Chen et al. *Advances in Difference Equations* (2017) 2017:279
DOI 10.1186/s13662-017-1331-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

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Existence of positive mild solutions for a class of fractional evolution equations on the half line

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Abstract

The equivalent integral equation of a new form for a class of fractional evolution equations is obtained by the method of Laplace transform, which is different from those given in the existing literature. By the monotone iterative method without the assumption of lower and upper solutions, we present some new results on the existence of positive mild solutions for the abstract fractional evolution equations on the half-line.

Keywords: fractional order; abstract evolution equations; mild solutions; monotone iterative method

1 Introduction

In this paper, we are concerned with the following fractional evolution equation in the Banach space E:

$$\begin{cases} {}^{C}D^{\alpha}_{0+}u(t) = Au(t) + f(t, u(t)), & t \in (0, +\infty), \\ u(0) = \mu u(\beta), \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha}_{0+}$ is the Caputo fractional derivative, $0 < \alpha < 1$, $\mu > 0$, $\beta > 0$, A is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ of operators on Banach E, and $f : [0, +\infty) \times E \to E$ satisfies certain conditions.

Fractional calculus, a generalization of the ordinary differentiation and integration, has played a significant role in science, economy, engineering, and other fields (see [1–3]). Today there is a large number of papers dealing with the fractional differential equations (see [4–16]) due to their various applications. One of the branches is the research on the theory about the evolution equations of fractional order, which comes from physics. Recently, fractional evolution equations have attracted increasing attention around the world, see [7–16] and the references therein. Among the existing literature, most of them are focused on the existence of the solutions on the finite interval, see [7–16].

In [8], El-Sayed investigated the Cauchy problem in a Banach space for a class of fractional evolution equations

$$\begin{cases} \frac{d^\alpha u}{d^\alpha t} = Au(t) + B(t)u(t), & t \in [0, T], \\ u(0) = u_0, \end{cases}$$

where $0 < \alpha \leq 1$, $T > 0$. The existence and uniqueness of the solution for the above Cauchy problem were studied for a wide class of the family of operators $\{B(t) : t \geq 0\}$.

As far as we know, for the first time, the equivalent integral equation of the above equation was given in terms of some probability densities by the method of Laplace transform. And since then, most of the research in this direction has been based on this paper. However, many of the previous papers about the existence of solutions of fractional evolution equations are only on the finite interval, and those presenting the existence results on the half-line are still few.

Motivated by [8, 9, 17], in this paper, we study the differential equation (1.1) under certain conditions on the unbounded domains. Here, by a method similar to that used in [8, 9], we give a corrected form of the equivalent integral equation of the main problem (1.1), which is different from those obtained in the existing literature. Employing the monotone iterative method, without the assumption of lower and upper solutions, we present some new results on the existence of positive mild solutions for the abstract evolution equations of fractional order. And to our best knowledge, there is not any paper to deal with the abstract problems of fractional order on the unbounded domains.

The rest of the paper is organized as follows. In Section 2, we introduce the definitions of fractional integral and fractional derivative, some results about fractional differential equations and some useful preliminaries. In Section 3, we obtain the existence result of the solution for problem (1.1) by the monotone iterative method. Then an example is given in Section 4 to demonstrate the application of our result.

2 Preliminaries

First of all, we present some fundamental facts on the fractional calculus theory which we will use in the next section.

Definition 2.1 ([1–3]) The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\nu} h(t) = D_{0+}^{-\nu} h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s)\, ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([1–3]) The Caputo fractional derivative of order $\nu > 0$ of a continuous function $h : (0, \infty) \to \mathbb{R}$ is given by

$${}^{C}D_{0+}^{\nu} h(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} h^n(s)\, ds, \tag{2.2}$$

where $n = [\nu] + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([1, 3]) *Assume that* ${}^{C}D_{0+}^{\nu}h(t) \in L^1(0,+\infty)$, $\nu > 0$. *Then we have*

$$I_{0+}^{\nu}\,{}^{C}D_{0+}^{\nu}h(t) = h(t) + C_1 + C_2 t + \cdots + C_N t^{N-1}, \quad t > 0, \tag{2.3}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, *where* N *is the smallest integer greater than or equal to* ν.

If h is an abstract function with values in the Banach space E, then the integrals appearing in Definition 2.1, Definition 2.2 and Lemma 2.1 are taken in Bochner's sense. And a measurable function h is Bochner integrable if the norm of h is Lebesgue integrable.

Now let us recall some definitions and standard facts about the cone.

Let P be a cone in the ordered Banach space E, which defines a partial order on E by $x \leq y$ if and only if $y - x \in P$. P is normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ is the zero element of the Banach space E. The infimum of all N with the property above is called the normal constant of P. For more details on the cone P, we refer readers to [18, 19].

Throughout the paper, we set E to be an ordered Banach space with the norm $\|\cdot\|$ and the partial order '$\leq$'. Let $P = \{x \in E \mid x \geq \theta\}$ be a positive cone, which is normal with normal constant N. Let $J = [0, +\infty)$. Set

$$\mathrm{BC}(J, E) = \bigl\{u(t) \mid u(t) \text{ is continuous and bounded on } J\bigr\}.$$

Obviously, $\mathrm{BC}(J, E)$ is a Banach space with the norm $\|u\|_{\mathrm{BC}} = \sup_{t\in J}\|u(t)\|$. Let

$$P_{\mathrm{BC}} = \bigl\{u \in \mathrm{BC}(J, E) \mid u(t) \geq \theta, t \in J\bigr\}.$$

It is easy to see that P_{BC} is also normal with the same normal constant N of the cone P. Besides, $\mathrm{BC}(J, E)$ is also an ordered Banach space with the partial order '$\leq$' induced by the positive cone P_{BC} (without confusion, we denote by '$\leq$' the partial order on both E and $\mathrm{BC}(J, E)$).

We denote by $[v, w]$ the order interval $\{u \in P_{\mathrm{BC}} \mid v \leq u \leq w, v, w \in \mathrm{BC}(J, E)\}$ on $\mathrm{BC}(J, E)$, and use $[v(t), w(t)]$ to denote the order interval $\{z \in E \mid v(t) \leq z \leq w(t)\}$ on E for $t \in J$.

Next, we give some facts about the semigroups of linear operators. These results can be found in [20, 21].

For a strongly continuous semigroup (i.e., C_0-semigroup) $\{T(t)\}_{t\geq 0}$, the infinitesimal generator of $\{T(t)\}_{t\geq 0}$ is defined by

$$Ax = \lim_{t\to 0^+}\frac{T(t)x - x}{t}, \quad x \in E.$$

We denote by $D(A)$ the domain of A, that is,

$$D(A) = \left\{x \in E \,\middle|\, \lim_{t\to 0^+}\frac{T(t)x - x}{t} \text{ exists}\right\}.$$

Lemma 2.2 ([20, 21]) *Let* $\{T(t)\}_{t\geq 0}$ *be a* C_0*-semigroup, then there exist constants* $C \geq 1$ *and* $\omega \in \mathbb{R}$ *such that* $\|T(t)\| \leq Ce^{\omega t}$, $t \geq 0$.

Lemma 2.3 ([20, 21]) *A linear operator A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ if and only if*

(i) *A is closed and $\overline{D(A)} = E$.*

(ii) *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and, for every $\lambda > 0$, we have*

$$\left\|R(\lambda, A)\right\| \leq \frac{1}{\lambda},$$

where

$$R(\lambda, A) := (\lambda I - A)^{-1} = \int_0^{+\infty} e^{-\lambda t} T(t)x\,dt, \quad x \in E.$$

Definition 2.3 ([20, 21]) A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is said to be uniformly exponentially stable if $\omega_0 < 0$, where ω_0 is the growth bound of $\{T(t)\}_{t\geq 0}$, which is defined by

$$\omega_0 = \inf\left\{\omega \in \mathbb{R} \mid \exists C \geq 1 \text{ such that } \left\|T(t)\right\| \leq Ce^{\omega t}, t \geq 0\right\}.$$

Definition 2.4 ([17]) A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is said to be positive on E if order inequality $T(t)x \geq \theta, x \in E$ and $t \geq 0$.

According to Lemma 2.2 and Definition 2.3, if $\{T(t)\}_{t\geq 0}$ is a uniformly exponentially stable C_0-semigroup with the growth bound ω_0, there exists a constant $C \geq 1$ such that $\|T(t)\| \leq Ce^{\omega t}, t \geq 0$, for any $\omega \in (0, |\omega_0|]$. Now, we define a norm in E by

$$\|x\|_\omega = \sup_{t\geq 0}\left\|e^{\omega t} T(t)x\right\|.$$

Evidently, $\|x\| \leq \|x\|_\omega \leq C\|x\|$, that is to say, the norms $\|\cdot\|_\omega$ and $\|\cdot\|$ are equivalent. We denote by $\|T(t)\|_\omega$ the norm of $T(t)$ induced by the norm $\|\cdot\|_\omega$, then

$$\left\|T(t)\right\|_\omega \leq e^{-\omega t}, \quad t \geq 0. \tag{2.4}$$

Also, we can define the equivalent norm on $\mathrm{BC}(J, E)$ by

$$\|u\|_{\mathrm{BC}_\omega} = \sup_{t\in J}\left\|u(t)\right\|_\omega, \quad u \in \mathrm{BC}(J, E).$$

3 Main results

In this section, we present the existence theorem for the abstract fractional differential equation on the half-line. In order to prove our main result, we need the following facts and lemmas.

Consider the one-sided stable probability density [9, 10, 22]

$$\psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty),$$

where $0 < \alpha < 1$.

From [9, 10, 22], the Laplace transform of the one-sided stable probability density $\psi_\alpha(\theta)$ is given by

$$\mathcal{L}\big[\psi_\alpha(\theta)\big] = \int_0^\infty e^{-\lambda\theta}\psi_\alpha(\theta)\,d\theta = e^{-\lambda^\alpha}, \quad 0 < \alpha < 1. \tag{3.1}$$

By Remark 2.8 in [10], for $0 \le \gamma \le 1$, one has

$$\int_0^\infty \theta^{-\alpha\gamma}\psi_\alpha(\theta)\,d\theta = \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}. \tag{3.2}$$

In the following, we assume that $\{T(t)\}_{t\ge 0}$ is a uniformly exponentially stable C_0-semigroup with the growth bound ω_0, and $\omega \in (0, |\omega_0|]$.

Lemma 3.1 *Define a linear operator* $V : E \longrightarrow E$ *as*

$$Vx = \mu \int_0^1 \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)} \frac{\psi_\alpha(\theta)}{\theta^\alpha} \tau^{-\alpha}(1-\tau)^{\alpha-1} T\left(\beta^\alpha \frac{(1-\tau)^\alpha}{\theta^\alpha}\right) x\,d\theta\,d\tau.$$

Then V *is bounded and* $\|V\|_\omega \le \mu$. *Besides, if* $0 < \mu < 1$, *then* $(I - V)^{-1}$ *is a linear bounded operator and*

$$\big\|(I-V)^{-1}\big\|_\omega \le \frac{1}{1-\mu}.$$

Proof Since

$$\begin{aligned}
\|Vx\|_\omega &\le \mu \int_0^1 \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)} \frac{\psi_\alpha(\theta)}{\theta^\alpha} \tau^{-\alpha}(1-\tau)^{\alpha-1} \left\| T\left(\frac{\beta^\alpha(1-\tau)^\alpha}{\theta^\alpha}\right)\right\|_\omega \|x\|_\omega\,d\theta\,d\tau \\
&\le \mu \left[\int_0^1 \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)} \frac{\psi_\alpha(\theta)}{\theta^\alpha} \tau^{-\alpha}(1-\tau)^{\alpha-1} e^{-\omega\frac{\beta^\alpha(1-\tau)^\alpha}{\theta^\alpha}}\,d\theta\,d\tau\right] \|x\|_\omega \\
&\le \mu \frac{\alpha}{\Gamma(1-\alpha)} \left[\int_0^1 \tau^{-\alpha}(1-\tau)^{\alpha-1} \left(\int_0^\infty \frac{\psi_\alpha(\theta)}{\theta^\alpha}\,d\theta\right) d\tau\right] \|x\|_\omega \\
&\le \mu \|x\|_\omega.
\end{aligned}$$

Hence, V is bounded and $\|V\|_\omega \le \mu$. □

Lemma 3.2 *Set*

$$(Qh)(t) = \int_0^t \int_0^\infty \alpha \frac{\psi_\alpha(\theta)}{\theta^\alpha} (t-s)^{\alpha-1} T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right) h(s)\,d\theta\,ds, \quad h \in \mathrm{BC}(J, E).$$

Then $Q : \mathrm{BC}(J,E) \longrightarrow \mathrm{BC}(J,E)$ *and*

$$\begin{aligned}
&\big\|(Qh)(t)\big\|_\omega \le \frac{1}{\omega} \frac{1}{\Gamma(\alpha+1)} \|h\|_{\mathrm{BC}_\omega}; \\
&\big\|(Qh)\big\|_{\mathrm{BC}_\omega} \le \frac{1}{\omega} \frac{1}{\Gamma(\alpha+1)} \|h\|_{\mathrm{BC}_\omega}.
\end{aligned}$$

Proof Since

$$(Qh)(t) = \int_0^t \int_0^\infty \alpha \frac{\psi_\alpha(\theta)}{\theta^\alpha}(t-s)^{\alpha-1} T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right) h(s)\,d\theta\,ds$$
$$= \int_0^1 \int_0^\infty \alpha \frac{\psi_\alpha(\theta)}{\theta^\alpha} t^\alpha (1-\tau)^{\alpha-1} T\left(\frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha}\right) h(t\tau)\,d\theta\,d\tau,$$

then

$$\begin{aligned}
\|(Qh)(t)\|_\omega &\le \int_0^1 \int_0^\infty \alpha \frac{\psi_\alpha(\theta)}{\theta^\alpha} t^\alpha (1-\tau)^{\alpha-1} \left\| T\left(\frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha}\right)\right\|_\omega \|h(t\tau)\|_\omega\,d\theta\,d\tau \\
&\le \int_0^1 \int_0^\infty \alpha \frac{\psi_\alpha(\theta)}{\theta^\alpha} t^\alpha (1-\tau)^{\alpha-1} e^{-\omega \frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha}} \|h(t\tau)\|_\omega\,d\theta\,d\tau \\
&\le \frac{1}{\omega}\|h\|_{\mathrm{BC}_\omega} \int_0^\infty \left[\left(\int_0^1 e^{-\omega(\frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha})}\, d\left(-\omega\frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha}\right)\right)\frac{\psi_\alpha(\theta)}{\theta^\alpha}\right] d\theta \\
&= \frac{1}{\omega}\|h\|_{\mathrm{BC}_\omega} \int_0^\infty \left(1 - e^{-\omega\frac{t^\alpha}{\theta^\alpha}}\right)\frac{\psi_\alpha(\theta)}{\theta^\alpha}\,d\theta \\
&\le \frac{1}{\omega}\frac{1}{\Gamma(\alpha+1)}\|h\|_{\mathrm{BC}_\omega}.
\end{aligned}$$

Therefore,

$$\|(Qh)\|_{\mathrm{BC}_\omega} \le \frac{1}{\omega}\frac{1}{\Gamma(\alpha+1)}\|h\|_{\mathrm{BC}_\omega}. \qquad \square$$

Lemma 3.3 *Let $h \in \mathrm{BC}(J,E)$ and $u_0 \in D(A)$. Then the linear fractional evolution equation*

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = Au(t) + h(t), & t \in (0,+\infty), \\ u(0) = u_0, \end{cases} \tag{3.3}$$

has a unique solution $u \in \mathrm{BC}(J,E)$ of the following form:

$$\begin{aligned}
u(t) &= \int_0^t \int_0^\infty \left[\alpha \frac{(t-s)^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta) T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\left(\frac{s^{-\alpha}}{\Gamma(1-\alpha)}u_0 + h(s)\right)\right] d\theta\,ds \\
&= \int_0^1 \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}\frac{\psi_\alpha(\theta)}{\theta^\alpha}\tau^{-\alpha}(1-\tau)^{\alpha-1} T\left(\frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha}\right) u_0\,d\theta\,d\tau \\
&\quad + (Qh)(t).
\end{aligned} \tag{3.4}$$

Proof In view of Definitions 2.1, 2.2 and Lemma 2.1, equation (3.3) can be rewritten by the equivalent integral equation as follows:

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[Au(s) + h(s)\right] ds. \tag{3.5}$$

Denote by $U(\lambda)$ and $H(\lambda)$ the Laplace transforms of $u(t)$ and $h(t)$, respectively, using a similar method as that in [7, 8], with the Laplace transform, then we can rewrite the above equation as

$$U(\lambda) = \frac{1}{\lambda}u_0 + \frac{1}{\lambda^\alpha}AU(\lambda) + \frac{1}{\lambda^\alpha}H(\lambda), \quad \lambda > 0. \tag{3.6}$$

Hence,

$$\left(\lambda^\alpha I - A\right)U(\lambda) = \lambda^{\alpha-1}u_0 + H(\lambda).$$

By virtue of (3.1) and Lemma 2.3, we obtain

$$\begin{aligned}
U(\lambda) &= \left(\lambda^\alpha I - A\right)^{-1}\lambda^{\alpha-1}u_0 + \left(\lambda^\alpha I - A\right)^{-1}H(\lambda) \\
&= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}T(s)u_0\,ds + \int_0^\infty e^{-\lambda^\alpha s}T(s)H(\lambda)\,ds \\
&= \lambda^{\alpha-1}\int_0^\infty\int_0^\infty e^{-\lambda s^{1/\alpha}\theta}\psi_\alpha(\theta)T(s)u_0\,d\theta\,ds \\
&\quad + \int_0^\infty\int_0^\infty e^{-\lambda s^{1/\alpha}\theta}\psi_\alpha(\theta)T(s)H(\lambda)\,d\theta\,ds \\
&= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda t}\left[\int_0^\infty \alpha\frac{t^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)u_0\,d\theta\right]dt \\
&\quad + \int_0^\infty e^{-\lambda t}\left[\int_0^t\int_0^\infty \alpha\frac{(t-s)^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)h(s)\,d\theta\,ds\right]dt.
\end{aligned}$$

By the definition of Laplace transforms and the convolution theorem, applying Lemma 3.2 and the inverse Laplace transforms, one can derive that

$$\begin{aligned}
u(t) &= \mathcal{L}^{-1}\left[\lambda^{\alpha-1}\right] * \mathcal{L}^{-1}\left[\int_0^\infty e^{-\lambda t}\left[\int_0^\infty \alpha\frac{t^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)u_0\,d\theta\right]dt\right] \\
&\quad + \mathcal{L}^{-1}\left[\int_0^\infty e^{-\lambda t}\left(\int_0^t\int_0^\infty \alpha\frac{(t-s)^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)h(s)\,d\theta\,ds\right)dt\right] \\
&= \frac{t^{-\alpha}}{\Gamma(1-\alpha)} * \left[\int_0^\infty \alpha\frac{t^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)u_0\,d\theta\right] \\
&\quad + \int_0^t\int_0^\infty \alpha\frac{(t-s)^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)h(s)\,d\theta\,ds \\
&= \int_0^t\int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}(t-s)^{-\alpha}\frac{s^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta)T\left(\frac{s^\alpha}{\theta^\alpha}\right)u_0\,d\theta\,ds + (Qh)(t) \\
&= \int_0^t\int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}s^{-\alpha}\frac{(t-s)^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)u_0\,d\theta\,ds + (Qh)(t) \\
&= \int_0^1\int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}\frac{\psi_\alpha(\theta)}{\theta^\alpha}\tau^{-\alpha}(1-\tau)^{\alpha-1}T\left(\frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha}\right)u_0\,d\theta\,d\tau + (Qh)(t).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^1\int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}\tau^{-\alpha}(1-\tau)^{\alpha-1}\frac{\psi_\alpha(\theta)}{\theta^\alpha}\left\| T\left(\frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha}\right)u_0\right\|_\omega d\theta\,d\tau + \left\|(Qh)(t)\right\|_\omega \\
&\quad \le \int_0^1\int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}\tau^{-\alpha}(1-\tau)^{\alpha-1}\frac{\psi_\alpha(\theta)}{\theta^\alpha}e^{-\omega(\frac{t^\alpha(1-\tau)^\alpha}{\theta^\alpha})}\|u_0\|_\omega\,d\theta\,d\tau + \left\|(Qh)(t)\right\|_\omega
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\alpha}{\Gamma(1-\alpha)}\|u_0\|_\omega \int_0^1 \tau^{-\alpha}(1-\tau)^{\alpha-1}\left(\int_0^\infty \frac{\psi_\alpha(\theta)}{\theta^\alpha}\,d\theta\right)d\tau + \left\|(Qh)(t)\right\|_\omega \\
&\leq \|u_0\|_\omega + \frac{1}{\omega}\frac{1}{\Gamma(\alpha+1)}\|h\|_{\mathrm{BC}_\omega}.
\end{aligned}
$$

Therefore, $u \in \mathrm{BC}(J,E)$. Then we complete the proof. □

Lemma 3.4 *Let $h \in \mathrm{BC}(J,E)$ and $u_0 \in D(A)$. Let $0 < \mu < 1$. Then the linear fractional evolution equation*

$$
\begin{cases}
{}^C D_{0+}^\alpha u(t) = Au(t) + h(t), & t \in (0,+\infty), \\
u(0) = \mu u(\beta),
\end{cases}
\tag{3.7}
$$

has a unique solution $u \in \mathrm{BC}(J,E)$ of the following form:

$$
\begin{aligned}
u(t) &= (L_A h)(t) \\
&:= \int_0^t \int_0^\infty \left[\alpha \frac{(t-s)^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta) T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\left(\frac{s^{-\alpha}}{\Gamma(1-\alpha)}\left[(I-V)^{-1}(Qh)(\beta)\right]\right.\right. \\
&\quad \left.\left. + h(s)\right)\right] d\theta\, ds \\
&= \int_0^1 \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}\frac{\psi_\alpha(\theta)}{\theta^\alpha}\tau^{-\alpha}(1-\tau)^{\alpha-1} T\left(t^\alpha \frac{(1-\tau)^\alpha}{\theta^\alpha}\right)\left[(I-V)^{-1}(Qh)(\beta)\right] d\theta\, d\tau \\
&\quad + (Qh)(t).
\end{aligned}
\tag{3.8}
$$

Also, L_A is a linear operator on the Banach space $\mathrm{BC}(J,E)$ *and*

$$
\|L_A\|_{\mathrm{BC}_\omega} \leq \frac{\kappa}{\omega},
$$

where

$$
\kappa := \frac{1}{\Gamma(\alpha+1)}\frac{1}{1-\mu}.
$$

Proof In view of Lemma 3.3, one can obtain

$$
u(\beta) = \int_0^\beta \int_0^\infty \left[\alpha \frac{(\beta-s)^{\alpha-1}}{\theta^\alpha}\psi_\alpha(\theta) T\left(\frac{(\beta-s)^\alpha}{\theta^\alpha}\right)\left(\frac{s^{-\alpha}}{\Gamma(1-\alpha)}u(0) + h(s)\right)\right] d\theta\, ds.
$$

From $u(0) = \mu u(\beta)$, we have

$$
\begin{aligned}
u(0) &= \mu \int_0^\beta \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}\frac{\psi_\alpha(\theta)}{\theta^\alpha} s^{-\alpha}(\beta-s)^{\alpha-1} T\left(\frac{(\beta-s)^\alpha}{\theta^\alpha}\right)u(0)\, d\theta\, ds + \mu(Qh)(\beta) \\
&= \mu \int_0^1 \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)}\frac{\psi_\alpha(\theta)}{\theta^\alpha}\tau^{-\alpha}(1-\tau)^{\alpha-1} T\left(\beta^\alpha \frac{(1-\tau)^\alpha}{\theta^\alpha}\right)u(0)\, d\theta\, d\tau\, \mu(Qh)(\beta) \\
&= Vu(0) + \mu(Qh)(\beta).
\end{aligned}
$$

Therefore,

$$
(I-V)u(0) = \mu(Qh)(\beta).
$$

So, we obtain

$$u(0) = \mu(I - V)^{-1}(Qh)(\beta).$$

Then (3.8) is followed.
By (3.8), one has

$$\begin{aligned}
\|(L_A h)(t)\|_\omega &\leq \int_0^1 \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)} \frac{\psi_\alpha(\theta)}{\theta^\alpha} \tau^{-\alpha}(1-\tau)^{\alpha-1} \left\| T\left(t^\alpha \frac{(1-\tau)^\alpha}{\theta^\alpha}\right) \right\|_\omega \\
&\quad \times \left\| \left[(I-V)^{-1}\mu(Qh)(\beta)\right] \right\|_\omega \, d\theta \, d\tau + \|(Qh)(t)\|_\omega \\
&\leq \int_0^1 \int_0^\infty \frac{\alpha}{\Gamma(1-\alpha)} \frac{\psi_\alpha(\theta)}{\theta^\alpha} \tau^{-\alpha}(1-\tau)^{\alpha-1} e^{-\omega \frac{t^\alpha (1-\tau)^\alpha}{\theta^\alpha}} \\
&\quad \times \left[\|(I-V)^{-1}\|_\omega \|\mu(Qh)(\beta)\|_\omega \right] d\theta \, d\tau + \|(Qh)(t)\|_\omega \\
&\leq \frac{1}{1-\mu} \frac{\mu}{\omega} \frac{1}{\Gamma(\alpha+1)} \|h\|_{\mathrm{BC}_\omega} + \frac{1}{\omega} \frac{1}{\Gamma(\alpha+1)} \|h\|_{\mathrm{BC}_\omega} \\
&= \frac{1}{\omega} \frac{1}{\Gamma(\alpha+1)} \frac{1}{1-\mu} \|h\|_{\mathrm{BC}_\omega} \\
&= \frac{\kappa}{\omega} \|h\|_{\mathrm{BC}_\omega}.
\end{aligned}$$

Therefore,

$$\|(L_A h)\|_{\mathrm{BC}_\omega} \leq \frac{\kappa}{\omega} \|h\|_{\mathrm{BC}_\omega}. \qquad \square$$

Now, we state the main result on the existence of the positive solutions to problem (1.1) in the following.

Theorem 3.1 *Let E be a Banach space, and P is its positive normal with N as the normal constant. Let $\{T(t)\}_{t\geq 0}$ be a uniformly exponentially stable C_0-semigroup with the growth bound ω_0 ($\omega_0 < 0$), and A is the infinitesimal generator of $\{T(t)\}_{t\geq 0}$. Let $0 < \mu < 1$. Provided that $f(t,u) : J \times E \longrightarrow E$ is continuous and $f(t,\theta) \geq \theta$ is bounded on J. If $f(t,u)$ satisfies the following conditions*:

(a) *There exists a constant $\mathcal{K}_1 < -\omega_0$ such that for*

$$f(t,y) - f(t,x) \leq \mathcal{K}_1(y-x), \quad \theta \leq x \leq y.$$

(b) *There exists a constant $\mathcal{K}_2 > \max\{-\mathcal{K}_1, \omega_0\}$ such that for*

$$f(t,y) - f(t,x) \geq -\mathcal{K}_2(y-x), \quad \theta \leq x \leq y.$$

(c)

$$0 < \frac{\mathcal{K}_1 + \mathcal{K}_2}{\mathcal{K}_2 - \omega_0} < \frac{1}{\kappa}.$$

Then problem (1.1) *has a unique positive mild solution in* $\mathrm{BC}(J,E)$.

Proof For simplicity of notation, we denote $f_0(t) = f(t,\theta)$, then we have $f_0(t) \in \mathrm{BC}(J,E)$ and $f_0(t) \geq \theta, t \in J$.

In the following, similar to the methods used in [17], we deduce the result of the theorem in four steps.

Step 1: Consider the abstract fractional differential equation

$$\begin{cases} {}^{C}D_{0+}^{\alpha} u(t) = Au(t) + \mathcal{K}_1 u(t) + f_0(t), & t \in (0,+\infty), \\ u(0) = \mu u(\beta). \end{cases} \tag{3.9}$$

By virtue of the theory of semigroups, we can get that $\{e^{\mathcal{K}_1 t} T(t)\}_{t\geq 0}$ is a uniformly exponentially stable C_0-semigroup on Banach E generated by $A + \mathcal{K}_1 I$. Besides, the semigroup is positive with the growth bound $\mathcal{K}_1 + \omega_0$ ($\mathcal{K}_1 + \omega_0 < 0$). In view of Lemma 3.4, equation (3.9) has a unique mild solution $\vartheta_0 \in \mathrm{BC}(J,E)$ and $\vartheta_0 \geq \theta$ as a result of $f_0(t) \geq \theta, t \in J$.

Step 2: For a given function $g \in \mathrm{BC}(J,E)$, consider the abstract fractional differential equation

$$\begin{cases} {}^{C}D_{0+}^{\alpha} u(t) + \mathcal{K}_2 u(t) = Au(t) + g(t), & t \in (0,+\infty), \\ u(0) = \mu u(\beta). \end{cases} \tag{3.10}$$

It is obvious that $A - \mathcal{K}_2 I$ generates a uniformly exponentially stable C_0-semigroup $\{e^{-\mathcal{K}_2 t} T(t)\}_{t\geq 0}$ on Banach E. Also, it is positive with the growth bound $-\mathcal{K}_2 + \omega_0$ ($-\mathcal{K}_2 + \omega_0 < 0$).

Based on Lemma 3.4, the unique mild solution of (3.10) is given by $u = L_{A-\mathcal{K}_2 I} g$, where $L_{A-\mathcal{K}_2 I} : \mathrm{BC}(J,E) \longrightarrow \mathrm{BC}(J,E)$ is a positive bounded linear operator (similar to the operator L_A) with the property that

$$\|L_{A-\mathcal{K}_2 I}\|_{\mathrm{BC}_\omega} \leq \frac{\kappa}{\mathcal{K}_2 - \omega_0}, \quad \text{for } \omega = \mathcal{K}_2 - \omega_0.$$

Combined with the first step, one can notice that ϑ_0 is the mild solution of problem (3.10) for $g = f_0 + \mathcal{K}_1 \vartheta_0 + \mathcal{K}_2 \vartheta_0$, so

$$\vartheta_0 = L_{A-\mathcal{K}_2 I}(f_0 + \mathcal{K}_1 \vartheta_0 + \mathcal{K}_2 \vartheta_0). \tag{3.11}$$

Step 3: Take $G(u) = f(t,u) + \mathcal{K}_2 u$. Evidently, $G(\theta) = f(t,\theta) = f_0(t) \geq \theta$ and $G : \mathrm{BC}(J,E) \longrightarrow \mathrm{BC}(J,E)$ is continuous due to conditions (a), (b) and the normality of the cone P_{BC}.

By condition (b), for $\theta \leq x \leq y$, one can obtain

$$G(y) - G(x) = f(t,y) + \mathcal{K}_2 y - f(t,x) - \mathcal{K}_2 x = f(t,y) - f(t,x) + \mathcal{K}_2(y-x) \geq \theta,$$

that is, G is an increasing operator on the positive cone P_{BC}.

Let $\upsilon_0 = \theta$. Taking account of a composition operator defined by $\mathcal{F} = L_{A-\mathcal{K}_2 I} \circ G$ on the order interval $[\theta, \vartheta_0]$, it is easy to see that the fixed point of $\mathcal{F}$ is the mild solution of problem (1.1). Now, our task is to demonstrate that the operator $\mathcal{F}$ has at least one fixed point.

Consider the following two sequences:

$$\vartheta_n = \mathcal{F}(\vartheta_{n-1}), \quad n = 1, 2, 3, \ldots, \tag{3.12}$$

and

$$\upsilon_n = \mathcal{F}(\upsilon_{n-1}), \quad n = 1, 2, 3, \ldots. \tag{3.13}$$

By condition (a), we have

$$f\bigl(t, \vartheta_0(t)\bigr) - f(t, \theta) \le \mathcal{K}_1 \vartheta_0(t),$$

then

$$f\bigl(t, \vartheta_0(t)\bigr) \le \mathcal{K}_1 \vartheta_0(t) + f_0(t).$$

So, we can get

$$G(\vartheta_0) = f\bigl(t, \vartheta_0(t)\bigr) + \mathcal{K}_2 \vartheta_0(t) \le \mathcal{K}_1 \vartheta_0(t) + \mathcal{K}_2 \vartheta_0(t) + f_0(t).$$

By the fact that G is an increasing operator on the cone P, therefore, we can obtain

$$\theta \le f_0(t) = G(\theta) \le G(\vartheta_0) \le \mathcal{K}_1 \vartheta_0 + \mathcal{K}_2 \vartheta_0 + f_0. \tag{3.14}$$

Combining (3.11), (3.14) and the positivity of the linear bounded operator $L_{A-\mathcal{K}_2 I}$, one can get

$$\theta \le L_{A-\mathcal{K}_2 I} \circ G(\theta) = \mathcal{F}(\theta) \le L_{A-\mathcal{K}_2 I} \circ G(\vartheta_0) = \mathcal{F}(\vartheta_0) \le L_{A-\mathcal{K}_2 I}(\mathcal{K}_1 \vartheta_0 + \mathcal{K}_2 \vartheta_0 + f_0) = \vartheta_0,$$

that is,

$$\theta = \upsilon_0 \le \vartheta_1 \le \vartheta_0. \tag{3.15}$$

As $\mathcal{F}$ is an increasing operator on the order interval $[\theta, \vartheta_0]$, by the definition of $\mathcal{F}$ and (3.15), we can get two sequences $\{\vartheta_n\}$ and $\{\upsilon_n\}$ ($n = 0, 1, 2, 3, \ldots$) satisfying

$$\theta = \upsilon_0 \le \upsilon_1 \le \upsilon_2 \le \cdots \le \upsilon_n \le \cdots \le \vartheta_n \le \cdots \le \vartheta_2 \le \vartheta_1 \le \vartheta_0.$$

According to the above facts, by condition (a), one can get that

$$\begin{aligned}
\theta \le \vartheta_n - \upsilon_n &= \mathcal{F}(\vartheta_{n-1}) - \mathcal{F}(\upsilon_{n-1}) \\
&= L_{A-\mathcal{K}_2 I} \circ G(\vartheta_{n-1}) - L_{A-\mathcal{K}_2 I} \circ G(\upsilon_{n-1}) \\
&= L_{A-\mathcal{K}_2 I}\bigl[f(\cdot, \vartheta_{n-1}) + \mathcal{K}_2 \vartheta_{n-1} - f(\cdot, \upsilon_{n-1}) - \mathcal{K}_2 \upsilon_{n-1}\bigr] \\
&\le (\mathcal{K}_1 + \mathcal{K}_2) L_{A-\mathcal{K}_2 I}(\vartheta_{n-1} - \upsilon_{n-1}).
\end{aligned}$$

Thus, by induction, we have

$$\theta \le \vartheta_n - \upsilon_n \le (\mathcal{K}_1 + \mathcal{K}_2)^n L_{A-\mathcal{K}_2 I}(\vartheta_0 - \upsilon_0) = (\mathcal{K}_1 + \mathcal{K}_2)^n L^n_{A-\mathcal{K}_2 I}(\vartheta_0).$$

On account of the fact that the cone P_{BC} is normal with the normal constant N, by virtue of condition (c), we get

$$\begin{aligned}
\|\vartheta_n - \upsilon_n\|_{\mathrm{BC}_\omega} &\le N(\mathcal{K}_1 + \mathcal{K}_2)^n \big\| L^n_{A-\mathcal{K}_2 I}(\vartheta_0)\big\|_{\mathrm{BC}_\omega} \\
&\le N(\mathcal{K}_1 + \mathcal{K}_2)^n \big\| L^n_{A-\mathcal{K}_2 I}\big\|_{\mathrm{BC}_\omega} \|\vartheta_0\|_{\mathrm{BC}_\omega} \\
&\le N(\mathcal{K}_1 + \mathcal{K}_2)^n \| L_{A-\mathcal{K}_2 I}\|^n_{\mathrm{BC}_\omega} \|\vartheta_0\|_{\mathrm{BC}_\omega} \\
&\le N(\mathcal{K}_1 + \mathcal{K}_2)^n \left(\frac{\kappa}{\mathcal{K}_2 - \omega_0}\right)^n \|\vartheta_0\|_{\mathrm{BC}_\omega} \\
&= N\left(\frac{\kappa(\mathcal{K}_1 + \mathcal{K}_2)}{\mathcal{K}_2 - \omega_0}\right)^n \|\vartheta_0\|_{\mathrm{BC}_\omega} \to 0, \quad n \to +\infty.
\end{aligned} \tag{3.16}$$

Thus, from (3.16), analogous to the nested interval method, there exists a unique $u^* \in \bigcap_{n=1}^{\infty}[\upsilon_n, \vartheta_n]$ such that $u^* = \lim_{n\to\infty} \vartheta_n = \lim_{n\to\infty} \upsilon_n$.

By (3.12) and (3.13), taking limit of $n \to \infty$, we obtain that

$$u^* = \mathcal{F}\big(u^*\big),$$

namely, u^* is a fixed point of the operator $\mathcal{F}$. Thus, u^* is a mild positive solution of problem (1.1).

Step 4: In this part, we certify the uniqueness of the mild solution for problem (1.1).

By using reduction to absurdity, suppose that u_1^* and u_2^* are two different positive mild solutions for the fractional evolution equation (1.1), thus, $\|u_1^* - u_2^*\|_{\mathrm{BC}_\omega} > 0$.

Replace ϑ_0 by u_1^* and u_2^* in (3.12), respectively. Following the same steps as above, for each u_i^* $(i = 1, 2)$, we can get that $u_i^* = \mathcal{F}(u_i^*)$, $\|u_i^* - \upsilon_n\|_{\mathrm{BC}_\omega} \to 0$ $(n \to \infty)$ and $\vartheta_n = u_i^*$ for each $n \in \mathbb{N}$ $(i = 1, 2)$. Therefore,

$$0 < \big\|u_1^* - u_2^*\big\|_{\mathrm{BC}_\omega} \le \big\|u_1^* - \upsilon_n\big\|_{\mathrm{BC}_\omega} + \big\|u_2^* - \upsilon_n\big\|_{\mathrm{BC}_\omega} \to 0, \quad n \to \infty,$$

which is a contradiction.

Hence, problem (1.1) has a unique positive solution. The proof is completed. □

4 Examples

To illustrate our main result, we present an example. Consider the following partial fractional differential equation.

Example 4.1

$$\begin{cases}
\partial_t^\alpha z(t,x) = \partial_x^2 z(t,x) + F(t, z(t,x)), & t \in [0, +\infty), \\
z(t,0) = z(t,\pi) = 0, & t \in [0, +\infty), \\
z(0,x) = \mu z(\beta, x), & x \in [0, \pi],
\end{cases} \tag{4.1}$$

where ∂_t^α is the Caputo fractional partial derivative of order $\alpha \in (0,1)$.

Set $E = L^2([0,\pi],\mathbb{R})$ and $Az = \partial_x^2 z$, according to [23], then $A : D(A) \longrightarrow E$ is a linear operator with domain $D(A) = \{u \in E \mid u' \in E, u(0) = u(\pi) = 0\}$. Besides, the operator A generates a uniformly exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ with the growth bound $\omega_0 \leq -1$.

Let $u(t) = z(t,\cdot), f(t,u(t)) = F(t,z(t,\cdot))$, then problem (4.1) can be written as

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = Au(t) + f(t,u(t)), & t \in (0,+\infty), \\ u(0) = \mu u(\beta). \end{cases} \tag{4.2}$$

Take $\alpha = 1/2$, $\mu = 3/4$, $\beta = 1$, then we can get

$$\kappa = \frac{1}{\Gamma(\alpha+1)}\frac{1}{1-\mu} = \frac{8}{\sqrt{\pi}}.$$

Consider the following function:

$$f(t,x) = \left(-\mathcal{K}_2 + \frac{1}{a^2(t)}\right)x,$$

where $a \in C[0,+\infty)$ is bounded and

$$\mathcal{K}_1 = 2\omega_0,$$
$$\mathcal{K}_2 = -\left(2 + \frac{2}{\kappa-1}\right)\omega_0.$$

It is easy for us to certify that

$$0 < \frac{\mathcal{K}_1 + \mathcal{K}_2}{\mathcal{K}_2 - \omega_0} = \frac{2}{3\kappa - 1} < \frac{1}{\kappa}.$$

Since

$$-\mathcal{K}_2 - \frac{\omega_0}{\kappa-1} = \left(2 + \frac{2}{\kappa-1}\right)\omega_0 - \frac{\omega_0}{\kappa-1} \leq 2\omega_0 = \mathcal{K}_1,$$

then, for $\theta \leq x \leq y$,

$$\begin{aligned} -\mathcal{K}_2 &\leq f(t,y) - f(t,x) \\ &= \left[-\mathcal{K}_2 + \frac{-\omega_0}{(1+a^2(t))(\kappa-1)}\right](y-x) \\ &\leq \left(-\mathcal{K}_2 - \frac{\omega_0}{\kappa-1}\right)(y-x) \\ &\leq \mathcal{K}_1(y-x). \end{aligned}$$

Noting that $f(t,\theta) = \theta$. Thereby, f satisfies the conditions of Theorem 3.1, we can conclude that problem (4.1) has a unique positive mild solution.

Acknowledgements
This research is supported by the Fundamental Research Funds for the Central Universities (No. 2014QNA52), Natural Science Foundation of Jiangsu Province of China (No. BK20140176), Shandong Provincial Natural Science Foundation (No. ZR2014AP011) and the NNSF (No. 11601508).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All of the authors contributed equally in writing this paper. All authors read and approved the final manuscript.

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Received: 1 July 2017 Accepted: 26 August 2017 Published online: 12 September 2017

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