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An optimized finite difference Crank-Nicolson iterative scheme for the 2D Sobolev equation

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Abstract

In this paper, we devote ourselves to establishing the unconditionally stable and absolutely convergent optimized finite difference Crank-Nicolson iterative (OFDCNI) scheme containing very few degrees of freedom but holding sufficiently high accuracy for the two-dimensional (2D) Sobolev equation by means of the proper orthogonal decomposition (POD) technique, analyzing the stability and convergence of the OFDCNI solutions and using the numerical simulations to verify the feasibility and effectiveness of the OFDCNI scheme.

MSC: 65M60; 65N30; 65N15

Keywords: optimized finite difference Crank-Nicolson iterative scheme; Sobolev equation; proper orthogonal decomposition; stability and convergence; numerical simulation

1 Introduction

For convenience and without loss of universality, we think about the following two-dimensional (2D) Sobolev equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon \frac{\partial \Delta u}{\partial t} - \gamma \Delta u = f(x,y,t), & (x,y,t) \in \Omega \times (0,T), \\ u(x,y,t) = Q(x,y,t), & (x,y,t) \in \partial\Omega \times (0,T], \\ u(x,y,0) = G(x,y), & (x,y) \in \Omega, \end{cases} \quad (1)$$

where $\Omega = (a,b) \times (c,d) \subset \mathbb{R}^2$ is a bounded open set with the boundary $\partial\Omega$, $u(x,y,t)$ is the unknown function, ε and γ are two known positive parameters, and $f(x,y,t)$ and $Q(x,y,t)$ as well as $G(x,y)$ are three given functions. The existence and uniqueness of the analytic solution for the Sobolev equation (1) were given in [1, 2].

The Sobolev equation plays an extremely important role in many numerical simulations of mathematical physics problems such as the fluid seepage through fractured rock or soil [3], the heat exchange in different media [4], and the moisture migration in soil [5]. However, because the Sobolev equation generally has complex known data or computational domains in the actual engineering applications, even if theoretically there exists the analytical solution, it can not be usually sought out so that one has to rely on the numerical methods. In nearly forty years, the Sobolev equation has been closely watched, there have

been many numerical research reports (see, *e.g.*, [6–14]). Among all numerical methods, the finite difference Crank-Nicolson (FDCN) scheme (see [11]) is regarded as one of the simplest and most convenient as well as the most easily programmed for calculating high accuracy numerical methods for solving the 2D Sobolev equation. However, the classical FDCN scheme for the 2D Sobolev equation is a macroscale system of equations containing lots of unknowns, *i.e.*, degrees of freedom so as to undertake very large computational load in the real-world engineering applications. Thus, an important issue is how to decrease the unknowns of the classical FDCN scheme so as to alleviate the truncated error amassing in the actual calculating procedure and retrench the calculating time but keep sufficiently high accuracy of numerical solutions.

A lot of numerical simulations (see, *e.g.*, [15–28]) have verified that the proper orthogonal decomposition (POD) technique is a very effective approach to decrease the degrees of freedom for numerical models and alleviate the truncated error amassing in the numerical calculation. But the most existing reduced-order models as mentioned above were built via the POD basis formulated with the classical numerical solutions at all time nodes, before computing the reduced-order numerical solutions at the same time nodes, which were some nugatory repeated calculations. Since 2014, some reduced-order extrapolating finite difference (FD) schemes based on the POD technique for PDEs have been established successively by Luo's team (see, *e.g.*, [29–32]) in order to avert the valueless repeated computations.

However, as far as we know, there has been not any paper that the POD technique is used to decrease the degrees of freedom in the classical FDCN scheme for the 2D Sobolev equation. Therefore, in this article, we use the POD technique to establish an optimized finite difference iterative (OFDCNI) scheme containing very few unknowns but holding sufficiently high accuracy for the 2D Sobolev equation, analyze the stability and convergence of the OFDCNI solutions, and verify the feasibility and effectiveness of the OFDCNI scheme by means of numerical simulations.

The major difference between the OFDCNI scheme and the existing POD-based reduced-order extrapolating FDCN schemes (see, *e.g.*, [29–32]) consists in that the Sobolev equation not only includes the time first-order derivative term and the spacial variables second-order derivative terms, but it also contains a mixed derivative term about time first-order and spacial variables second-order so that either the establishment of the OFDCNI scheme or the analysis of the stability and convergence of the OFDCNI solutions faces more difficulties and needs more skills than the existing reduced-order extrapolating FD schemes as mentioned, but the Sobolev equation has some specific applications. Fortunately, we adopt the vector and matrix analysis approaches to analyze the stability and convergence of the classical FDCN and OFDCNI solutions such that the theoretical analysis not only becomes much simpler and more convenient but the numerical simulations in computer can also be easily implemented. Especially, the OFDCNI scheme has fully second-order accuracy, is unconditionally stable and absolutely convergent, and is only built by the POD basis constituted with the classical FDCN solutions over the initial very short time span so that it also has not repeated calculation like in [29–32]. Hence, it is development and improvement over the existing ones mentioned above.

The rest of the article is arranged as follows. The classical FDCN scheme for the 2D Sobolev equation is posed in Section 2. The OFDCNI scheme based on the POD technique for the 2D Sobolev equation is built in Section 3. The stability and convergence of the

OFDCNI solutions is deduced in Section 4. In Section 5, some numerical simulations are used to verify the feasibility and effectiveness of the OFDCNI scheme. Finally, some main conclusions are generalized in Section 6.

2 The classical FDCN scheme for the 2D Sobolev equation

Let Δt be the time step and Δx and Δy be, separately, the spacial steps in x and y directions, $u_{i,j}^n$ denote the classical FDCN approximations of u at points (x_i, y_j, t_n) $(x_i = a + i\Delta x, y_j = c + j\Delta y, t_n = n\Delta t, 0 \le i \le I \equiv [(b-a)/\Delta x], 0 \le j \le J \equiv [(d-c)/\Delta y]$, and $0 \le n \le N \equiv [T/\Delta t]$, where $[R]$ represents the integer part of the real number R).

By approximating to the derivatives of (1) by means of the following difference quotient:

$$\frac{\partial u}{\partial t} = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + O(\Delta t) \approx \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{k+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + O(\Delta x^2) \approx \frac{u_{k+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1}^n - u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2} + O(\Delta y^2) \approx \frac{u_{i,j+1}^n - u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2},$$

we obtain the following classical FDCN scheme:

$$u_{i,j}^{n+1} - \frac{\varepsilon + 0.5\gamma\,\Delta t}{\Delta x^2}\left(u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\right) - \frac{\varepsilon + 0.5\gamma\,\Delta t}{\Delta y^2}\left(u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}\right)$$

$$= (0.5\gamma\,\Delta t - \varepsilon)\left[\frac{1}{\Delta x^2}\left(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n\right) + \frac{1}{\Delta y^2}\left(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n\right)\right] + u_{i,j}^n$$

$$+ \Delta t f_{i,j}^n, \quad i = 1,2,\dots,I-1, j = 1,2,\dots,J-1, n = 0,1,2,\dots,N-1 \quad (2)$$

with boundary conditions

$$u_{0,j}^n = Q(0, j\Delta y, n\Delta t), \qquad u_{I,j}^n = Q(I\Delta x, j\Delta y, n\Delta t), \quad j = 0,1,\dots,J,$$

$$u_{i,0}^n = Q(i\Delta x, 0, n\Delta t), \qquad u_{i,J}^n = Q(i\Delta x, J\Delta y, n\Delta t), \quad i = 0,1,\dots,I$$

and initial conditions

$$u_{i,j}^0 = G(i\Delta x, j\Delta y), \quad i = 1,2,\dots,I-1, j = 1,2,\dots,J-1,$$

where $f_{i,j}^n = f(x_i, y_j, t_n)$. Then the matrix forms of (2) are written as follows:

$$\boldsymbol{A}\boldsymbol{U}^{n+1} = \boldsymbol{A}_1\boldsymbol{U}^n + \Delta t\boldsymbol{F}^n, \quad n = 0,1,\dots,N-1, \quad (3)$$

where $\boldsymbol{U}^n = (u_{0,0}^n, u_{1,0}^n, \dots, u_{I,0}^n, u_{0,1}^n, u_{1,1}^n, \dots, u_{I,1}^n, \dots, u_{I,J}^n)^T$, $\boldsymbol{F}^n = (f_{0,0}^n, f_{1,0}^n, \dots, f_{I,0}^n, f_{0,1}^n, f_{1,1}^n, \dots, f_{I,1}^n, \dots, f_{I,J}^n)^T$, $\boldsymbol{U}^0 = (G(x_0, y_0), G(x_1, y_0), \dots, G(x_I, y_0), G(x_0, y_1), G(x_1, y_1), \dots, G(x_I, y_1), \dots, G(x_I, y_J))^T$, $\boldsymbol{A} = \boldsymbol{I} + (\varepsilon + 0.5\gamma\,\Delta t)\Delta x^{-2}\boldsymbol{B} + (\varepsilon + 0.5\gamma\,\Delta t)\Delta y^{-2}\boldsymbol{C}$, $\boldsymbol{A}_1 = \boldsymbol{I} + (\varepsilon - 0.5\gamma\,\Delta t)\Delta x^{-2}\boldsymbol{B} +$

$(\varepsilon - 0.5\gamma\,\Delta t)\Delta y^{-2}\boldsymbol{C}$, $\boldsymbol{I}$ is the unit matrix, and

$$
\boldsymbol{B} = \begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & \cdots & -1 & 2
\end{bmatrix},
\qquad
\boldsymbol{C} = \begin{bmatrix}
2 & \overbrace{\cdots}^{I \text{ zeros}} & -1 & 0 & \cdots & 0 \\
\vdots & 2 & \cdots & \ddots & \ddots & 0 \\
-1 & \vdots & \ddots & \cdots & \ddots & 0 \\
0 & \ddots & \vdots & \ddots & \vdots & -1 \\
0 & \ddots & \ddots & \cdots & 2 & \vdots \\
0 & 0 & 0 & -1 & \underbrace{\cdots}_{I \text{ zeros}} & 2
\end{bmatrix}.
$$

Define the norm of matrix $\hat{\boldsymbol{A}}$ by $\|\hat{\boldsymbol{A}}\|_{2,2} = \sup_{\boldsymbol{x}\neq 0}\|\hat{\boldsymbol{A}}\boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2$, where $\|\boldsymbol{x}\|_2 = [\sum_{i=1}^{M} x_i^2]^{1/2}$ denotes the l^2 norm of vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_M)$ and $M = (I+1)(J+1)$. For the classical FDCN scheme, we have the following results.

Theorem 1 *The classical FDCN scheme* (3) *has a unique set of solutions* $\{\boldsymbol{U}^n\}_{n=1}^{N}$ *and the classical FDCN solutions* $\boldsymbol{U}^n$ $(n = 1, 2, \ldots, N)$ *are unconditionally stable and absolutely convergent. When* $u \in C((0, T]; H^4(\Omega)) \cap C^2((0, T])$ *is the exact solution for the 2D Sobolev equation, the following error estimates hold*:

$$
\left\|\tilde{\boldsymbol{U}}^n - \boldsymbol{U}^n\right\|_2 = O\left(\Delta x^2, \Delta y^2, \Delta t^2\right), \tag{4}
$$

where $n = 1, 2, \ldots, N$ *and* $\tilde{\boldsymbol{U}}^n = (u(x_0, y_0, t_n), u(x_1, y_0, t_n), \ldots, u(x_I, y_0, t_n), u(x_0, y_1, t_n), u(x_1, y_1, t_n), \ldots, u(x_I, y_1, t_n), \ldots, u(x_I, y_J, t_n))^T$ $(n = 1, 2, \ldots, N)$ *are formed with the analytic solution of the Sobolev equation* (1).

Proof Because $\boldsymbol{B}$ and $\boldsymbol{C}$ are two positive definite matrices, it is easily known that the matrix $\boldsymbol{A}$ is positive definite. Therefore, (3) has a unique set of solutions $\{\boldsymbol{U}^n\}_{n=1}^{N}$ satisfying

$$
\boldsymbol{U}^n = \boldsymbol{A}^{-1}\boldsymbol{A}_1\boldsymbol{U}^{n-1} + \Delta t\boldsymbol{A}^{-1}\boldsymbol{F}^{n-1}, \quad n = 1, 2, \ldots, N. \tag{5}
$$

By the computing formulas of eigenvalues (see, e.g., [33], Theorem 1.3.2), we obtain the eigenvectors of $\boldsymbol{A}^{-1}\boldsymbol{A}_1$ as follows:

$$
\tilde{\lambda}_j = \frac{1 + 4(\varepsilon - 0.5\gamma\,\Delta t)(\Delta x^{-2} + \Delta y^{-2})\sin^2\theta_j}{1 + 4(\varepsilon + 0.5\gamma\,\Delta t)(\Delta x^{-2} + \Delta y^{-2})\sin^2\theta_j}, \tag{6}
$$

where $\theta_j = (2j - 1)\pi/(4M)$, $j = 1, 2, \ldots, M$. Thus, the spectral radius $\rho(\boldsymbol{A}^{-1}\boldsymbol{A}_1)$ of $\boldsymbol{A}^{-1}\boldsymbol{A}_1$ satisfies

$$
\rho\left(\boldsymbol{A}^{-1}\boldsymbol{A}_1\right) = \max_{1 \leq j \leq M} |\tilde{\lambda}_j|
$$

$$
= \max_{1 \leq j \leq M}\left|\frac{1 + 4(\varepsilon - 0.5\gamma\,\Delta t)(\Delta x^{-2} + \Delta y^{-2})\sin^2\theta_j}{1 + 4(\varepsilon + 0.5\gamma\,\Delta t)(\Delta x^{-2} + \Delta y^{-2})\sin^2\theta_j}\right| \leq 1. \tag{7}
$$

Therefore, the classical FDCN solutions $\{\boldsymbol{U}^n\}_{n=1}^{N}$ are unconditionally stable. Furthermore, by Lax's stability theorem (see, *e.g.*, [33, 34]), we easily deduce that the FDCN solutions

$\{\boldsymbol{U}^n\}_{n=1}^N$ are absolutely convergent. By Taylor's formulas or the above discrete process, we easily deduce the error estimates (4), which completes the proof of Theorem 1. □

Remark 1 In this study, we adopt a simpler explicit FDCN scheme to discrete the 2D Sobolev equation, but the ideas and approaches here can be easily extended to other FDCN schemes, for example, a staggered direction FDCN scheme or an implicit FDCN scheme.

If f, G, Q, Δt, Δx, Δy, and parameters ε and γ are given, then we can gain the set of FDCN solutions $\{\boldsymbol{U}^1, \boldsymbol{U}^2, \ldots, \boldsymbol{U}^N\}$ by solving (3). A subset $\{\boldsymbol{U}^i\}_{i=1}^L$ (usually $L \ll N$), called the snapshots, is extracted from the initial L solution vectors of $\{\boldsymbol{U}^1, \boldsymbol{U}^2, \ldots, \boldsymbol{U}^N\}$.

3 The OFDCNI scheme for the 2D Sobolev equation

3.1 The constitution of the POD basis

For $M = (I+1)(J+1)$, let $\boldsymbol{A}_u = (\boldsymbol{U}^1, \boldsymbol{U}^2, \ldots, \boldsymbol{U}^L) \in \mathbb{R}^{M \times L}$, $\lambda_j > 0$ $(j = 1, 2, \ldots, r = \mathrm{rank}(\boldsymbol{A}_u))$ be the positive eigenvalues of $\boldsymbol{A}_u \boldsymbol{A}_u^T$ arrayed non-increasingly and $\boldsymbol{U}_u = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_r) \in \mathbb{R}^{M \times r}$ be the orthonormal eigenvectors of $\boldsymbol{A}_u \boldsymbol{A}_u^T$ corresponding to the positive eigenvalues. Then the POD basis $\boldsymbol{\Phi} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_d)$ $(d \le r)$ consists of the initial d vectors in $\boldsymbol{U}_u$ and has the following property (see, *e.g.*, [23]):

$$\left\| \boldsymbol{A}_u - \boldsymbol{\Phi} \boldsymbol{\Phi}^T \boldsymbol{A}_u \right\|_{2,2} = \sqrt{\lambda_{d+1}}. \tag{8}$$

Further, we have

$$
\begin{aligned}
\left\| \boldsymbol{U}^n - \boldsymbol{\Phi} \boldsymbol{\Phi}^T \boldsymbol{U}^n \right\|_2 &= \left\| \left(\boldsymbol{A}_u - \boldsymbol{\Phi} \boldsymbol{\Phi}^T \boldsymbol{A}_u \right) \boldsymbol{\varepsilon}_n \right\|_2 \\
&\le \left\| \boldsymbol{A}_u - \boldsymbol{\Phi} \boldsymbol{\Phi}^T \boldsymbol{A}_u \right\|_{2,2} \|\boldsymbol{\varepsilon}_n\|_2 \le \sqrt{\lambda_{d+1}},
\end{aligned} \tag{9}
$$

where $n = 1, 2, \ldots, L$, and $\boldsymbol{\varepsilon}_n$ $(n = 1, 2, \ldots, L)$ are the unit vectors with nth component being 1. Therefore, $\boldsymbol{\Phi}_u = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_d)$ is a series of optimal basis.

Remark 2 Because the degree L of the matrix $\boldsymbol{A}_u^T \boldsymbol{A}_u$ is far smaller than the degree M of the matrix $\boldsymbol{A}_u \boldsymbol{A}_u^T$, *i.e.*, the number of extracted snapshots L is much smaller than that of the spacial mesh points M, but their positive eigenvalues λ_i $(i = 1, 2, \ldots, r)$ are the same. Thus, we may first compute the eigenvalues λ_i $(i = 1, 2, \ldots, r)$ for the matrix $\boldsymbol{A}_u^T \boldsymbol{A}_u$ and the corresponding eigenvectors $\boldsymbol{\psi}_i$ $(i = 1, 2, \ldots, r)$, and then, by means of the formula $\boldsymbol{\varphi}_i = \boldsymbol{A}_u \boldsymbol{\psi}_i / \sqrt{\lambda_i}$ $(i = 1, 2, \ldots, r)$, we can acquire the eigenvectors $\boldsymbol{\varphi}_i$ $(i = 1, 2, \ldots, r)$ corresponding to the positive eigenvalues λ_i $(i = 1, 2, \ldots, r)$ for the matrix $\boldsymbol{A}_u \boldsymbol{A}_u^T$. Thus, we can conveniently find out the POD basis.

3.2 The formulation of the OFDCNI scheme for 2D Sobolev equation

In Section 3.1, we have acquired the initial L OFDCNI solutions $\boldsymbol{U}_d^n = \boldsymbol{\Phi}_u \boldsymbol{\Phi}_u^T \boldsymbol{U}^n =: \boldsymbol{\Phi}_u \boldsymbol{\beta}_d^n$ $(n = 1, 2, \ldots, L)$, where $\boldsymbol{U}_d^n = (u_{d,0,0}^n, \ldots, u_{d,I,0}^n, u_{d,0,1}^n, \ldots, u_{d,I,1}^n, \ldots, u_{d,I,J}^n)^T$ and $\boldsymbol{\beta}_d^n = (\beta_1^n, \beta_2^n, \ldots, \beta_d^n)^T$. If the solution vectors $\boldsymbol{U}^n$ $(n = L+1, L+2, \ldots, N)$ for the classical FDCN scheme (3) are approximated by $\boldsymbol{U}_d^n = \boldsymbol{\Phi}_u \boldsymbol{\beta}_d^n$ $(n = L+1, L+2, \ldots, N)$, *i.e.*, $\boldsymbol{U}^n$ are replaced with $\boldsymbol{U}_d^n = \boldsymbol{\Phi}_u \boldsymbol{\beta}_d^n$ $(n = L+1, L+2, \ldots, N)$, we can build the OFDCNI scheme based on the

POD basis as follows:

$$
\begin{cases}
\boldsymbol{\Phi}\boldsymbol{\beta}_d^n = \boldsymbol{\Phi}_u \boldsymbol{\Phi}_u^T \boldsymbol{U}^n, & 1 \le n \le L; \\
\boldsymbol{A}\boldsymbol{\Phi}_u \boldsymbol{\beta}_d^{n+1} = \boldsymbol{A}_1 \boldsymbol{\Phi}_u \boldsymbol{\beta}_d^n + \Delta t \boldsymbol{F}^n, & L \le n \le N - 1, \\
\boldsymbol{U}_d^n = \boldsymbol{\Phi}\boldsymbol{\beta}_d^n, & n = 1, 2, \dots, N,
\end{cases}
\tag{10}
$$

where $\boldsymbol{U}^n$ ($n = 1, 2, \dots, L$) are the initial L known classical FDCN solutions to (3) and the matrices $\boldsymbol{A}$ and $\boldsymbol{A}_1$ are given in (3). OFDCNI scheme (10) is simplified into

$$
\begin{cases}
\boldsymbol{\beta}_d^n = \boldsymbol{\Phi}_u^T \boldsymbol{U}^n, & 1 \le n \le L; \\
\boldsymbol{\beta}_d^{n+1} = \boldsymbol{\Phi}_u^T \boldsymbol{A}^{-1} \boldsymbol{A}_1 \boldsymbol{\Phi}_u \boldsymbol{\beta}_d^n + \Delta t \boldsymbol{\Phi}_u^T \boldsymbol{A}^{-1} \boldsymbol{F}^n, & L \le n \le N - 1, \\
\boldsymbol{U}_d^n = \boldsymbol{\Phi}_u \boldsymbol{\beta}_d^n, & n = 1, 2, \dots, N.
\end{cases}
\tag{11}
$$

Remark 3 Because the classical FDCN scheme (3) includes $M = (I+1)(J+1)$ unknowns at each time node, whereas OFDCNI scheme (11) at the same node only includes d unknowns ($d \ll M$), we can expressly realize the merit of OFDCNI scheme (11).

4 The existence, stability, and convergence of the OFDCNI solutions

In the following, we devote ourselves to deducing the existence, stability, and convergence of the OFDCNI solutions. We have the following main results.

Theorem 2 *Under the conditions of Theorem* 1, *OFDCNI scheme* (11) *has a unique set of solutions* $\{\boldsymbol{U}_d^n\}_{n=1}^N$, *the solutions* $\boldsymbol{U}_d^n$ ($n = 1, 2, \dots, N$) *are unconditionally stable and absolutely convergent, and the following error estimates hold*:

$$
\left\| \boldsymbol{U}^n - \boldsymbol{U}_d^n \right\|_2 \le E(n) \sqrt{\lambda_{(d+1)}},
\tag{12}
$$

where $E(n) = 0$ ($1 \le n \le L$) *and* $E(n) = \exp[4(n-L)\gamma \Delta t(\Delta x^{-2} + \Delta y^{-2})]$ ($L + 1 \le n \le N$). *Moreover, if the vectors* $\tilde{\boldsymbol{U}}^n = (u(x_0, y_0, t_n), u(x_1, y_0, t_n), \dots, u(x_I, y_0, t_n), u(x_0, y_1, t_n), u(x_1, y_1, t_n), \dots, u(x_I, y_1, t_n), \dots, u(x_I, y_J, t_n))^T$ ($n = 1, 2, \dots, N$) *consist of the analytic solution of the Sobolev equation* (1), *then we have the following error estimates*:

$$
\left\| \tilde{\boldsymbol{U}}^n - \boldsymbol{U}_d^n \right\|_2 = O\bigl(\Delta x^2, \Delta y^2, \Delta t^2, E(n)\sqrt{\lambda_{(d+1)}}\bigr).
\tag{13}
$$

Proof By using $\boldsymbol{U}_d^n = \boldsymbol{\Phi}\boldsymbol{\beta}_d^n$ ($n = 1, 2, \dots, N$), OFDCNI scheme (10) is reverted into the following form:

$$
\boldsymbol{U}_d^n = \boldsymbol{\Phi}_u \boldsymbol{\Phi}_u^T \boldsymbol{U}^n, \quad n = 1, 2, \dots, L;
\tag{14}
$$

$$
\boldsymbol{A}\boldsymbol{U}_d^{n+1} = \boldsymbol{A}_1 \boldsymbol{U}_d^n + \Delta t \boldsymbol{F}^n, \quad L \le n \le N - 1.
\tag{15}
$$

Because the classical FDCN solutions $\boldsymbol{U}^n$ ($n = 1, 2, \dots, L$) are known and stable, when $n = 1, 2, \dots, L$, from (14), we obtain unique solutions $\boldsymbol{U}_d^n = \boldsymbol{\Phi}_u \boldsymbol{\Phi}_u^T \boldsymbol{U}^n$ ($n = 1, 2, \dots, L$) that are stable since $\|\boldsymbol{U}_d^n\|_2 \le \|\boldsymbol{U}^n\|_2$. On the other hand, when $n = L + 1, 2 + 2, \dots, N$, because the matrix $\boldsymbol{A}$ is positive definite, OFDCNI scheme (15) has a unique set of solutions $\{\boldsymbol{U}_d^n\}_{n=L+1}^N$. By using the same arguments as those in Theorem 1, we easily deduce that the OFDCNI

solutions $\{U_d^n\}_{n=L+1}^N$ are unconditionally stable. Thus, OFDCNI scheme (11) has a unique set of stable solutions $\{U_d^n\}_{n=1}^N$. Furthermore, by Lax's stability theorem (see, *e.g.*, [5, 32]), it is deduced that the OFDCNI solutions $\{U_d^n\}_{n=1}^N$ are absolutely convergent.

When $n = 1, 2, \ldots, L$, from (9), we immediately obtain the following error estimates:

$$\left\| U^n - U_d^n \right\|_2 = \left\| U^n - \Phi_u \Phi_u^T U^n \right\|_2 \le \sqrt{\lambda_{(d+1)}}, \quad n = 1, 2, \ldots, L. \tag{16}$$

Let $e_n = U^n - U_d^n$. Because $\|A^{-1}\|_{2,2} < 1$ and $\|B\|_{2,2} = \|C\|_{2,2} < 4$, when $n = L+1, L+2, \ldots, N$, from (3) and (15), we have

$$\begin{aligned}
\|e_{n+1}\|_2 &= \left\| A^{-1}\left[A - \gamma\,\Delta t\left(\Delta x^{-2} B + \Delta y^{-2} C\right)\right] e_n \right\|_2 \\
&\le \|e_n\|_2 + 4\gamma\,\Delta t\left(\Delta x^{-2} + \Delta y^{-2}\right)\|e_n\|_2.
\end{aligned} \tag{17}$$

By summing (17) from L to $n - 1$ and using (16) and Gronwall's inequality (see, *e.g.*, [33, 34]), we have

$$\|e_n\|_2 \le \sqrt{\lambda_{(d+1)}} + 4\gamma\,\Delta t\left(\Delta x^{-2} + \Delta y^{-2}\right)\sum_{i=L}^{n-1} \|e_i\|_2 \le E(n)\sqrt{\lambda_{(d+1)}}, \tag{18}$$

where $E(n) = \exp[4(n - L)\gamma\,\Delta t(\Delta x^{-2} + \Delta y^{-2})]$ $(n = L+1, L+2, \ldots, N)$. Combining (16) with (18) yields (12), and combining Theorem 1 with (12) yields (13), which accomplishes the demonstration of Theorem 2. □

Remark 4 The error factors $\sqrt{\lambda_{d+1}}$ and $E(n) = \exp[4(n - L)\gamma\,\Delta t(\Delta x^{-2} + \Delta y^{-2})]$ $(n = L+1, L+2, \ldots, M)$ in Theorem 2 are induced by the reduced-order of the classical FDCN scheme and the iteration, respectively, which could be acted as the suggestions of choosing the number d of POD bases, *i.e.*, as long as we choose d such that $E(N)\sqrt{\lambda_{d+1}} = O(\Delta x^2, \Delta y^2, \Delta t^2)$.

5 Numerical simulations

In this section, we provide some numerical simulations to verify the superiority of the OFDCNI scheme for the 2D Sobolev equation.

In the 2D Sobolev equation (1), we chose the computational domain as $\Omega = \{(x, y) : 0 \le x \le 2, 0 \le y \le 2\}$, $f(x, y, t) = 2\pi^2 e^{-\pi t} \sin \pi x \sin \pi y$, $Q(x, y, t) = 0$, $G(x, y) = \sin \pi x \sin \pi y$, $\varepsilon = 10^{-8}$, and $\gamma = 1$. The spatial steps are chosen as $\Delta x = \Delta y = 0.01$ and the time step is chosen as $\Delta t = 0.0001$.

First, the snapshots were extracted from the initial classical FDCN solutions U^n ($n = 1, 2, \ldots, 20$) for the classical FDCN scheme (3). And then, the snapshot matrix A_u was compiled and the eigenvalues and the corresponding eigenvectors of $A_u A_u^T$ were computed. It was achieved by estimation that $\sqrt{\lambda_6} \le 10^{-4}$, which implies that as long as the initial five eigenvectors of matrix $A_u A_u^T$ are chosen as the POD basis, the accuracy requirement could be satisfied. In the end, the OFDCNI solutions at $t = 0.1, 1.0, 2.0$ were computed out via OFDCNI scheme (11) with five POD bases, and the classical FDCN solutions were also computed out by means of the classical FDCN scheme (3) at the same time nodes, which were exhibited in Figures 1 to 6, respectively.

Figure 1 The classical FDCN solution at *t* = 0.1.

Figure 2 The OFDCNI solution at *t* = 0.1.
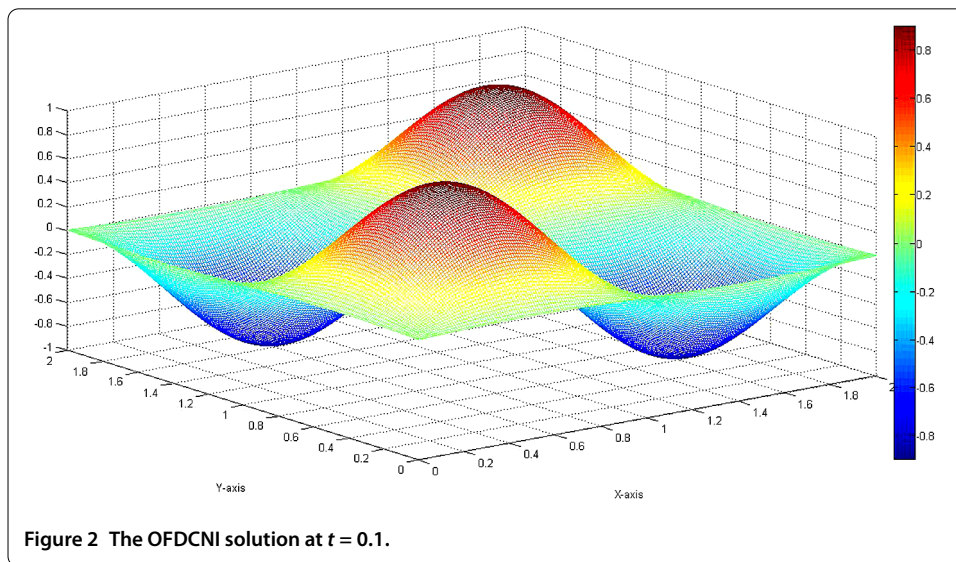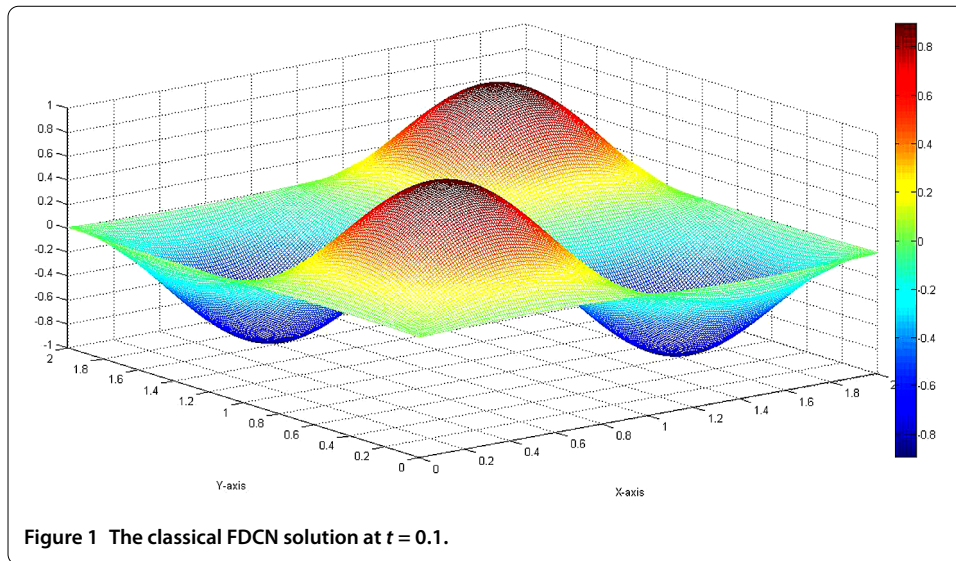
Comparing the numerical conclusions of the classical FDCN scheme with the OFDCNI scheme summarized the following conclusions. The charts in Figures 1 and 2 are basically identical at *t* = 0.1. However, the classical FDCN solution appeared with a little dispersion (see Figure 3) at *t* = 1; whereas the OFDCNI solution was still stable and smooth (see Figure 4). Especially, the classical FDCN solution emerged with very great dispersion (see Figure 5) at *t* = 2 due to the truncated error amassing, but the OFDCNI solution yet remained stable and smooth (see Figure 6). It implies that the OFDCNI solutions were much better than the classical FDCN solutions and shows that the OFDCNI scheme is much more efficient and advanced than the classical FDCN scheme for solving the 2D Sobolev equation.

Due to the OFDCNI scheme greatly decreasing the degrees of freedom, the consuming time of the OFDCNI scheme is far less than that of the classical FDCN scheme in the above numerical simulations. For example, the execution time for the OFDCNI scheme with five POD bases was only 8 seconds at *t* = 2, while that of the classical FDCN scheme was 96

Figure 3 The classical FDCN solution at *t* = 1.

**Figure 4 The OFDCNI solution at *t* = 1.**
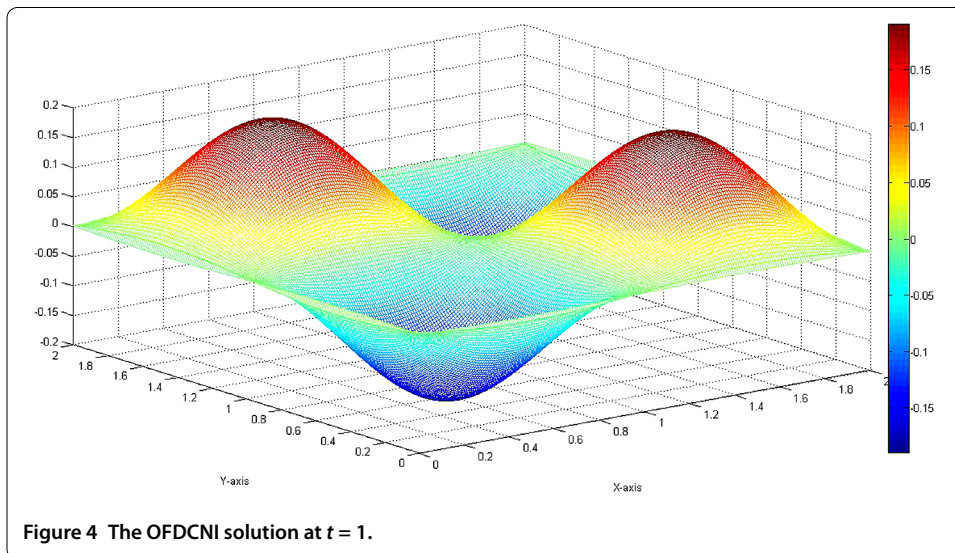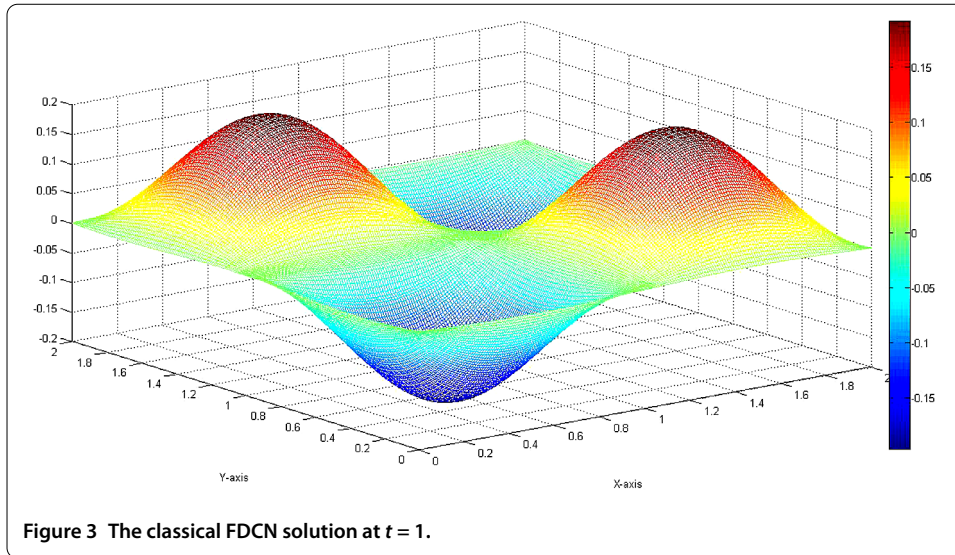
seconds in the same PC, but the errors of both solutions did not exceed $O(10^{-4})$, which also shows that the numerical computational results were in line with the theoretical ones.

In addition, in the above numerical simulations, we only used the initial few given classical FDCN solutions over a very short time span $[0, T_0]$ ($T_0 \ll T$) as the snapshots to constitute the POD basis and build the OFDCNI scheme before calculating the OFD-CNI solutions over the total time span $[0, T]$. When one solves the real-world engineering problems, one may use the recorded data (over a very short time span $[0, T_0]$) to constitute the POD basis, to build the OFDCNI scheme, and to predict future physical phenomena and changes (over a time span $[T_0, T]$). Therefore, the OFDCNI scheme holds very extensive applied prospect.

6 Conclusions

In this article, we have established the OFDCNI scheme based on the POD technique for the 2D Sobolev equation. First, the snapshots are extracted from the initial few FDCN so-

Figure 5 The classical FDCN solution at *t* **= 2.**

Figure 6 The OFDCNI solution at *t* **= 2.**
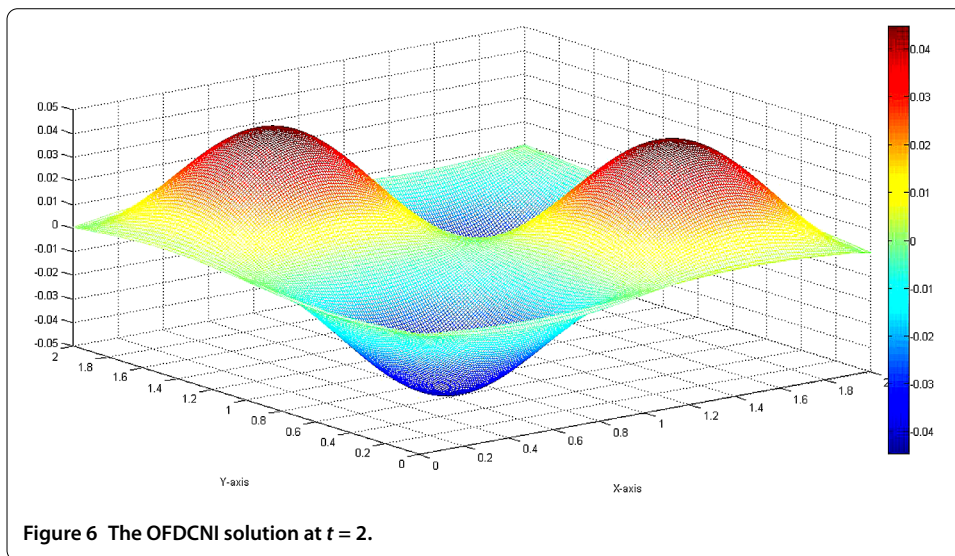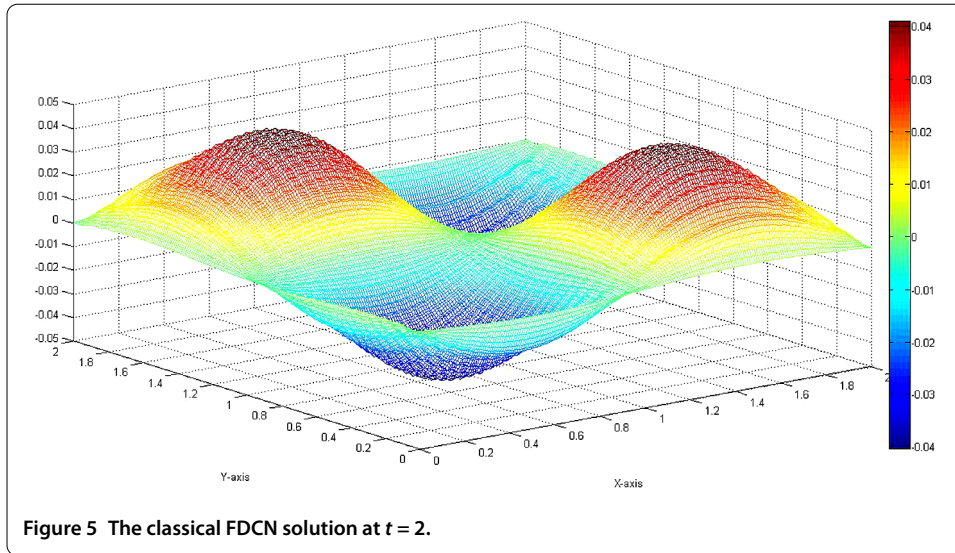
lutions for the 2D Sobolev equation. And then, the POD basis is constituted by the snapshots, and the OFDCNI scheme having the fully second-order accuracy and containing very few unknowns is established by replacing the unknown FDCN solution vectors with the linear-combination of the POD basis. Finally, the stability and convergence of the OFDCNI solutions are deduced. The numerical simulations have exhibited that the OFDCNI solutions are far better than the classical ones. This implies that the OFDCNI scheme is highly efficient and reliable for solving the 2D Sobolev equation.

Even though we only discuss the OFDCNI scheme for the 2D Sobolev equation on the domain $\overline{\Omega} = [a,b] \times [c,d]$, the approach here can be extended to more general domains, even extended to more complicated engineering problems. Therefore, the technique here has very extensive applications.

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