 \cdot 10^{-7}$		$9.3 \cdot 10^{-6}$	$8.1 \cdot 10^{-7}$
6		$3.3 \cdot 10^{-8}$	$2.7 \cdot 10^{-9}$		$3.9 \cdot 10^{-8}$	$1.9 \cdot 10^{-9}$		$6.1 \cdot 10^{-8}$	$9.7 \cdot 10^{-9}$

Table 2 Comparison of best errors of Example 1

	γ
	1.8
Method in [14] ($M = 6, \tau = \frac{1}{300}$)	$1.8 \cdot 10^{-9}$
SCCM ($M = 6$)	$3.9 \cdot 10^{-8}$
SCPGM ($M = 6$)	$1.9 \cdot 10^{-9}$

Table 3 Comparison between best errors of Example 2

	γ			
	1.2	**1.4**	**1.6**	**1.8**
Method in [19] ($\Delta x = 0.01$)	$0.5667 \cdot 10^{-5}$	$0.4444 \cdot 10^{-5}$	$0.3561 \cdot 10^{-5}$	$0.4848 \cdot 10^{-5}$
SCCM ($M = 14$)	$2.54 \cdot 10^{-16}$	$7.28 \cdot 10^{-16}$	$5.77 \cdot 10^{-15}$	$2.78 \cdot 10^{-15}$
SCPGM ($M = 14$)	$3.57 \cdot 10^{-15}$	$5.36 \cdot 10^{-15}$	$2.22 \cdot 10^{-14}$	$3.47 \cdot 10^{-14}$

subject to the nonhomogeneous boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = e^{-t}, \quad t \in [0,\tau],$$

where $\nu = 0.2$, $\sigma = \frac{\Gamma(5-\gamma)}{24}$, and the initial condition

$$u(x,0) = x^4, \quad x \in [0,1].$$

The exact solution of (6.3) is $u(x,t) = e^{-t}x^4$. In Table 3 we compare our results with the results obtained in [19].

From the results in Table 3, we conclude that our spectral method is more accurate than the fractional difference method used in [19].

Example 3 Consider the space fractional diffusion problem [14]

$$\begin{aligned} &\frac{\partial u}{\partial t} = A\frac{\partial^\sigma u}{\partial |x|^\sigma} + B\frac{\partial^\gamma u}{\partial |x|^\gamma} + f(x,t), \\ &\gamma \in (1,2], \qquad \sigma \in (0,1], \qquad (x,t) \in (0,1) \times (0,\tau), \end{aligned} \tag{6.4}$$

subject to the homogeneous boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in [0,\tau],$$

and the initial condition

$$u(x,0) = 0, \quad x \in [0,1],$$

Table 4 Maximum pointwise error of Example 3 ($M = 7$)

(γ,σ)	E_C	E_{PG}	(γ,σ)	E_C	E_{PG}	(γ,σ)	E_C	E_{PG}
(1.4,0.2)	$3.7 \cdot 10^{-9}$	$4.2 \cdot 10^{-9}$	(1.5,0.6)	$5.7 \cdot 10^{-9}$	$8.2 \cdot 10^{-9}$	(1.7,0.4)	$6.1 \cdot 10^{-9}$	$8.4 \cdot 10^{-9}$

Table 5 Comparison between different errors of Example 3 ($t = 2$)

x	E_C ($M = 7$)	E_{PG} ($M = 7$)	[14] ($M = 7$, $\tau = 0.002$)
0.1	$1.7 \cdot 10^{-9}$	$7.5 \cdot 10^{-9}$	$0.94 \cdot 10^{-7}$
0.3	$4.5 \cdot 10^{-9}$	$5.9 \cdot 10^{-9}$	$5.48 \cdot 10^{-7}$
0.5	$3.8 \cdot 10^{-9}$	$6.2 \cdot 10^{-9}$	$7.02 \cdot 10^{-7}$
0.7	$5.4 \cdot 10^{-9}$	$1.4 \cdot 10^{-9}$	$2.89 \cdot 10^{-6}$
0.9	$1.88 \cdot 10^{-9}$	$7.2 \cdot 10^{-9}$	$6.06 \cdot 10^{-6}$

where $f(x,t)$ is chosen such that the exact solution of equation (6.4) is $u(x,t) = t^{\gamma} e^{\sigma t} x^2(1-x)^2$. We take $A = B = 2$. We apply SCCM and SCPGM with $M = 7$. In Table 4, we list the maximum pointwise error for different values of σ and γ, while in Table 5, we compare our results with those obtained by the method developed in [14] for the case corresponding to $\sigma = 0.4$, $\gamma = 1.7$, and $\alpha = \beta = 0$.

Example 4 Consider the following Riesz fractional reaction dispersion equation [20, 21]:

$$\begin{aligned} &\frac{\partial u}{\partial t} + u = \frac{\partial^{\alpha} u}{\partial |x|^{\alpha}} + f(x,t), \\ &\alpha \in (1,2), \qquad (x,t) \in (0,\ell) \times (0,\tau), \end{aligned} \tag{6.5}$$

subject to the boundary conditions

$$u(0,t) = u(\ell,t) = 0, \quad t \in [0,\tau],$$

and the initial condition

$$u(x,0) = x^2(\ell - x)^2, \quad x \in [0,\ell],$$

where

$$f(x,t) = \frac{x^{2-\nu}(\ell^2(\nu-4)(\nu-3) + 6\ell(\nu-4)x + 12x^2) + (\ell - x)^{2-\nu}(\ell^2(\nu-1)\nu - 6\ell\nu x + 12x^2)}{\cos(\frac{\pi}{2}\nu)\Gamma(5-\nu)} e^{-t}.$$

We apply the Petrov-Galerkin Tchebyshev collocation method (PGCCM) to equation (6.5). In Table 6, we compare our results with the results obtained by [20] and [21]. The results in this table demonstrate that our method is more accurate than the methods developed in [20] and [21].

Example 5 Consider the Riesz space fractional diffusion problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = -\frac{\partial^{\beta} u}{\partial |x|^{\beta}} + \frac{\partial^{\gamma} u}{\partial |x|^{\gamma}} + f(x,t), \\ &\gamma \in (1,2], \qquad \beta \in (0,1], \qquad (x,t) \in (0,1) \times (0,\tau), \end{aligned} \tag{6.6}$$

Table 6 Comparison between different errors of Example 4 for the case $M = 9$

x	PGCCM			[20]			[21]		
	$\nu = 1.2$	$\nu = 1.5$	$\nu = 1.8$	$\nu = 1.2$	$\nu = 1.5$	$\nu = 1.8$	$\nu = 1.2$	$\nu = 1.5$	$\nu = 1.8$
0.2	$1 \cdot 10^{-12}$	$2 \cdot 10^{-12}$	$1 \cdot 10^{-12}$	$2 \cdot 10^{-12}$	$4 \cdot 10^{-11}$	$4 \cdot 10^{-11}$	$2 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$7 \cdot 10^{-4}$
0.4	$3 \cdot 10^{-12}$	$5 \cdot 10^{-12}$	$1 \cdot 10^{-12}$	$1 \cdot 10^{-10}$	$9 \cdot 10^{-11}$	$1 \cdot 10^{-10}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$6 \cdot 10^{-4}$
0.6	$4 \cdot 10^{-12}$	$6 \cdot 10^{-12}$	$1 \cdot 10^{-12}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$1 \cdot 10^{-10}$	$9 \cdot 10^{-4}$	$7 \cdot 10^{-4}$	$4 \cdot 10^{-4}$
0.8	$2 \cdot 10^{-12}$	$1 \cdot 10^{-12}$	$2 \cdot 10^{-12}$	$1 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$8 \cdot 10^{-4}$	$6 \cdot 10^{-4}$	$3 \cdot 10^{-4}$
1.0	$7 \cdot 10^{-12}$	$3 \cdot 10^{-12}$	$2 \cdot 10^{-12}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$7 \cdot 10^{-4}$	$6 \cdot 10^{-4}$	$3 \cdot 10^{-4}$
1.2	$4 \cdot 10^{-12}$	$2 \cdot 10^{-12}$	$2 \cdot 10^{-12}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$3 \cdot 10^{-10}$	$8 \cdot 10^{-4}$	$6 \cdot 10^{-4}$	$3 \cdot 10^{-4}$
1.4	$2 \cdot 10^{-12}$	$3 \cdot 10^{-12}$	$2 \cdot 10^{-12}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$9 \cdot 10^{-4}$	$7 \cdot 10^{-4}$	$4 \cdot 10^{-4}$
1.6	$3 \cdot 10^{-12}$	$3 \cdot 10^{-12}$	$1 \cdot 10^{-12}$	$5 \cdot 10^{-11}$	$1 \cdot 10^{-10}$	$6 \cdot 10^{-11}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$6 \cdot 10^{-4}$
1.8	$5 \cdot 10^{-12}$	$4 \cdot 10^{-12}$	$3 \cdot 10^{-12}$	$3 \cdot 10^{-11}$	$4 \cdot 10^{-11}$	$3 \cdot 10^{-11}$	$2 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$7 \cdot 10^{-4}$
Max	$7 \cdot 10^{-12}$	$6 \cdot 10^{-12}$	$3 \cdot 10^{-12}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$3 \cdot 10^{-10}$	-	-	-

Table 7 Pointwise error of Example 5 for the case $M = 14$ at $t = 0.3$

x	$\beta = 0.7, \gamma = 1.7$	$\beta = 0.8, \gamma = 1.8$	$\beta = 0.9, \gamma = 1.9$
0.0	0	0	0
0.2	$2.2 \cdot 10^{-12}$	$7.8 \cdot 10^{-13}$	$3.5 \cdot 10^{-15}$
0.4	$2.3 \cdot 10^{-12}$	$2.5 \cdot 10^{-13}$	$5.7 \cdot 10^{-15}$
0.6	$4.7 \cdot 10^{-12}$	$3.1 \cdot 10^{-13}$	$7.4 \cdot 10^{-15}$
0.8	$4.1 \cdot 10^{-12}$	$2.4 \cdot 10^{-13}$	$4.2 \cdot 10^{-15}$
1.0	0	0	0

subject to the boundary conditions

$$u(0,t) = e^{-t}, \qquad u(1,t) = 2e^{-t}, \quad t \in [0,\tau],$$

and the initial condition

$$u(x,0) = x^2 + 1, \quad x \in [0,1],$$

where $f(x,t)$ is chosen such that the exact solution of equation (6.6) is $u(x,t) = \frac{(x^2+1)}{e^t}$. We apply the SCPGM with $M = 14$. In Table 7, we list the pointwise error for different values of β and γ.

7 Concluding remarks

FADEs are used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium. In this research article, two efficient spectral methods are presented and analyzed to solve two kinds of space fractional linear advection-dispersion problems. The spectral collocation and Petrov-Galerkin methods are employed to obtain semi-analytic solutions for the FADE. Efficient and highly accurate solutions are obtained with a small number of retained modes.

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