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A new technique to solve the initial value problems for fractional fuzzy delay differential equations

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Abstract

Using some recent results of fixed point of weakly contractive mappings on the partially ordered space, the existence and uniqueness of solution for interval fractional delay differential equations (IFDDEs) in the setting of the Caputo generalized Hukuhara fractional differentiability are studied. The dependence of the solution on the order and the initial condition of IFDDE is shown. A new technique is proposed to find the exact solutions of IFDDE by using the solutions of interval integer order delay differential equation. Finally, some examples are given to illustrate the applications of our results.

Keywords: fractional interval-valued differential equations; fractional interval-valued delay differential equations; weakly contractive mapping

1 Introduction

Fractional calculus and fractional differential equations are a field of increasing interest due to their applicability to the analysis of phenomena, and they play an important role in a variety of fields such as rheology, viscoelasticity, electrochemistry, diffusion processes, etc. Usually applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. One can find applications of fractional differential equations in signal processing and in the complex dynamic in biological tissues (see [1–3]). To observe some basic information and results of various type of fractional differential equations, one can see the papers and monographs of Samko et al. [4], Podlubny [5] and Kilbas et al. [6].

Interval analysis and interval differential equation were proposed as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena in which uncertainties or vagueness pervade. In the recent time this theory has been developed in theoretical directions, and a wide number of applications of this theory have been considered (see, for instance, [7–12]). Recently, the issue of fuzzy fractional calculus and fuzzy fractional differential equations has emerged as the significant subject, and this new theory has become very attractive to many scientists. The concept of fuzzy type Riemann-Liouville differentiability based on the Hukuhara differentiability was initiated by Agarwal et al. in [13, 14] with some applications to fractional order initial value problem of fuzzy differential equation. By using the Hausdorff measure of non-compactness and under compactness type conditions, authors proved the existence of solution of fuzzy fractional integral equation. Following this direction, the concepts of fuzzy fractional differentiability have been developed and extended in some papers to investigate some results on the existence and uniqueness of solutions to fuzzy differential equations, and have been considered in a wide number of applications of this theory (see, for instance, [15–28]).

It is well known that the Banach fixed point theorem is a useful tool in mathematics and plays an important role in finding solutions to nonlinear, differential and integral equations, among others. In the recent time this theorem has been extended and generalized by several authors in various ways, for instance, the results on the existence of a fixed point in partially ordered sets for first order ordinary differential equations, Fredholm and Volterra type integral equations, among others, have been studied. In particular, some applications of fixed points in partially ordered sets to resolution of matrix equation were presented by Ran and Reurings in [29], and the applicability of the existence of a unique fixed point for mappings defined in partially ordered sets to the study of the existence of a unique solution for periodic boundary condition problems for integer order ordinary differential equations was shown by Nieto and Rodríguez-López in [30, 31]. In [32] Harjani and Sadarangani presented some fixed point generalized theorems involving altering distance functions in the ordered metric spaces, and this result was used to investigate the existence problem of solution to first and second order ordinary differential equations. Besides, Villamizar-Roa et al. [33] used some more generalized fixed point results of weakly contractive mappings in a partial order metric space of fuzzy-valued functions to investigate the existence and uniqueness of fuzzy solutions of the initial-valued problem for integer order fuzzy differential equation in the setting of generalized Hukuhara derivatives. By employing the weakly contractive mapping in the partially ordered space of fuzzy functions, Long et al. [34–37] studied the existence and uniqueness of weak solution for some classes of fuzzy fractional partially differential equations under Caputo gH-differentiability without the Lipschitzian right-hand side. Briefly in this paper some recent results of fixed point of weakly contractive mappings on the partially ordered space are used to investigate the existence and uniqueness of solution for interval fractional delay differential equations in the setting of the Caputo generalized Hukuhara fractional differentiability. We focus on the following initial value problem for an interval-valued delay differential equations with Caputo generalized Hukuhara fractional differentiability under the form

$$\begin{cases} ({}^{C}\mathcal{D}^{\alpha}_{a^+}X)(t) = F(t, X(t), X_t), & t \in [a, b], \\ X(t) = \varphi(t-a), & t \in [a-\sigma, a], \end{cases} \tag{1.1}$$

where ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ is the Caputo derivative of order $\alpha \in (0, 1]$, F is a continuous interval function on $[a, b]$, and the initial function φ is continuous on $[a-\sigma, a]$. Our aims are to

- give the existence and uniqueness theorem of solution for a general form of the interval fractional integral equation by using some recent results of fixed point of weakly contractive mappings on partially ordered sets, and use these results in order to investigate the existence and uniqueness result of solution for problem (1.1).
- show that the solutions of the initial value problem (1.1) depend continuously on the initial condition, the order and the right-hand side of equation.

- propose a new technique to find the exact solutions of problem (1.1) by using the solutions of interval integer order delay differential equation.

This paper is organized as follows. In Section 2, some basic concepts and notations about fractional derivative for interval functions are introduced. Besides, some properties concerning the partially ordered space of interval functions and a fixed point theorem in partial metric spaces are presented. In Section 3, we prove the existence and uniqueness of solution for a general form of the interval fractional integral equation and use this result to investigate the existence and uniqueness results of solutions for problem (1.1). Finally, a new technique to find the exact solutions of problem (1.1) is provided and two examples are given to illustrate this technique.

2 Preliminaries

In this section, we present basic notations and necessary preliminaries used throughout the paper. Some of them were detailed further in [8–10, 20, 38–40] and the references therein. In the following, we denote the space of all nonempty compact intervals of the real line $\mathbb{R}$ by $K_C(\mathbb{R})$. Let $A = [\underline{A}, \overline{A}]$, $B = [\underline{B}, \overline{B}] \in K_C(\mathbb{R})$, then the usual interval operations, i.e., Minkowski addition, Minkowski difference and scalar multiplication, are defined by $A + B = [\underline{A} + \underline{B}, \overline{A} + \overline{B}]$, $A - B = A + (-1)B = [\underline{A} - \overline{B}, \overline{A} - \underline{B}]$ and

$$\lambda A = \lambda[\underline{A}, \overline{A}] = \begin{cases} [\lambda\underline{A}, \lambda\overline{A}] & \text{if } \lambda > 0, \\ \mathbf{0} & \text{if } \lambda = 0, \\ [\lambda\overline{A}, \lambda\underline{A}] & \text{if } \lambda < 0. \end{cases}$$

It is well known that with respect to the above operations, $K_C(\mathbb{R})$ is a quasilinear space (Markov [41]). Let $A = [\underline{A}, \overline{A}]$, $B = [\underline{B}, \overline{B}] \in K_C(\mathbb{R})$, then the generalized Hukuhara difference (or gH-difference for short) of two intervals A, B is defined as follows [10, 41]:

$$[\underline{A}, \overline{A}] \ominus_g [\underline{B}, \overline{B}] = \big[\min\{\underline{A} - \underline{B}, \overline{A} - \overline{B}\}, \max\{\underline{A} - \underline{B}, \overline{A} - \overline{B}\}\big].$$

The width of $A \in K_C(\mathbb{R})$ is defined by $w(A) = \overline{A} - \underline{A}$. Then it is easy to see that $w(-A) = w(A)$, $w(A + B) = w(A) + w(B)$ and $w(A \ominus_g B) = |w(A) - w(B)|$. The Hausdorff-Pompeiu metric H in $K_C(\mathbb{R})$ is defined as follows:

$$H[A, B] = \max\big\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\big\}. \tag{2.1}$$

It is well known that $(K_C(\mathbb{R}), H)$ is a complete, separable and locally compact metric space. Some properties of the Pompeiu-Hausdorff metric are well known in (see [42]). As $(K_C(\mathbb{R}), +, \cdot, H)$ is a quasilinear metric space, then the concepts of continuity and limit for interval-valued functions are understood in the sense of the metric H. Next, for $X, Y \in K_C(\mathbb{R})$ with $X = [\underline{X}, \overline{X}]$, $Y = [\underline{Y}, \overline{Y}]$, we can define the partial orders $\preceq$ and $\succeq$ as $X \preceq Y$ ($X \succeq Y$) if and only if $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$ ($\underline{X} \geq \underline{Y}$ and $\overline{X} \geq \overline{Y}$). Some interesting properties on the partial orderings $\preceq$ and $\succeq$ are presented in [40].

Lemma 2.1 ([40, 43]) *On $K_C(\mathbb{R})$ the following properties hold*:

(i) *If $(X)_{n\in\mathbb{N}} \subset K_C(\mathbb{R})$ is a nondecreasing sequence such that $X_n \to X$ in $K_C(\mathbb{R})$, then $X_n \preceq X$ for all $n \in \mathbb{N}$.*

(ii) *Every pair of elements of* $K_C(\mathbb{R})$ *has a lower bound or an upper bound.*

The partial ordering $\preceq$ and $\succeq$ can be extended to the space of interval functions as follows:

$$X \preceq Y \text{ if and only if } \underline{X}(t) \leq \underline{Y}(t) \text{ and } \overline{X}(t) \leq \overline{Y}(t), \quad \forall t \in [a,b].$$

In this paper we call $C([a,b],K_C(\mathbb{R}))$ the set of continuous interval functions on $[a,b]$. It is well known that $C([a,b],K_C(\mathbb{R}))$ is a complete metric space with respect to the metric $H_C[X,Y] = \|X \ominus_g Y\|_C$, where $\|X\|_C := \sup_{a\leq t\leq b} H[X(t),\mathbf{0}]$.

Lemma 2.2 ([40, 43]) *Let* $(K_C(\mathbb{R}),\preceq)$ *be a partial ordered space, then we have the following properties*:

(i) $(C([a,b],K_C(\mathbb{R})),\preceq)$ *is a partial ordered space.*

(ii) *If* $(X)_{n\in\mathbb{N}} \subset C([a,b],K_C(\mathbb{R}))$ *is a nondecreasing sequence such that* $X_n \to X$ *in* $C([a,b],K_C(\mathbb{R}))$, *then* $X_n \preceq X$ *for all* $n \in \mathbb{N}$.

(iii) *Every pair of elements of* $C([a,b],K_C(\mathbb{R}))$ *has a lower bound or an upper bound.*

Let an interval function $X : [a,b] \to K_C(\mathbb{R})$, then X is called w-increasing (w-decreasing) on $[a,b]$ if $t \mapsto w(X(t))$ is nondecreasing (nonincreasing) on $[a,b]$. We say that X is w-monotone on $[a,b]$ if X is w-increasing or w-decreasing on $[a,b]$.

Generalized Hukuhara fractional derivative. [10] Let $t_0 \in [a,b]$, then the generalized Hukuhara derivative (gH-derivative for short) of an interval mapping $X : [a,b] \to K_C(\mathbb{R})$ at t_0 is the function $X'(t_0) \in K_C(\mathbb{R})$ given by

$$X'(t_0) = \lim_{h\to 0} \frac{X(t_0+h) \ominus_g X(t_0)}{h}. \tag{2.2}$$

We recall some definitions of Riemann-Liouville and Caputo derivatives for interval-valued functions and some necessary results are given to use in the next section. Some of them were detailed further in [38, 39] and the references therein.

Let $L([a,b],K_C(\mathbb{R}))$ denote the space of all Lebesgue integrable interval-valued functions on the bounded interval $[a,b]$. Let $\alpha \in (0,1]$, then the interval-valued Riemann-Liouville integral of interval-valued function $X \in L([a,b],K_C(\mathbb{R}))$ is defined by

$$\left(\Im^{\alpha}_{a^+} X\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} X(s)\, ds \quad \text{for } t \in [a,b] \tag{2.3}$$

and the Riemann-Liouville gH-fractional derivative of X is defined by

$$\left({}^{RL}\mathcal{D}^{\alpha}_{a^+} X\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} X(s)\, ds\right)' \quad \text{for } t \in [a,b].$$

Remark 2.1 ([16]) Let $X, Y \in C([a,b],K_C(\mathbb{R}))$ and $A \in K_C(\mathbb{R})$ and also $X \preceq Y$, then for every $\alpha \in (0,1]$, we have

(i) $(\Im^{\alpha}_{a^+} X)(t) \preceq (\Im^{\alpha}_{a^+} Y)(t), \forall t \in [a,b]$.

(ii) $A \ominus (-1)(\Im^{\alpha}_{a^+}X)(t) \preceq A \ominus (-1)(\Im^{\alpha}_{a^+}Y)(t), \forall t \in [a,b]$, provided $A \ominus (-1)(\Im^{\alpha}_{a^+}X)(t)$ and $A \ominus (-1)(\Im^{\alpha}_{a^+}Y)(t)$ are well defined.

Let $X \in L([a,b], K_C(\mathbb{R}))$ be an interval-valued function such that the Riemann-Liouville gH-fractional derivative ${}^{RL}\mathcal{D}^{\alpha}_{a^+}X$, $\alpha \in (0,1]$ exists on $[a,b]$. In this case the interval-valued Caputo fractional derivative (or Caputo gH-fractional derivative) of order $\alpha \in (0,1]$ of X is defined as follows:

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+}X\right)(t) := \left({}^{RL}\mathcal{D}^{\alpha}_{a^+}\left[X(\cdot) \ominus_g X(a)\right]\right)(t) \quad \text{for } t \in [a,b].$$

Generalized fixed point theorems. In the sequel we recall some generalized fixed point theorem in partially ordered space (see [32]) that will be used in the next section to analyze the existence of solutions for a general form of the interval fractional integral equation.

Let $C(\mathbb{R}^+, \mathbb{R}^+)$ denote the space of all continuously nonnegative functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, then a real-valued function $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ is called an altering distance function on $[0,\infty)$ if ψ is nondecreasing and satisfies the following property: $\psi(t) = 0$ if and only if $t = 0$. Some examples for altering distance functions on $[0,\infty)$ are $t^2, \ln(1+t)$ and $t^2 - \ln(1+t^2)$. Let $(\mathbb{X}, \preceq)$ be a partially ordered set and $f : \mathbb{X} \to \mathbb{X}$. The function f is said to be monotone nonincreasing (or nondecreasing) if $u \preceq v$ for $u, v \in \mathbb{X}$, then $f(v) \preceq f(u)$ or $(f(u) \preceq f(v))$, respectively.

Theorem 2.1 ([32]) *Assume that* $(\mathbb{X}, \preceq)$ *is a partially ordered set such that every pair of elements of* X *has a lower bound or an upper bound, and there exists a metric* d *on* $\mathbb{X}$ *such that* $(\mathbb{X}, \preceq)$ *is a complete metric space. Let* $f : \mathbb{X} \to \mathbb{X}$ *be a monotone nondecreasing function satisfying the weakly contractive condition*

$$\psi\left(d\left(f(u), f(v)\right)\right) \leq \psi\left(d(u,v)\right) - \xi\left(d(u,v)\right)$$

for all $u \succeq v$ *and for some altering distance functions* ψ *and* ξ. *In addition, suppose that* $\mathbb{X}$ *satisfies the following property*:

(i) *if a nondecreasing sequence* $(u_n)_{n\in\mathbb{N}} \subset \mathbb{X}$ *converges to* u *in* $\mathbb{X}$, *then* $u_n \preceq u$ *for all* $n \in \mathbb{N}$, *or*

(ii) *if a nonincreasing sequence* $(u_n)_{n\in\mathbb{N}} \subset \mathbb{X}$ *converges to* u *in* $\mathbb{X}$, *then* $u \preceq u_n$ *for all* $n \in \mathbb{N}$.

Then, if there exists $u_0 \in \mathbb{X}$ *satisfying case* (i) *such that* $u_0 \preceq f(u_0)$ *or there exists* $u_0 \in \mathbb{X}$ *satisfying case* (ii) *such that* $u_0 \succeq f(u_0)$, *then* f *has a unique fixed point. Moreover, if* u^* *is the fixed point of* f, *then it holds* $\lim_{n\to\infty} f^n(u) = u^*$ *for all* $u \in \mathbb{X}$.

3 Main results

In this section, our aim is to give existence and uniqueness theorems of solutions for a general form of the interval-valued delay fractional integral equation by using some recent results of fixed point of weakly contractive mappings on partially ordered sets, and in the next section, we use these results in order to investigate the existence and uniqueness results of solutions for an interval-valued delay fractional differential equation. For a positive number σ, we denote by C_σ the space $C([-\sigma, 0], K_C(\mathbb{R}))$ equipped with the metric defined by

$$H_\sigma[X,Y] = \sup_{t\in[-\sigma,0]} H\big[X(t),Y(t)\big].$$

Define $I = [a,b], J = [a-\sigma,a] \cup I = [a-\sigma,b]$. Then, for each $t \in I$, we denote by X_t the element of C_σ defined by $X_t(s) = X(t+s)$, $s \in [-\sigma,0]$.

Interval-valued delay fractional integral equation: Consider the following interval-valued delay fractional integral equation:

$$\begin{cases} X(t) \ominus_g X(a) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(s,X(s),X_s)\,ds, & t \in [a,b], \\ X(t) = \varphi(t-a), & t \in [a-\sigma,a], \end{cases} \tag{3.1}$$

where $F : [a,b] \times K_C(\mathbb{R}) \times C_\sigma \to K_C(\mathbb{R}), \varphi(t-a) \in K_C(\mathbb{R})$. We say that a continuous interval function $X : [a,b] \to K_C(\mathbb{R})$ is a solution to the interval fractional integral equation (3.1) if it satisfies equation (3.1). Let us suppose that $X \in C([a,b],K_C(\mathbb{R}))$ is w-monotone on $[a,b]$ and satisfies (3.1). As X is w-monotone on $[a,b]$, then it follows that $X(t) \ominus_g X(a)$ is w-increasing on $[a,b]$. Hence, from (3.1) it follows that the right-hand side of (3.1) must be w-increasing on $[a,b]$ (see [38]). Furthermore, we observe that if a continuous interval function X is a unique w-monotone solution of (3.1) on $[a,b]$, then the function $Y(t) := X(t) \ominus_g X(a)$ is w-increasing on $[a,b]$. In addition, the function Y may create two solutions of (3.1): a unique w-increasing solution of (3.1) and a unique w-decreasing solution of (3.1) on $[a,b]$.

Remark 3.1 If $X \in C([a,b],K_C(\mathbb{R}))$ is such that $w(X(t)) \geq w(X(a))$ for all $t \in [a,b]$, then (3.1) can be rewritten as

$$\begin{cases} X(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(s,X(s),X_s)\,ds, & t \in [a,b], \\ X(t) = \varphi(t-a), & t \in [a-\sigma,a]. \end{cases}$$

If $X \in C([a,b],K_C(\mathbb{R}))$ is such that $w(X(t)) \leq w(X(a))$ for all $t \in [a,b]$, then (3.1) can be rewritten as

$$\begin{cases} X(t) = \varphi(0) \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(s,X(s),X_s)\,ds, & t \in [a,b], \\ X(t) = \varphi(t-a), & t \in [a-\sigma,a]. \end{cases}$$

Definition 3.1 A w-monotone interval function $X^L \in C([a,b],K_C(\mathbb{R}))$ is a lower solution for (3.1) if

$$\begin{cases} X^L(t) \ominus_g X^L(a) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(s,X^L(s),X^L_s)\,ds, & t \in [a,b], \\ X^L(t) = \varphi^L(t-a) \preceq \varphi(t-a), & t \in [a-\sigma,a]. \end{cases} \tag{3.2}$$

A w-monotone interval-valued function $X^U \in C([a,b],K_C(\mathbb{R}))$ is an upper solution for (3.1) if it satisfies the reverse inequalities of (3.2).

In the following, for given $k > 0$, we consider the set $\mathbb{B}_k$ of all continuous interval functions $X \in C([a-\sigma,b],K_C(\mathbb{R}))$ such that $\sup_{t\in[a-\sigma,b]}\{H[X(t),\mathbf{0}]\exp(-k(t+\sigma))\} < \infty$. On $\mathbb{B}_k$ we can define the following metric:

$$\begin{aligned} H_k[X,Y] &= \sup_{t\in[a-\sigma,b]}\bigl\{H\bigl[X(t),Y(t)\bigr]\exp\bigl(-k(t+\sigma)\bigr)\bigr\}, \\ &\quad X,Y \in C\bigl([a-\sigma,b],K_C(\mathbb{R})\bigr), \end{aligned} \tag{3.3}$$

where $k > 0$ is large enough such that $k > 2^{1/\alpha}$. It is well known that $(C([a,b],K_C(\mathbb{R})),H_k)$ is a complete metric space.

Theorem 3.1 *Let $F \in C([a,b] \times K_C(\mathbb{R}) \times C_\sigma, K_C(\mathbb{R}))$ and suppose that $F(t,A,B)$ is nondecreasing in A and B for each $t \in [a,b]$, that is, if $A \succeq C$ and $B \succeq D$, then $F(t,A,B) \succeq F(t,C,D)$. Moreover, assume that the following conditions are satisfied*:

(A1) *there exists a w-monotone lower solution $X^L \in C([a,b],K_C(\mathbb{R}))$ for problem* (3.1);

(A2) *$F(t,A,B)$ is weakly contractive for comparable elements, that is, for some altering distance functions $\mathbb{T}_1$ and $\mathbb{T}_2$, it holds*

$$\begin{aligned} &\mathbb{T}_1\bigl(H\bigl[F(t,A,B),F(t,C,D)\bigr]\bigr) \\ &\quad\le \bigl[\mathbb{T}_1\bigl(H[A,B]\bigr) - \mathbb{T}_2\bigl(H[A,B]\bigr)\bigr] + \bigl[\mathbb{T}_1\bigl(H_\sigma[A,B]\bigr) - \mathbb{T}_2\bigl(H_\sigma[A,B]\bigr)\bigr], \end{aligned}$$

if $A \succeq C, B \succeq D$ and $t \in [a,b]$. Then there exists a unique w-monotone solution X for problem (3.1) *in some intervals $[a,\mathbb{T}]$, with $\mathbb{T} \le b$.*

Proof Let $\mathcal{X}(t) := X(t) \ominus_g X(a), t \in [a-\sigma,b]$. We define the operator $\mathbb{Q} : C([a-\sigma,b], K_C(\mathbb{R})) \to C([a-\sigma,b],K_C(\mathbb{R}))$ by

$$(\mathbb{Q}\mathcal{X})(t) = \begin{cases} \varphi(t-a) \ominus_g \varphi(0), & t \in [a-\sigma,a], \\ \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F(s,X(s),X_s)\,ds, & t \in [a,b]. \end{cases}$$

We check that the conditions in Theorem 2.1 are satisfied. Indeed, let $X \succeq Y$ and $X_t \succeq Y_t$ on $[a,b]$, then we have $(\mathbb{Q}\mathcal{X})(a) = (\mathbb{Q}\mathcal{Y})(a), t \in [a-\sigma,a]$, and for $t \in [a,b]$

$$F(t,X,X_t) \succeq F(t,Y,Y_t).$$

From the result of Remark 2.1-(i), we obtain

$$\begin{aligned} \mathbb{Q}\mathcal{X}(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\bigl(s,X(s),X_s\bigr)\,ds \succeq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\bigl(s,Y(s),Y_s\bigr)\,ds \\ &= (\mathbb{Q}\mathcal{Y})(t). \end{aligned}$$

Then $\mathbb{Q}\mathcal{X} \succeq \mathbb{Q}\mathcal{Y}$ whenever $X \succeq Y, X_t \succeq Y_t$ on $[a,b]$, and consequently, the operator $\mathbb{Q}$ is nondecreasing. Now, condition (A2) shows that

$$H\bigl[F\bigl(t,X(t),X_t\bigr),F\bigl(t,Y(t),Y_t\bigr)\bigr] \le H\bigl[X(t),Y(t)\bigr] + H_\sigma[X_t,Y_t] \tag{3.4}$$

for all $X \succeq Y, X_t \succeq Y_t$ for $t \in [a,b]$. Indeed, from (A2) we get

$$\mathbb{T}_1\bigl(H\bigl[F\bigl(t,X(t),X_t\bigr),F\bigl(t,Y(t),Y_t\bigr)\bigr]\bigr) \le \mathbb{T}_1\bigl(H\bigl[X(t),Y(t)\bigr]\bigr) + \mathbb{T}_1\bigl(H_\sigma[X_t,Y_t]\bigr) \tag{3.5}$$

for all $X \succeq Y, X_t \succeq Y_t$. If inequality (3.4) is not true, then for all $X \succeq Y, X_t \succeq Y_t$ we have

$$H\bigl[X(t),Y(t)\bigr] + H_\sigma[X_t,Y_t] < \max H\bigl[F\bigl(t,X(t),X_t\bigr),F\bigl(t,Y(t),Y_t\bigr)\bigr].$$

Then, since $\mathbb{T}_1$ is nondecreasing, for all $X \succeq Y, X_t \succeq Y_t$ it holds

$$\mathbb{T}_1\bigl(H\bigl[X(t),Y(t)\bigr]\bigr) + \mathbb{T}_1\bigl(H_\sigma[X_t,Y_t]\bigr) \le \mathbb{T}_1\bigl(H\bigl[F\bigl(t,X(t),X_t\bigr),F\bigl(t,Y(t),Y_t\bigr)\bigr]\bigr).$$

Therefore, from (3.5),

$$\mathbb{T}_1\bigl(H\bigl[X(t),Y(t)\bigr]\bigr) + \mathbb{T}_1\bigl(H_\sigma[X_t,Y_t]\bigr) = \mathbb{T}_1\bigl(H\bigl[F\bigl(t,X(t),X_t\bigr),F\bigl(t,Y(t),Y_t\bigr)\bigr]\bigr)$$

for all $X \succeq Y, X_t \succeq Y_t$. From (A2) it follows that $0 \le -(\mathbb{T}_2(H[X(t),Y(t)]) + \mathbb{T}_2(H_\sigma[X_t,Y_t]))$, and therefore,

$$\mathbb{T}_2\bigl(H\bigl[X(t),Y(t)\bigr]\bigr) = \mathbb{T}_2\bigl(H_\sigma[X_t,Y_t]\bigr) = 0.$$

As $\mathbb{T}_2$ is an altering distance function, we have that $H[X(t),Y(t)] = 0, H_\sigma[X_t,Y_t] = 0$ for all $X \succeq Y, X_t \succeq Y_t$. This infers a contradiction, that is, $H[F(t,X(t),X_t),F(t,Y(t),Y_t)] = 0$. Thus, inequality (3.4) is true. Next, for $X \succeq Y, X_t \succeq Y_t$ and $t \in [a,b]$, we get

$$\begin{aligned} &H\bigl[(\mathbb{Q}\mathcal{X})(t),(\mathbb{Q}\mathcal{Y})(t)\bigr] \\ &\quad \le \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Bigl(H\bigl[X(s),Y(s)\bigr] + \sup_{\theta\in[s-\sigma,s]} H\bigl[X(\theta),Y(\theta)\bigr]\Bigr)\,ds. \end{aligned}$$

By the definition of metric (3.3), it follows that $H[X(s),Y(s)] \le H_k[X,Y]e^{k(s+\sigma)}$ for all $t \ge a-\sigma$ and $\sup_{\theta\in[s-\sigma,s]} H[X(\theta),Y(\theta)] \le H_k[X,Y]e^{k(s+\sigma)}$ for all $s \ge a$. Then, for all $t \ge a$, we obtain

$$\begin{aligned} &H\bigl[(\mathbb{Q}\mathcal{X})(t),(\mathbb{Q}\mathcal{Y})(t)\bigr] \\ &\quad \le \frac{2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}H_k[X,Y]e^{k(s+\sigma)}\,ds, \end{aligned}$$

and so

$$\begin{aligned} H_k[\mathbb{Q}\mathcal{X},\mathbb{Q}\mathcal{Y}] &\le \frac{2H_k[X,Y]}{\Gamma(\alpha)}\sup_{t\in[a,b]}\int_a^t (t-s)^{\alpha-1}e^{k(s-t)}\,ds \\ &\le \frac{2H_k[X,Y]}{k^\alpha\Gamma(\alpha)}\sup_{t\in[a,b]}\int_0^{k(t-a)} u^{\alpha-1}e^{-u}\,du \\ &\le \frac{2H_k[X,Y]}{k^\alpha}. \end{aligned}$$

Therefore, it holds that

$$\begin{aligned}\mathbb{T}_1\big(H_k[\mathbb{Q}\mathcal{X},\mathbb{Q}\mathcal{Y}]\big) &\le \mathbb{T}_1\bigg(\frac{2H_k[X,Y]}{k^\alpha}\bigg)\\ &= \mathbb{T}_1\big(H_k[X,Y]\big) - \bigg[\mathbb{T}_1\big(H_k[X,Y]\big) - \mathbb{T}_1\bigg(\frac{2H_k[X,Y]}{k^\alpha}\bigg)\bigg].\end{aligned}$$

Then, if $\mathbb{T}_2(t) = \mathbb{T}_1(t) - \mathbb{T}_1(2t/k^\alpha)$, it follows that

$$\mathbb{T}_1\big(H_k[\mathbb{Q}\mathcal{X},\mathbb{Q}\mathcal{Y}]\big) \le \mathbb{T}_1\big(H_k[X,Y]\big) - \mathbb{T}_2\big(H_k[X,Y]\big)$$

for all $X \succeq Y$. Finally, using the existence of the lower solution, we check that $\mathcal{X}$ is such that $\mathcal{X}^L \preceq \mathbb{Q}\mathcal{X}^L$. Indeed, since $X^L(t) = \xi(t-a) \preceq \varphi(t-a)$ for $t \in [a-\sigma,a]$ and for $t \in [a,a+p]$,

$$X^L(t) \ominus_g X(a) \preceq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\big(s,X^L(s),X_s^L\big)\,ds,$$

it follows that

$$\begin{cases}\mathcal{X}^L(t) := X^L(t) \ominus_g \varphi(0) \preceq \varphi(t-a) \ominus_g \varphi(0) = \mathbb{Q}\mathcal{X}^L(t), & t \in [a-\sigma,a],\\ \mathcal{X}^L(t) \preceq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F(s,X^L(s),X_s^L)\,ds = \mathbb{Q}\mathcal{X}^L(t), & t \in [a,a+p].\end{cases}$$

As the operator $\mathbb{Q}$ satisfies all hypotheses of Theorem 2.1, $\mathbb{Q}$ has a fixed point in $C([a-\sigma,b],K_C(\mathbb{R}))$. Moreover, since every pair of interval-valued functions in $C([a-\sigma,b],K_C(\mathbb{R}))$ has an upper bound (see Lemma 2.2), the operator $\mathbb{Q}$ has a unique fixed point $\mathcal{X}$ and $\mathcal{X}$ is the unique solution to (3.1). □

Remark 3.2 The conclusion of Theorem 3.1 is still valid if the existence of a w-monotone lower solution for problem (3.1) is replaced by the existence of a w-monotone upper solution for problem (3.1).

Interval-valued delay fractional differential equation: Let us consider again the interval-valued delay fractional differential equation with Caputo generalized Hukuhara fractional differentiability under the form

$$\begin{cases}({}^C\mathcal{D}_{a^+}^\alpha X)(t) = F(t,X(t),X_t), & t \in [a,b]\\ X(t) = \varphi(t-a), & t \in [a-\sigma,a].\end{cases} \tag{3.6}$$

Denote by $C^{1,F}([a,b],K_C(\mathbb{R}))$ the space of interval-valued functions which are continuous Caputo gH-fractional differentiable on $[a,b]$. A solution $X \in C([a-\sigma,b],K_C(\mathbb{R})) \cap C^{1,F}([a,b],K_C(\mathbb{R}))$ of (3.6) is said to be w-monotone if it is w-increasing or w-decreasing on $[a,b]$.

Lemma 3.1 *Let F be interval functions such that $F \in C([a,b],K_C(\mathbb{R}))$ for any $X \in K_C(\mathbb{R})$. Then a w-monotone interval function $X \in C([a-\sigma,b],K_C(\mathbb{R}))$ is a solution of initial value problem* (3.6) *if and only if X satisfies the interval fractional integral equation* (3.1) *and the* *interval-valued function* $t \mapsto \Im^{\alpha}_{a^+}\mathbb{F}(t)$ *is w-increasing on* $[a,b]$, *where*

$$\mathbb{F}(t) := F\bigl(t, X(t), X_t\bigr), \quad t \in [a,b]. \tag{3.7}$$

Proof The proof of this lemma is similar to the proof of Lemma 3.1 in [38]. □

Corollary 3.1 *If a w-monotone interval function X is a solution of* (3.1) *such that the function* $t \mapsto \Im^{\alpha}_{a^+}\mathbb{F}(t)$ *is w-increasing on* $[a,b]$, *then X is a w-monotone solution of* (3.6).

Definition 3.2 A w-monotone interval function $X^U \in X \in C([a-\sigma,b],K_C(\mathbb{R})) \cap C^{1,F}([a,b],K_C(\mathbb{R}))$ is an upper solution for (3.6) if

$$\begin{aligned} &\bigl({}^C\mathcal{D}^{\alpha}_{a^+}X^U\bigr)(t) \succeq F\bigl(t, X^U(t), X^U_t\bigr), \quad t \in [a,b], \\ &X^U(t) = \xi(t-a) \succeq \varphi(t-a), \quad t \in [a-\sigma,a]. \end{aligned} \tag{3.8}$$

A w-monotone interval-valued function $X^L \in C([a-\sigma,b],K_C(\mathbb{R})) \cap C^{1,F}([a,b],K_C(\mathbb{R}))$ is a lower solution for (3.6) if it satisfies the reverse inequalities of (3.8).

Corollary 3.2 *Let* $F \in C([a,b],K_C(\mathbb{R}))$ *and suppose that* $F(t,A,B)$ *is nondecreasing in* A,B *for each* $t \in [a,b]$, *that is, if* $A \succeq C, B \succeq D$, *then* $F(t,A,B) \succeq F(t,C,D)$. *Moreover, assume that the following conditions are satisfied*:

(A3) *there exists a w-monotone upper solution* $X^U \in C([a-\sigma,b],K_C(\mathbb{R})) \cap C^{1,F}([a,b],K_C(\mathbb{R}))$ *for problem* (3.6);

(A4) *for an altering distance function* $\mathbb{T}_3$, *it holds*

$$\begin{aligned} &H\bigl[F(t,A,B),F(t,C,D)\bigr] \\ &\quad \le \bigl(H[A,B] + H_{\sigma}[B,D]\bigr) - \bigl(\mathbb{T}_3\bigl(H[A,B]\bigr) + \mathbb{T}_3\bigl(H_{\sigma}[B,D]\bigr)\bigr) \end{aligned}$$

if $A \succeq C, B \succeq D$ *and* $t \in [a,b]$. *Then there exists a unique w-monotone solution X for problem* (3.6) *in some intervals* $[a,\mathbb{T}]$, *with* $\mathbb{T} \le b$.

Proof In the same way as the proof of Theorem 3.1, let $\mathcal{X} := X(t) \ominus_g X(a), t \in [a-\sigma,b]$, and we define the operator $\mathbb{P} : C([a-\sigma,b],K_C(\mathbb{R})) \to C([a-\sigma,b],K_C(\mathbb{R}))$ by $(\mathbb{P}\mathcal{X})(t) = \varphi(t-a) \ominus_g \varphi(0), t \in [a-\sigma,a]$ and

$$(\mathbb{P}\mathcal{X})(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\bigl(s, X(s), X_s\bigr)\,ds, \quad t \in [a,b].$$

From the proof of Theorem 3.1 it is easy to see that from hypothesis of the nondecreasing property of F with respect to the second and third variables, the operator $\mathbb{P}$ is nondecreasing, that is, $\mathbb{P}\mathcal{X} \succeq \mathbb{P}\mathcal{Y}$ whenever $X \succeq Y$. On the other hand, hypothesis (A4) is implied from hypothesis (A2) with considering $\mathbb{T}_1(u) = u$. Therefore, we can easily infer that the operator $\mathbb{P}$ is contractive-like. Finally, hypothesis (A3) infers that $\mathcal{X}^U \succeq \mathbb{P}\mathcal{X}^U$. Indeed, since $\mathcal{X}^U$ is an upper solution and $X^U(t) \succeq X(t), t \in [a-\sigma,a]$, from Lemma 3.1 for $t \in [a,b]$ we get

$$\begin{aligned} X^{\mathcal{U}}(t) \ominus_g \varphi(0) &\succeq \left(\Im^{\alpha}_{a^+} {}^C\mathcal{D}^{\alpha}_{a^+} X\right)(t) \\ &\succeq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\left(s, X^{\mathcal{U}}(s), X^{\mathcal{U}}_s\right) ds \\ &= \left(\mathbb{P}\mathcal{X}^{\mathcal{U}}\right)(t), \quad t \in [a,b]. \end{aligned}$$

Thus $\mathcal{X}^{\mathcal{U}} \succeq \mathbb{P}\mathcal{X}^{\mathcal{U}}$. We see that the operator $\mathbb{P}$ verifies all the hypotheses of Theorem 2.1. In consequence, $\mathbb{P}$ has a fixed point in $C([a-\sigma,b],K_C(\mathbb{R}))$. Furthermore, the space $C([a-\sigma,b],K_C(\mathbb{R}))$ satisfies that every pair of elements of $C([a-\sigma,b],K_C(\mathbb{R}))$ has an upper bound (see Lemma 2.2). It follows that $\mathbb{P}$ has a unique fixed point. The proof is complete. □

In the following corollary, we analyze the dependence of the solution on the order and the initial condition for problem (3.6).

Theorem 3.2 *Let F satisfy the assumptions of Corollary* 3.2, *and let* $\alpha \in (0,1)$, $\delta > 0$ *such that* $0 < \alpha - \delta < \alpha < 1$. *For* $t \in [a,\mathbb{T}]$, *assume that X and Z are the solutions of the initial value problem* (3.6) *and*

$$\begin{aligned} &\left({}^C\mathcal{D}^{\alpha-\delta}_{a^+} Z\right)(t) = F\left(t, Z(t), Z_t\right), \quad t \in [a,b], \\ &Z(t) = \psi(t-a), \quad t \in [a-\sigma,a], \end{aligned} \tag{3.9}$$

respectively. Then the following holds:

$$H\left[X(t), Z(t)\right] \leq B(t) + \int_a^t \sum_{i=1}^{\infty} \left(\frac{2}{\Gamma(\alpha)}\Gamma(\alpha-\delta)\right)^i \frac{(t-s)^{i(\alpha-\delta)-1}}{\Gamma(i(\alpha-\delta))} B(s)\, ds,$$

where

$$\begin{aligned} B(t) := {}& H\left[X(a), Z(a)\right] \\ &+ \left| \frac{(t-a)^{\alpha-\delta}}{\alpha-\delta} \left(\frac{1}{\Gamma(\alpha-\delta)} - \frac{1}{\Gamma(\alpha)}\right)\right| \sup_{t\in[0,\mathbb{T}]} F\left(t, Y(t), Y_t\right) \\ &+ \left| \frac{(t-a)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\right| \sup_{t\in[0,\mathbb{T}]} F\left(t, X(t), X_t\right). \end{aligned}$$

Proof From Lemma 3.1, the solutions of the initial value problems (3.6) and (3.9) are given by

$$X(t) \ominus_g \varphi(0) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\left(s, X(s), X_s\right) ds, \quad t \in [a,\mathbb{T}]$$

and

$$Z(t) \ominus_g \psi(0) = \frac{1}{\Gamma(\alpha-\delta)} \int_a^t (t-s)^{\alpha-\delta-1} F\left(s, Z(s), Z_s\right) ds, \quad t \in [a,\mathbb{T}],$$

respectively. Observe that for $t \in \mathbb{T}$,

$$\begin{aligned}
&H\big[X(t), Z(t)\big] \\
&\quad \le H\big[\varphi(0), \psi(0)\big] \\
&\qquad + H\bigg[\frac{1}{\Gamma(\alpha-\delta)}\int_a^t (t-s)^{\alpha-\delta-1} F\big(s, Z(s), Z_s\big)\,ds, \\
&\qquad \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1} F\big(s, Z(s), Z_s\big)\,ds\bigg] \\
&\qquad + H\bigg[\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1} F\big(s, Z(s), Z_s\big)\,ds, \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1} F\big(s, X(s), X_s\big)\,ds\bigg] \\
&\qquad + H\bigg[\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1} F\big(s, X(s), X_s\big)\,ds, \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} F\big(s, X(s), X_s\big)\,ds\bigg] \\
&\quad \le H\big[\varphi(0), \psi(0)\big] + \bigg|\frac{(t-a)^{\alpha-\delta}}{\alpha-\delta}\bigg(\frac{1}{\Gamma(\alpha-\delta)} - \frac{1}{\Gamma(\alpha)}\bigg)\bigg| \sup_{t\in[a,\mathbb{T}]} F\big(t, Z(t), Z_t\big) \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1}\Big(H\big[X(s), Z(s)\big] + \sup_{\theta\in[s-\sigma,s]} H\big[X(\theta), Z(\theta)\big]\Big)\,ds \\
&\qquad + \bigg|\frac{(t-a)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\bigg| \sup_{t\in[a,\mathbb{T}]} F\big(t, X(t), X_t\big).
\end{aligned}$$

Putting $k(s) = \sup_{\theta\in[s-\sigma,s]} H[X(\theta), Z(\theta)]$ for any $s \in [a, \mathbb{T}]$, we have, by generalized Gronwall's inequality (see Theorem 1 in [44]), that

$$H\big[X(t), Z(t)\big] \le B(t) + \int_a^t \sum_{i=1}^{\infty}\bigg(\frac{2}{\Gamma(\alpha)}\Gamma(\alpha-\delta)\bigg)^i \frac{(t-s)^{i(\alpha-\delta)-1}}{\Gamma(i(\alpha-\delta))} B(s)\,ds. \qquad \square$$

Remark 3.3 Under the hypothesis of Corollary 3.2, if $\delta = 0$, then we get the following estimate:

$$H\big[X(t), Z(t)\big] \le H\big[\varphi(0), \psi(0)\big] \sum_{i=0}^{\infty} \frac{2^i (t-a)^{i\alpha}}{\Gamma(i\alpha + i)}.$$

In the sequel, we show that the solutions of initial value problem (3.6) depend continuously on the initial condition, the order and the right-hand side of equation.

Theorem 3.3 *Let F, G satisfy the assumptions of Corollary* 3.2, *and let $\alpha \in (0,1)$, $\delta > 0$ such that $0 < \alpha - \delta < \alpha < 1$. For $t \in [a, \mathbb{T}]$, assume that X and Z are the solutions of initial value problem* (3.6) *and*

$$\big({}^C\mathcal{D}_{a^+}^{\alpha-\delta} Z\big)(t) = G\big(t, Z(t), Z_t\big), \quad t \in [a, b], \qquad Z(t) = \psi(t-a), \quad t \in [a-\sigma, a], \tag{3.10}$$

respectively. Assume also that there exists a positive constant ε such that

$$H\big[F(t, A, B), G(t, A, B)\big] \le \varepsilon, \quad t \in [a, \mathbb{T}].$$

Then the following holds:

$$H\big[X(t),Z(t)\big] \le C(t) + \int_a^t \sum_{i=1}^{\infty} \left(\frac{2}{\Gamma(\alpha)}\Gamma(\alpha-\delta)\right)^i \frac{(t-s)^{i(\alpha-\delta)-1}}{\Gamma(i(\alpha-\delta))} C(s)\,ds,$$

where

$$\begin{aligned} C(t) &:= H\big[\varphi(0),\psi(0)\big] + \left|\frac{(t-a)^{\alpha-\delta}}{\alpha-\delta}\left(\frac{1}{\Gamma(\alpha-\delta)} - \frac{1}{\Gamma(\alpha)}\right)\right| \sup_{t\in[0,\mathbb{T}]} F\big(t,Y(t),Y_t\big) \\ &\quad + \left|\frac{(t-a)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\right| \sup_{t\in[0,\mathbb{T}]} F\big(t,X(t),X_t\big) + \frac{\varepsilon(t-a)^{\alpha-\sigma}}{\Gamma(\alpha)(\alpha-\sigma)}. \end{aligned}$$

Proof Let X, Z denote the solutions of problems (3.6) and (3.10), respectively. For $t \in [a,\mathbb{T}]$, we get

$$\begin{aligned} H\big[F(t,X,X_t),G(t,Z,Z_t)\big] &\le H\big[F(t,X,X_t),G(t,X,X_t)\big] + H\big[G(t,X,X_t),G(t,Z,Z_t)\big] \\ &\le \varepsilon + H[X,Z] + H_\sigma[X_t,Z_t]. \end{aligned}$$

Therefore, we obtain the following estimate:

$$\begin{aligned} &H\big[X(t),Z(t)\big] \\ &\quad\le H\big[\varphi(0),\psi(0)\big] \\ &\qquad + H\bigg[\frac{1}{\Gamma(\alpha-\delta)}\int_a^t (t-s)^{\alpha-\delta-1} G\big(s,Z(s),Z_s\big)\,ds, \\ &\qquad \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1} G\big(s,Z(s),Z_s\big)\,ds\bigg] \\ &\qquad + H\bigg[\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1} G\big(s,Z(s),Z_s\big)\,ds, \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1} F\big(s,X(s),X_s\big)\,ds\bigg] \\ &\qquad + H\bigg[\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1} F\big(s,X(s),X_s\big)\,ds, \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} F\big(s,X(s),X_s\big)\,ds\bigg] \\ &\quad\le C(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-\delta-1}\Big(H\big[X(s),Z(s)\big] + \sup_{\theta\in[s-\sigma,s]} H\big[X(\theta),Z(\theta)\big]\Big)\,ds. \end{aligned}$$

Putting $k(s) = \sup_{\theta\in[s-\sigma,s]} H[X(\theta),Z(\theta)]$ for any $s \in [a,\mathbb{T}]$ we obtain, by generalized Gronwall's inequality (see Theorem 1 in [44]), that

$$H\big[X(t),Z(t)\big] \le C(t) + \int_a^t \sum_{i=1}^{\infty} \left(\frac{2}{\Gamma(\alpha)}\Gamma(\alpha-\delta)\right)^i \frac{(t-s)^{i(\alpha-\delta)-1}}{\Gamma(i(\alpha-\delta))} C(s)\,ds. \qquad \square$$

The following corollary shows a new technique to find the exact solutions of interval-valued delay fractional differential equation by using the solutions of interval-valued delay integer order differential equation.

Corollary 3.3 *Assume that the conditions of Corollary* 3.2 *hold. Then a solution of* (3.6), X_{FO}, *is given by*

$$\begin{cases} X_{FO}(t) = X_{IO}(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}), & t \in [a,b] \\ X_{FO}(t) = \varphi(t-a), & t \in [a-\sigma,a], \end{cases}$$

where $X_{IO}(v)$ *is a solution of IVP of the interval-valued delay integer order differential equation*

$$\begin{cases} X'_{IO}(v) = F^*(v, X_{IO}(v), X_{IO,v}), & v \in [0,(b-a)^\alpha/\Gamma(\alpha+1)] \\ X_{IO}(v) = \varphi(t-a-[(t-a)^\alpha - v\Gamma(\alpha+1)]^{1/\alpha}), & v \in [-\sigma,0], \end{cases} \tag{3.11}$$

where $F^*(v, X_{IO}(v), X_{IO,v}) = F(k(t,v), X_{FO}(k(t,v)), X_{FO,k(t,v)})$, *and* $k(t,v) := t-([t-a]^\alpha - v\Gamma(\alpha+1))^{1/\alpha}$.

Proof From Corollary 3.2, we infer that the solution of problem (3.6), X_{FO}, exists and is given by

$$\begin{cases} X_{FO}(t) \ominus_g X_{FO}(a) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(s, X_{FO}(s), X_{FO,s})\, ds, & t \in [a,b] \\ X_{FO}(t) = \varphi(t-a), & t \in [a-\sigma,a]. \end{cases} \tag{3.12}$$

Let $s = t - [(t-a)^\alpha - v\Gamma(\alpha+1)]^{1/\alpha}, t \in [a,b]$. Then the interval delay fractional integral equation (3.12) can be written as

$$\begin{aligned} X_{FO}(t) \ominus_g X_{FO}(a) &= \int_0^{(t-a)^\alpha/\Gamma(\alpha+1)} F\bigl(k(s,v), X_{FO}\bigl(k(s,v)\bigr), X_{FO,k(s,v)}\bigr)\, dv \\ &= \int_0^{(t-a)^\alpha/\Gamma(\alpha+1)} F^*\bigl(v, X_{IO}(v), X_{IO,v}\bigr)\, dv. \end{aligned} \tag{3.13}$$

On the other hand, from the interval differential equation (3.11) we obtain

$$X_{IO}(v) \ominus_g X_{IO}(0) = \int_0^v F^*\bigl(v, X_{IO}(v), X_{IO,v}\bigr)\, dv, \tag{3.14}$$

where $v \in [0,(b-a)^\alpha/\Gamma(\alpha+1)]$. From (3.13), (3.14), $X_{FO}(a) = X_{IO}(0) = \varphi(0)$ and as $0 \le (t-a)^\alpha/\Gamma(\alpha+1) \le (b-a)^\alpha/\Gamma(\alpha+1)$, we get

$$\begin{aligned} X_{FO}(t) \ominus_g \varphi(0) &= X_{IO}(v) \ominus_g \varphi(0) \\ &= X_{IO}\left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\right) \ominus_g \varphi(0). \end{aligned}$$

The proof is completed. □

Example 3.4 Consider the fractional order initial value problem for interval-valued delay differential equation given by

$$\begin{cases} ({}^C\mathcal{D}_{0^+}^{0.5} X)(t) = \lambda_1 X(t-1) - \lambda_2 t, & t \in [0,1] \\ X(t) = [t-1,t], & t \in [-1,0], \end{cases} \tag{3.15}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$. Using the result of Corollary 3.3 for this example, we get

$$F^*\big(\nu, X_{IO}(\nu), X_{IO,\nu}\big) = \lambda_1 X\big(k(t,\nu)-1\big) - \lambda_2\big(2\sqrt{t}\Gamma(3/2)\nu - \nu^2\Gamma^2(3/2)\big).$$

The corresponding differential equation of this fractional initial value problem (3.15) is

$$\begin{cases} X'_{IO}(\nu) = \lambda_1 X_{IO}(k(t,\nu)-1) - \lambda_2(2\sqrt{t}\Gamma(3/2)\nu - \nu^2\Gamma^2(3/2)), & \nu \in [0, 1/\Gamma(3/2)] \\ X_{IO}(\nu) = [k(t,\nu)-1, k(t,\nu)], & \nu \in [-1, 0], \end{cases}$$

where $k(t,\nu) = (2\sqrt{t}\Gamma(3/2)\nu - \nu^2\Gamma^2(3/2))$. Then, by using the method of steps, we get the following problem:

$$\begin{cases} X'_{IO}(\nu) = \lambda_1[k(t,\nu)-2, k(t,\nu)-1] - \lambda_2 k(t,\nu), & \nu \in [0, 1/\Gamma(3/2)] \\ X_{IO}(0) = [-1, 0]. \end{cases}$$

Case 1. $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and X_{IO} is w-increasing. Then we obtain the solution $X_{IO}(\nu) = [\underline{X}_{IO}(\nu), \overline{X}_{IO}(\nu)]$, where

$$\underline{X}_{IO}(\nu) = (\lambda_1 - \lambda_2)\left(\sqrt{t}\Gamma(3/2)\nu^2 - \frac{\Gamma(3/2)\nu^3}{3}\right) - 2\lambda_1\nu - 1,$$

$$\overline{X}_{IO}(\nu) = (\lambda_1 - \lambda_2)\left(\sqrt{t}\Gamma(3/2)\nu^2 - \frac{\Gamma(3/2)\nu^3}{3}\right) - \lambda_1\nu.$$

From the result of Corollary 3.3, the solution of the given fractional order (3.17) is $X_{FO}(t) = X_{IO}(\frac{t^{0.5}}{\Gamma(1.5)}) = [\underline{X}_{FO}(t), \overline{X}_{FO}(t)]$, where

$$\underline{X}_{FO}(t) = (\lambda_1 - \lambda_2)\frac{2t^{3/2}}{3\Gamma(3/2)} - \frac{2\lambda_1\sqrt{t}}{\Gamma(3/2)} - 1, \qquad \overline{X}_{FO}(t) = (\lambda_1 - \lambda_2)\frac{2t^{3/2}}{3\Gamma(3/2)} - \frac{\lambda_1\sqrt{t}}{\Gamma(3/2)}.$$

Case 2. $\lambda_1, \lambda_2 \in \mathbb{R}^-$ and X_{IO} is w-decreasing. Similar to Case 1, we obtain

$$\big[\underline{X}_{FO}(t), \overline{X}_{FO}(t)\big] = (\lambda_1 - \lambda_2)\frac{2t^{3/2}}{3\Gamma(3/2)} + \left[-\frac{2\lambda_1\sqrt{t}}{\Gamma(3/2)} - 1, -\frac{\lambda_1\sqrt{t}}{\Gamma(3/2)}\right].$$

In the sequel we consider an example which corresponds to the interval version of the problem of the fish population size over time, and the propositional harvesting model in uncertain environment is presented to show the efficiency of the approach. First of all, we recall the framework of the fish population growth model in the situation where quantity of fish is precisely described. Under simplified conditions such as a constant environment (and with no migration), it can be shown that the change in population size p through time t (the time horizon is from zero to $b > 0$) will depend on three factors including birth rate, death rate and harvest rate, and given by

$$\frac{dp(t)}{dt} = \beta p(t) - \big(m + cp(t)\big)p(t) - h(t), \quad p(0) = z_0, \tag{3.16}$$

where $\beta p(t)$ is the birth rate, $(m + cp(t))p(t)$ is the death rate (here, the natural mortality coefficient m is augmented by the term $cp(t)$ which accounts for overcrowding),

$h(t)$ is the harvest rate which depends on time t, and β, m, c are negative proportionality constants. The symbol p_0 denotes the initial population size and p denotes the current population size. If the partial information in the classical population model (3.16) may be known or in the parameters used in the above model may be uncertainty, then model (3.16) can be appropriate by interval or fuzzy theory. Therefore, the corresponding to (3.16) model incorporating uncertainty could be the interval fish population growth model by using the concept of interval Caputo fractional derivative. Furthermore, in the classical population model (3.16), it is considered that the birth rate changes immediately as soon as a change in the number of individuals is produced. However, the members of the population must reach a certain degree of development to give birth to new individuals and this suggests an introduction of a delay term into the problem.

Example 3.5 From the crisp problem (3.16), in this example we choose a form (representation) for the corresponding interval fractional order initial value problem with delay as follows:

$$\begin{cases} ({}^{C}\mathcal{D}_{0^+}^{0.5}P)(t) + H(t) = (\beta - m)P(t) + P(t-1), & t \in [0,1] \\ P(t) = [t+1, t+2] \in K_C(\mathbb{R}), & t \in [-1,0], \end{cases} \tag{3.17}$$

where $(\beta - m) \in \mathbb{R}\backslash\{0\}$ and $H(t) := [0, ht^2] \in K_C(\mathbb{R})$ is the harvest rate, for $h > 0$, and there is no overcrowding, i.e., $c = 0$. Using the result of Corollary 3.3 for this example, we get

$$F^*\bigl(\nu, P_{IO}(\nu), P_{IO,\nu}\bigr) = \lambda_1 P_{IO}(\nu) + P_{IO}\bigl(k(t,\nu) - 1\bigr) \ominus \bigl[0, hk^2(t,\nu)\bigr],$$

where $\lambda_1 := \beta - m$. The corresponding differential equation of this fractional initial value problem (3.17) is

$$\begin{cases} P'_{IO}(\nu) = \lambda_1 P_{IO}(\nu) + P_{IO}(k(t,\nu) - 1) \ominus [0, hk^2(t,\nu)], & \nu \in [0, 1/\Gamma(3/2)] \\ P_{IO}(\nu) = [k(t,\nu) + 1, k(t,\nu) + 2], & \nu \in [-1, 0], \end{cases}$$

where $k(t,\nu) = (2\sqrt{t}\Gamma(3/2)\nu - \nu^2\Gamma^2(3/2))$. Then, by using the method of steps, we get the following problem:

$$\begin{cases} P'_{IO}(\nu) = \lambda_1 P_{IO}(\nu) + [k(t,\nu), k(t,\nu) - hk^2(t,\nu) + 1], & \nu \in [0, 1/\Gamma(3/2)] \\ P_{IO}(0) = [1, 2]. \end{cases}$$

Case 1. $\lambda_1 \in \mathbb{R}^+$ and P_{IO} is w-increasing. Then we obtain the solution $P_{IO}(\nu) = [\underline{P}_{IO}(\nu), \overline{P}_{IO}(\nu)]$, where

$$\begin{aligned} \underline{P}_{IO}(\nu) = {} & \frac{\Gamma(3/2)}{\lambda_1^3}\bigl[\Gamma(3/2)\lambda_1\bigl(\lambda_1\nu^2 + 2\nu\bigr) - 2\lambda_1\sqrt{t}(\lambda_1\nu + 1) + 2\Gamma(3/2)\bigr], \\ & + e^{\lambda_1\nu}\left[1 + \frac{\Gamma(3/2)}{\lambda_1^3}\bigl(2\lambda_1\sqrt{t} - 2\Gamma(3/2)\bigr)\right], \end{aligned}$$

$$\begin{aligned}\overline{P}_{IO}(v) = &\Bigg[2 + \frac{\Gamma(3/2)}{\lambda_1^5}\Bigg(-24h\Gamma^3(3/2) - 2\Gamma(3/2)\lambda_1^2 + \frac{\lambda_1^4}{\Gamma(3/2)}\\
&+ 2\lambda_1^3\sqrt{t} - 8h\Gamma(3/2)\lambda_1^2 t + 24h\Gamma^2(3/2)\lambda_1\sqrt{t}\Bigg)\Bigg]e^{\lambda_1 v}\\
&+ \Bigg[\frac{\Gamma(3/2)}{\lambda_1^5}\big(24h\Gamma^3(3/2) + 2\Gamma(3/2)\lambda_1^2 - \lambda_1^4/\Gamma(3/2) - 2\lambda_1^3\sqrt{t}\\
&+ 8h\Gamma(3/2)\lambda_1^2 t - 24h\Gamma^2(3/2)\lambda_1\sqrt{t}\big)\\
&+ \frac{\Gamma(3/2)v}{\lambda_1^4}\big(-2\Gamma(3/2)\lambda_1^2 + 24h\Gamma^3(3/2) - 2\lambda_1^3\sqrt{t}\\
&- 24h\Gamma^2(3/2)\lambda_1\sqrt{t} - 8h\Gamma(3/2)\lambda_1^2 t\big)\\
&+ \frac{\Gamma(3/2)v^2}{\lambda_1^3}\big(\Gamma(3/2)\lambda_1^2 + 12h\Gamma^3(3/2) - 12\Gamma^2(3/2)\lambda_1 h\sqrt{t} + 4h\Gamma(3/2)\lambda_1^2 t\big)\\
&+ \frac{\Gamma(3/2)v^3}{\lambda_1^2}\big(4h\Gamma^3(3/2) - 4h\Gamma^2(3/2)\lambda_1\sqrt{t}\big) + \frac{\Gamma^4(3/2)hv^4}{\lambda_1}\Bigg].\end{aligned}$$

From the result of Corollary 3.3, the solution of the given fractional order (3.17) is $P_{FO}(t) = P_{IO}(\frac{t^{0.5}}{\Gamma(1.5)}) = [\underline{P}_{FO}(t), \overline{P}_{FO}(t)]$, where

$$\begin{aligned}\underline{P}_{FO}(t) = &\frac{1}{\lambda_1^3}\left(-\lambda_1^2 t + \frac{\pi}{2}\right) + \exp\left(\frac{2\lambda_1\sqrt{t}}{\sqrt{\pi}}\right)\left(1 + \frac{1}{\lambda_1^3}(\lambda_1\sqrt{\pi t} - \pi/2)\right),\\
\overline{P}_{FO}(t) = &\exp\left(\frac{2\lambda_1\sqrt{t}}{\sqrt{\pi}}\right)\\
&\times\left(2 + \frac{1}{\lambda_1^5}\left[-\frac{3h}{2}\pi^2 - \pi\frac{\lambda_1^2}{2} + \lambda_1^4 + \lambda_1^3\sqrt{\pi t} - 2h\pi\lambda_1^2 t + 3h\pi\lambda_1\sqrt{\pi t}\right]\right)\\
&+ \left[\frac{1}{\lambda_1^5}\left(\frac{3h}{2}\pi^2 + \pi\frac{\lambda_1^2}{2} - \lambda_1^4 - \lambda_1^3\sqrt{\pi t} + 2h\pi\lambda_1^2 t - 3h\pi\lambda_1\sqrt{\pi t}\right)\right.\\
&+ \frac{1}{\lambda_1^4}\big(3h\pi\sqrt{\pi t} - \lambda_1^2\sqrt{\pi t} - 2\lambda_1^3 t - 6h\pi\lambda_1 t - 4h\sqrt{\pi}\lambda_1^2 t^{3/2}\big)\\
&\left.+ \frac{1}{\lambda_1^3}\big(\lambda_1^2 t + 3h\pi t - 6h\sqrt{\pi}\lambda_1 t^{3/2} + 4h\lambda_1^2 t^2\big) + \frac{4h}{\lambda_1^2}\big(2\sqrt{\pi}t^{3/2} - \lambda_1 t^2\big) + \frac{h}{\lambda_1}t^2\right].\end{aligned}$$

In our numerical simulations, we use the value of parameters $\lambda_1 = 1$ and $h = 0.1$. The w-increasing solution of (3.17) is shown in Figure 1.

Case 2. $\lambda_1 \in \mathbb{R}^-$ and P_{IO} is w-decreasing. Similar to Case 1, from the result of Corollary 3.3, the solution of the given fractional order (3.17) is $P_{FO}(t) = P_{IO}(\frac{t^{0.5}}{\Gamma(1.5)}) = [\underline{P}_{FO}(t), \overline{P}_{FO}(t)]$, where

$$\begin{aligned}\underline{P}_{FO}(t) = &\exp\left(\frac{2\lambda_1\sqrt{t}}{\sqrt{\pi}}\right)\\
&\times\left(1 + \frac{1}{\lambda_1^5}\left[-\frac{3h}{2}\pi^2 - \pi\frac{\lambda_1^2}{2} + \lambda_1^4 + \lambda_1^3\sqrt{\pi t} - 2h\pi\lambda_1^2 t + 3h\pi\lambda_1\sqrt{\pi t}\right]\right)\\
&+ \left[\frac{1}{\lambda_1^5}\left(\frac{3h}{2}\pi^2 + \pi\frac{\lambda_1^2}{2} - \lambda_1^4 - \lambda_1^3\sqrt{\pi t} + 2h\pi\lambda_1^2 t - 3h\pi\lambda_1\sqrt{\pi t}\right)\right.\end{aligned}$$

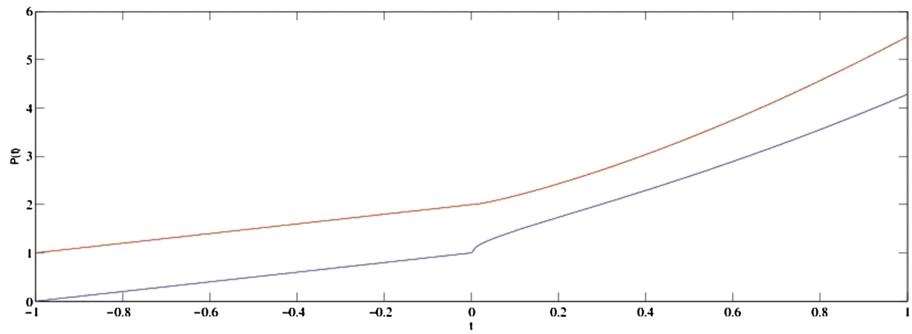

Figure 1 The *w*-increasing solution of Example 3.5 in Case 1.

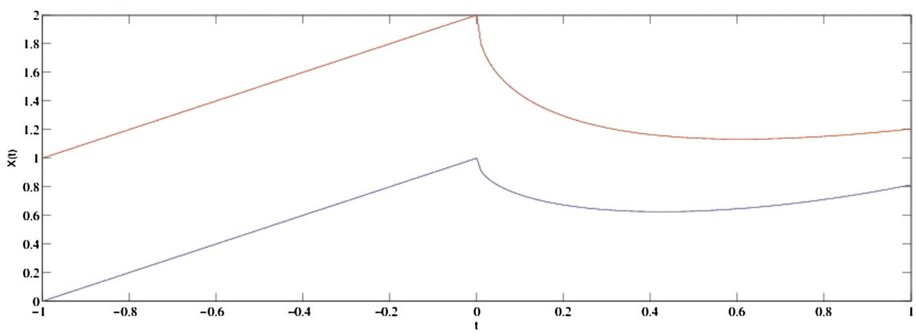

Figure 2 The *w*-decreasing solution of Example 3.5 in Case 2.

$$
\begin{aligned}
&+ \frac{1}{\lambda_1^4}\bigl(3h\pi\sqrt{\pi t} - \lambda_1^2\sqrt{\pi t} - 2\lambda_1^3 t - 6h\pi\lambda_1 t - 4h\sqrt{\pi}\lambda_1^2 t^{3/2}\bigr) \\
&+ \frac{1}{\lambda_1^3}\bigl(\lambda_1^2 t + 3h\pi t - 6h\sqrt{\pi}\lambda_1 t^{3/2} + 4h\lambda_1^2 t^2\bigr) + \frac{4h}{\lambda_1^2}\bigl(2\sqrt{\pi}t^{3/2} - \lambda_1 t^2\bigr) + \frac{h}{\lambda_1}t^2\Biggr],
\end{aligned}
$$

$$
\overline{P}_{FO}(t) = \frac{1}{\lambda_1^3}\left(-\lambda_1^2 t + \frac{\pi}{2}\right) + \exp\left(\frac{2\lambda_1\sqrt{t}}{\sqrt{\pi}}\right)\left(2 + \frac{1}{\lambda_1^3}\bigl(\lambda_1\sqrt{\pi t} - \pi/2\bigr)\right).
$$

In our numerical simulations, we use the value of parameters $\lambda_1 = -2$ and $h = 0.4$. The w-decreasing solution of (3.17) is shown in Figure 2.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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