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Nonlocal boundary value problems for second-order nonlinear Hahn integro-difference equations with integral boundary conditions

Umaphon Sriphanomwan[1], Jessada Tariboon[1,4*], Nichaphat Patanarapeelert[1], Sotiris K Ntouyas[2,3] and Thanin Sitthiwirattham[5]

*Correspondence:
jessada.t@sci.kmutnb.ac.th
[1]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand
[4]Centre of Excellence in Mathematics, CHE, Sri Ayutthaya Rd., Bangkok, 10400, Thailand
Full list of author information is available at the end of the article

Abstract

In this paper, we study a boundary value problem for second-order nonlinear Hahn integro-difference equations with nonlocal integral boundary conditions. Our problem contains two Hahn difference operators and a Hahn integral. The existence and uniqueness of solutions is obtained by using the Banach fixed point theorem, and the existence of at least one solution is established by using the Leray-Schauder nonlinear alternative and Krasnoselskii's fixed point theorem. Illustrative examples are also presented to show the applicability of our results.

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Keywords: Hahn difference equations; boundary value problems; existence; uniqueness; fixed point theorems

1 Introduction

A quantum calculus substitute the classical derivative by a difference operator, which allows one to deal with sets of non-differentiable functions. There are many different types of quantum difference operators such as h-calculus, q-calculus, Hahn's calculus, forward quantum calculus and backward quantum calculus. These operators are also found in many applications of mathematical areas such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations and the theory of relativity. Some recent results in quantum calculus can be found in [1–8] and the references cited therein.

Hahn [9] introduced his difference operator $D_{q,\omega}$ (see Definition 2.1) where $q \in (0,1)$ and $\omega > 0$ are fixed, which unifies (in the limit) the two best-known and most-used quantum difference operators: the Jackson q-difference derivative D_q, where $q \in (0,1)$ (*cf.* [10–12]); and the forward difference D_ω where $\omega > 0$ (*cf.* [2, 13, 14]). The Hahn difference operator is a successful tool for constructing families of orthogonal polynomials and investigating some approximation problems (*cf.* [15–17]).

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The right inverse of the Hahn difference operator was introduced by Aldwoah [18, 19] who defined the right inverse of $D_{q,\omega}$ in the terms of both the Jackson q-integral containing the right inverse of D_q and the Nörlund sum involving the right inverse of Δ_ω.

Malinowska and Torres [20, 21] introduced the Hahn quantum variational calculus, while Malinowska and Martins [22] studied the generalized transversality conditions for the Hahn quantum variational calculus. Hamza *et al.* [23, 24] studied the theory of linear Hahn difference equations, and investigated the existence and uniqueness results of the initial value problems with Hahn difference equations using the method of successive approximations.

Recently, Sitthiwirattham [25] initiated the study of boundary value problems for Hahn difference equations by considering the boundary value problem consisting of the nonlinear Hahn difference equation supplemented with nonlocal three-point boundary conditions of the form:

$$\begin{aligned} &D^2_{q,\omega}x(t) + f\big(t, x(t), D_{p,\theta}x(pt+\theta)\big) = 0, \quad t \in [\omega_0, T]_{q,\omega}, \\ &x(\omega_0) = \varphi(x), \\ &x(T) = \lambda x(\eta), \qquad \eta \in (\omega_0, T)_{q,\omega}, \end{aligned} \tag{1.1}$$

where $0 < q < 1$, $0 < \omega < T$, $\omega_0 := \frac{\omega}{1-q}$, $1 \leq \lambda < \frac{T-\omega_0}{\eta-\omega_0}$, $p = q^m$, $m \in \mathbb{N}$, $\theta = \omega(\frac{1-p}{1-q})$, $f : [\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $\varphi : C([\omega_0, T]_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ is a given functional. He proved existence results for (1.1) by using the Banach and Krasnoselskii fixed point theorems and also gave some numerical examples.

In this paper, motivated by the above papers, we continue the study of boundary value problems for Hahn difference equations by considering the nonlinear boundary value problem for Hahn integro-difference equations with nonlocal integral boundary conditions of the form

$$\begin{aligned} &D^2_{q,\omega}x(t) = f\big(t, x(t), D_{p,\theta}x(pt+\theta), \Psi_{p,\theta}x(pt+\theta)\big), \quad t \in [\omega_0, T]_{q,\omega}, \\ &x(\omega_0) = x(T), \\ &x(\eta) = \mu \int_{\omega_0}^{T} g(s)x(s)\, d_{q,\omega}s, \quad \eta \in (\omega_0, T)_{q,\omega}, \end{aligned} \tag{1.2}$$

where $0 < q < 1$, $0 < \omega < T$, $\omega_0 := \frac{\omega}{1-q}$, $\mu \int_{\omega_0}^{T} g(r)\, d_{q,\omega}r \neq 1$, $\mu \in \mathbb{R}$, $p = q^m$, $m \in \mathbb{N}$, $\theta = \omega(\frac{1-p}{1-q})$, $f \in C([\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $g \in C([\omega_0, T]_{q,\omega}, \mathbb{R}^+)$ are given functions, and for $\varphi \in C([\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}, [0, \infty))$

$$\Psi_{p,\theta}x(t) := \int_{\omega_0}^{t} \varphi(t, ps+\theta)x(ps+\theta)\, d_{p,\theta}s. \tag{1.3}$$

Existence and uniqueness results are proved by using fixed point theorems. Also many special cases and examples are presented.

The paper is organized as follows: In Section 2, we briefly recall some definitions and lemmas that are used in this research. In Section 3, we prove an existence and uniqueness result by using the Banach fixed point theorem and two existence result via the Leray-Schauder nonlinear alternative and Krasnoselskii's fixed point theorem, respectively.

2 Preliminaries

In the following, there are notations, definitions, and lemmas which are used in proving the main results.

Definition 2.1 ([9]) For $0 < q < 1$, $\omega > 0$ and f defined on an interval $I \subseteq \mathbb{R}$ containing $\omega_0 := \frac{\omega}{1-q}$, the Hahn difference of f is defined by

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega} \quad \text{for } t \neq \omega_0,$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$, provided that f is differentiable at ω_0. We call $D_{q,\omega}f$ the q, ω-derivative of f, and say that f is q, ω-differentiable on I.

This operator unifies and generalizes two well-known difference operators. The first is Jackson q-difference operator defined by

$$D_{q,0}f(t) = \begin{cases} \frac{f(t)-f(qt)}{t(1-q)}, & t \neq 0, \\ f'(0), & t = 0, \end{cases} \tag{2.1}$$

provident that $f'(0)$ exists. Here f is supposed to be defined on a q-geometric set $A \subset \mathbb{R}$, for which $qt \in A$ whenever $t \in A$.

The second operator is the forward difference operator

$$D_{1,\omega}f(t) = \frac{f(t+\omega) - f(t)}{\omega}, \tag{2.2}$$

where $\omega > 0$ is fixed.

Letting $a, b \in I \subseteq \mathbb{R}$ with $a < \omega_0 < b$ and $[k]_q = \frac{1-q^k}{1-q}$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define the q, ω-interval by

$$\begin{aligned}[a,b]_{q,\omega} &:= \left\{q^k a + \omega[k]_q : k \in \mathbb{N}_0\right\} \cup \left\{q^k b + \omega[k]_q : k \in \mathbb{N}_0\right\} \cup \{\omega_0\} \\ &= [a,\omega_0]_{q,\omega} \cup [\omega_0, b]_{q,\omega} \\ &= (a,b)_{q,\omega} \cup \{a,b\} = [a,b)_{q,\omega} \cup \{b\} = (a,b]_{q,\omega} \cup \{a\}.\end{aligned}$$

Example 2.1 The interval $[\frac{1}{2}, 20]_{\frac{1}{4},3}$ can be expressed by

$$\left[\frac{1}{2}, 20\right]_{\frac{1}{4},3} = \left\{\frac{1}{2}, \frac{25}{8}, \frac{121}{32}, \frac{505}{128}, \frac{2{,}041}{512}, \frac{8{,}185}{2{,}048}, \dots\right\} \cup \left\{20, 8, 5, \frac{17}{4}, \frac{65}{16}, \frac{257}{64}, \dots\right\} \cup \{4\}.$$

An essential function which plays an important role in Hahn's calculus is $h(t) = qt + \omega$. This function is normally taken to be defined on an interval I, which contains the number $\omega_0 = \frac{\omega}{1-q}$. Note that h is a contraction, $h(I) \subseteq I$, $h(t) < t$ for $t > \omega_0$, $h(t) > t$ for $t < \omega_0$, and $h(\omega_0) = \omega_0$. One can see that the kth-order iteration of $h(t)$ is given by $h^k(t) = q^k t + \omega[k]_q$, $t \in I$. Observe that, for each $s \in [a,b]_{q,\omega}$, the sequence $\{q^k s + \omega[k]_q\}_{k=0}^{\infty}$ is uniformly convergent to ω_0.

If f is q,ω-differentiable n times on q,ω-interval $I_{q,\omega}$, we define the higher-order derivatives by

$$D^n_{q,\omega}f(s) := D_{q,\omega}D^{n-1}_{q,\omega}f(s),$$

where $D^0_{q,\omega}f(s) := f(s)$, $s \in I_{q,\omega} \subset \mathbb{R}$.

Next, we introduce the right inverse of the operator $D_{q,\omega}$, the so-called q,ω-integral operator.

Definition 2.2 ([18]) Let I be any closed interval of $\mathbb{R}$ containing a, b and ω_0. Assuming that $f : I \to \mathbb{R}$ is a given function, we define the q,ω-integral of f from a to b by

$$\int_a^b f(t)\,d_{q,\omega}t := \int_{\omega_0}^b f(t)\,d_{q,\omega}t - \int_{\omega_0}^a f(t)\,d_{q,\omega}t,$$

where

$$\int_{\omega_0}^t f(s)\,d_{q,\omega}s = \mathcal{I}_{q,\omega}f(t) = \bigl[t(1-q)-\omega\bigr]\sum_{k=0}^{\infty} q^k f\bigl(tq^k + \omega[k]_q\bigr), \quad x \in I,$$

is the convergent series at $x = a$ and $x = b$. f is called q,ω-integrable on $[a,b]$ and the sum to the right hand side of the above equation will be called the Jackson-Nörlund sum.

We note that the actual domain of the function f is $[a,b]_{q,\omega} \subset I$.

The following lemma is the *fundamental theorem of Hahn calculus*.

Lemma 2.1 ([18]) *Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^x f(t)\,d_{q,\omega}t, \quad x \in I,$$

then F is continuous at ω_0. Furthermore, $D_{q,\omega_0}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega}F(t)\,d_{q,\omega}t = F(b) - F(a) \quad \text{for all } a, b \in I.$$

Next, we give some auxiliary lemmas for simplifying calculations.

Lemma 2.2 ([25]) *Let $0 < q < 1$, $\omega > 0$ and $x : I \to \mathbb{R}$ be continuous at ω_0. Then*

$$\int_{\omega_0}^t \int_{\omega_0}^r x(s)\,d_{q,\omega}s\,d_{q,\omega}r = \int_{\omega_0}^t \int_{qs+\omega}^t x(s)\,d_{q,\omega}r\,d_{q,\omega}s.$$

Remark 2.1 Observe that

$$\int_{\omega_0}^t \int_{qs+\omega}^t x(s)\,d_{q,\omega}r\,d_{q,\omega}s = \int_{\omega_0}^t \bigl(t-(qs+\omega)\bigr)x(s)\,d_{q,\omega}s.$$

Lemma 2.3 ([25]) *Let* $0 < q < 1$ *and* $\omega > 0$, *then*

$$\int_{\omega_0}^{t} d_{q,\omega}s = t - \omega_0 \quad \text{and} \quad \int_{\omega_0}^{t} \bigl[t - (qs + \omega)\bigr]\, d_{q,\omega}s = \frac{(t - \omega_0)^2}{1 + q}.$$

Lemma 2.4 *Let* $0 < q < 1$ *and* $\omega > 0$, *then the following equation holds*:

$$\underbrace{\int_{\omega_0}^{t} \cdots \int_{\omega_0}^{t}}_{n \text{ times}} d_{q,\omega}s \cdots d_{q,\omega}r = \frac{(t - \omega_0)^n}{[n]_q!}, \tag{2.3}$$

where $[n]_q! = \prod_{k=1}^{n} \frac{1-q^k}{1-q}$.

Proof Mathematical induction will be used in our proof as follows. For $n = 1$, we have

$$\int_{\omega_0}^{t} d_{q,\omega}s = \bigl[t(1 - q) - \omega\bigr] \sum_{k=0}^{\infty} q^k = (t - \omega_0),$$

which means that equation (2.3) is true for $n = 1$.

Suppose that equation (2.3) holds for $n = k$. Hence, for $n = k + 1$, we have

$$\begin{aligned}
\underbrace{\int_{\omega_0}^{t} \cdots \int_{\omega_0}^{t}}_{k+1 \text{ times}} d_{q,\omega}s \cdots d_{q,\omega}r &= \int_{\omega_0}^{t} \frac{(r - \omega_0)^k}{[k]_q!}\, d_{q,\omega}r \\
&= \frac{[t(1 - q) - \omega]}{[k]_q!} \sum_{k=0}^{\infty} q^k \bigl(tq^k + \omega[k]_q - \omega_0\bigr)^k \\
&= \frac{(1 - q)(t - \omega_0)}{[k]_q!} \sum_{k=0}^{\infty} q^k \bigl(tq^k - \omega_0 q^k\bigr)^k \\
&= \frac{(1 - q)(t - \omega_0)}{[k]_q!} \sum_{k=0}^{\infty} q^k q^{k^2} (t - \omega_0)^k \\
&= \frac{(1 - q)(t - \omega_0)^{k+1}}{[k]_q!(1 - q^{k+1})} \\
&= \frac{(t - \omega_0)^{k+1}}{[k + 1]_q!}.
\end{aligned}$$

Thus, equation (2.3) holds for $n = k + 1$. By the principle of induction, equation (2.3) is true for all $n \in \mathbb{N}$. □

The following lemma deals with the linear variant of problem (1.2) and gives a presentation of the solution.

Lemma 2.5 *Let* $\mu \int_{\omega_0}^{T} g(r)\, d_{q,\omega}r \neq 1$ *and* $h \in C([\omega_0, T]_{q,\omega}, \mathbb{R})$ *be a given function. Then the function* x *is a solution of the problem*

$$D_{q,\omega}^2 x(t) = h(t), \quad t \in [\omega_0, T]_{q,\omega}, \tag{2.4}$$

$$x(\omega_0) = x(T), \qquad x(\eta) = \mu \mathcal{I}_{q,\omega}(gx)(T), \quad \eta \in (\omega_0, T)_{q,\omega}, \tag{2.5}$$

if and only if

$$\begin{aligned} x(t) &= \int_{\omega_0}^{t} \bigl[t-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s - \frac{(t-\omega_0)}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s \\ &\quad + \frac{1}{\Omega}\biggl[\mu\int_{\omega_0}^{T}\int_{\omega_0}^{r} g(r)\bigl[r-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s\,d_{q,\omega}r \\ &\quad - \int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s + \frac{1}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s \\ &\quad \times\biggl(\eta-\omega_0-\mu\int_{\omega_0}^{T} g(r)(r-\omega_0)\,d_{q,\omega}r\biggr)\biggr], \end{aligned} \tag{2.6}$$

where

$$\Omega = 1-\mu\int_{\omega_0}^{T} g(r)\,d_{q,\omega}r. \tag{2.7}$$

Proof By Lemma 2.2 and Remark 2.1, a general solution for (2.4) can be written as

$$\begin{aligned} x(t) &= \int_{\omega_0}^{t}\int_{\omega_0}^{r} h(s)\,d_{q,\omega}s\,d_{q,\omega}r + C_1(t-\omega_0) + C_2 \\ &= \int_{\omega_0}^{t}\int_{qs+\omega}^{t} h(s)\,d_{q,\omega}r\,d_{q,\omega}s + C_1(t-\omega_0) + C_2 \\ &= \int_{\omega_0}^{t}\bigl[t-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s + C_1(t-\omega_0) + C_2, \end{aligned} \tag{2.8}$$

for $t \in [\omega_0, T]_{q,\omega}$. Taking the q, ω-integral for (2.8), we obtain, for $t \in [\omega_0, T]_{q,\omega}$,

$$\begin{aligned} \mathcal{I}_{q,\omega}x(t) &= \int_{\omega_0}^{t}\int_{\omega_0}^{r}\bigl[r-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s\,d_{q,\omega}r + C_1\int_{\omega_0}^{t}(r-\omega_0)\,d_{q,\omega}r \\ &\quad + C_2\int_{\omega_0}^{t} d_{q,\omega}r. \end{aligned} \tag{2.9}$$

From the boundary conditions of (2.5), we obtain

$$C_1 = -\frac{1}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s, \tag{2.10}$$

$$\begin{aligned} C_2 &= \frac{1}{\Omega}\biggl[\mu\int_{\omega_0}^{T}\int_{\omega_0}^{r} g(r)\bigl[r-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s\,d_{q,\omega}r - \int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s \\ &\quad - \frac{\mu}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s\int_{\omega_0}^{T} g(r)(r-\omega_0)\,d_{q,\omega}s \\ &\quad + \frac{\eta-\omega_0}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s\biggr], \end{aligned} \tag{2.11}$$

where Ω is defined as (2.7). Substituting the constants C_1, C_2 into (2.8), we obtain (2.6).

On the other hand, by taking the second-order q, ω-derivative to (2.6), we have (2.4). It is easy to check that equation (2.6) satisfies (2.5). This completes the proof. □

3 Existence and uniqueness results

In this section, we present the existence and uniqueness of solutions for problem (1.2). Let

$$X = \{x | x \in C([\omega_0, T]_{q,\omega}, \mathbb{R}) \text{ and } D_{p,\theta}x \in C([\omega_0, T]_{q,\omega}, \mathbb{R})\}$$

be the Banach space of all continuous functions $x : [\omega_0, T]_{q,\omega} \to \mathbb{R}$ with the norm defined by

$$\|x\|_X = \|x\| + \|D_{p,\theta}x\|,$$

where $\|x\| = \max\{|x(t)| : t \in [\omega_0, T]_{q,\omega}\}$ and $\|D_{p,\theta}x\| = \max\{|D_{p,\theta}x(pt + \theta)| : t \in [\omega_0, T]_{q,\omega}\}$. We define an operator $\mathcal{F} : X \to X$ by

$$\begin{aligned}
(\mathcal{F}x)(t) &= \int_{\omega_0}^{t} [t - (qs + \omega)] f(s, x(s), D_{p,\theta}x(ps + \theta), \Psi_{p,\theta}x(ps + \theta))\, d_{q,\omega}s \\
&\quad - \frac{(t - \omega_0)}{T - \omega_0} \int_{\omega_0}^{T} [T - (qs + \omega)] f(s, x(s), D_{p,\theta}x(ps + \theta), \Psi_{p,\theta}x(ps + \theta))\, d_{q,\omega}s \\
&\quad + \frac{1}{\Omega} \biggl[\mu \int_{\omega_0}^{T} \int_{\omega_0}^{r} g(r) [r - (qs + \omega)] f(s, x(s), D_{p,\theta}x(ps + \theta), \Psi_{p,\theta}x(ps + \theta))\, d_{q,\omega}s\, d_{q,\omega}r \\
&\quad - \int_{\omega_0}^{\eta} [\eta - (qs + \omega)] f(s, x(s), D_{p,\theta}x(ps + \theta), \Psi_{p,\theta}x(ps + \theta))\, d_{q,\omega}s \\
&\quad + \frac{1}{T - \omega_0} \int_{\omega_0}^{T} [T - (qs + \omega)] f(s, x(s), D_{p,\theta}x(ps + \theta), \Psi_{p,\theta}x(ps + \theta))\, d_{q,\omega}s \\
&\quad \times \biggl(\eta - \omega_0 - \mu \int_{\omega_0}^{T} g(r)(r - \omega_0)\, d_{q,\omega}r \biggr) \biggr],
\end{aligned} \tag{3.1}$$

where $\Omega \neq 0$ is defined by (2.3), $p = q^m$, $m \in \mathbb{N}$ and $\theta = \omega(\frac{1-p}{1-q})$.

Obviously, problem (1.2) has solutions if and only if the operator $\mathcal{F}$ has fixed points.

3.1 Existence and uniqueness result via Banach's fixed point theorem

Our first result concerns existence and uniqueness of solutions of problem (1.2) and is based on Banach's fixed point theorem.

Theorem 3.1 *Let*

$$\varphi_0 = \max\{\varphi(t, ps + \theta) : (t, ps + \theta) \in [\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}\}. \tag{3.2}$$

Assume that:

(H_1) *there exist functions* $h_i \in C([\omega_0, T]_{q,\omega}, \mathbb{R}^+)$, $i = 1, 2, 3$ *such that*

$$|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq h_1(t)|x_1 - y_1| + h_2(t)|x_2 - y_2| + h_3(t)|x_3 - y_3|$$

for each $t \in [\omega_0, T]_{q,\omega}$ *and* $x_i, y_i, z_i \in \mathbb{R}$, $i = 1, 2, 3$;

(H_2) *for each* $t \in [\omega_0, T]_{q,\omega}$, $0 < g(t) < N$;
(H_3) $\mathfrak{S} := \Phi_0(\Phi_1 + \Phi_2) < 1$,

where

$$\begin{aligned} \Phi_0 &= \|h_1\| + \|h_2\| + \|h_3\|\varphi_0(T-\omega_0), \\ \Phi_1 &= \frac{2(T-\omega_0)^2}{1+q} + \frac{1}{(1+q)|\Omega|}\biggl[\frac{|\mu|N(T-\omega_0)^3(2+2q+q^2)}{(1+q)(1+q+q^2)} \\ &\quad + (\eta-\omega_0)(T+\eta-2\omega_0)\biggr], \\ \Phi_2 &= \frac{(T-\omega_0)(1+p+p^2)}{1+q}. \end{aligned} \tag{3.3}$$

Then problem (1.2) *has a unique solution on* $[\omega_0, T]_{q,\omega}$.

Proof Denote $\hat{f}_x(t) := f(t, x(t), D_{p,\theta}x(pt+\theta), \Psi_{p,\theta}x(pt+\theta))$. Then we have

$$\begin{aligned} \bigl|\hat{f}_x(t) - \hat{f}_y(t)\bigr| &\le h_1(t)\bigl|x(t)-y(t)\bigr| + h_2(t)\bigl|D_{p,\theta}x(pt+\theta) - D_{p,\theta}y(pt+\theta)\bigr| \\ &\quad + h_3(t)\bigl|\Psi_{p,\theta}x(pt+\theta) - \Psi_{p,\theta}y(pt+\theta)\bigr| \\ &\le \|h_1\|\|x-y\| + \|h_2\|\|D_{p,\theta}x - D_{p,\theta}y\| + \|h_3\|\varphi_0(T-\omega_0)\|x-y\| \\ &\le \|h_1\|\|x-y\|_X + \|h_2\|\|x-y\|_X + \|h_3\|\varphi_0(T-\omega_0)\|x-y\|_X \\ &= \bigl(\|h_1\| + \|h_2\| + \|h_3\|\varphi_0(T-\omega_0)\bigr)\|x-y\|_X. \end{aligned}$$

Using Lemma 2.3, for each $t \in [\omega_0, T]_{q,\omega}$ and $x, y \in X$ we have

$$\begin{aligned} &\bigl|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\bigr| \\ &\quad \le \int_{\omega_0}^{t}\bigl[t-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s + \frac{(t-\omega_0)}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s \\ &\qquad + \frac{1}{|\Omega|}\biggl[\biggl|\mu\int_{\omega_0}^{T}\int_{\omega_0}^{r} g(r)\bigl[r-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s\,d_{q,\omega}r \\ &\qquad - \int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s\biggr| \\ &\qquad + \frac{1}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s\biggl|\eta-\omega_0-\mu\int_{\omega_0}^{T} g(r)(r-\omega_0)\,d_{q,\omega}r\biggr|\biggr] \\ &\quad \le \int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s + \int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s \\ &\qquad + \frac{1}{|\Omega|}\biggl[|\mu|N\int_{\omega_0}^{T}\int_{\omega_0}^{r}\bigl[r-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s\,d_{q,\omega}r \\ &\qquad + \int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s \\ &\qquad + \frac{1}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)-\hat{f}_y(t)\bigr|\,d_{q,\omega}s\biggl((\eta-\omega_0) \\ &\qquad + |\mu|N\int_{\omega_0}^{T}(r-\omega_0)\,d_{q,\omega}r\biggr)\biggr] \end{aligned}$$

$$\begin{aligned}
&\leq \Bigg\{ \frac{2(T-\omega_0)^2}{1+q} + \frac{1}{|\Omega|}\Bigg[\frac{|\mu| N(T-\omega_0)^3}{(1+q)(1+q+q^2)} + \frac{(\eta-\omega_0)^2}{1+q} \\
&\quad + \frac{T-\omega_0}{1+q}\bigg((\eta-\omega_0) + \frac{|\mu|N(T-\omega_0)^2}{1+q}\bigg)\Bigg]\Bigg\} \\
&\quad \times \big(\|h_1\| + \|h_2\| + \|h_3\|\varphi_0(T-\omega_0)\big)\|x-y\|_X \\
&\leq \Bigg\{ \frac{2(T-\omega_0)^2}{1+q} + \frac{1}{(1+q)|\Omega|}\Bigg[\frac{|\mu|N(T-\omega_0)^3(2+2q+q^2)}{(1+q)(1+q+q^2)} \\
&\quad + (\eta-\omega_0)(T+\eta-2\omega_0)\Bigg]\Bigg\}\big(\|h_1\| + \|h_2\| + \|h_3\|\varphi_0(T-\omega_0)\big)\|x-y\|_X \\
&= \Phi_0\Phi_1\|x-y\|_X.
\end{aligned}$$

Taking the p,θ-derivative for (3.1) where $p = q^m$, $m \in \mathbb{N}$ and $\theta = \omega(\frac{1-p}{1-q})$, we obtain

$$\begin{aligned}
&\big| (D_{p,\theta}\mathcal{F}x)(pt+\theta) - (D_{p,\theta}\mathcal{F}y)(pt+\theta)\big| \\
&\leq \Bigg| \frac{1}{[-(pt+\theta)(1-p)+\theta]}\Bigg\{\bigg(\int_{\omega_0}^{p(pt+\theta)+\theta} \big[p(pt+\theta)+\theta-(qs+\omega)\big]\big|\hat{f}_x(t)-\hat{f}_y(t)\big|\, d_{q,\omega}s \\
&\quad - \int_{\omega_0}^{pt+\theta} \big[(pt+\theta)-(qs+\omega)\big]\big|\hat{f}_x(t)-\hat{f}_y(t)\big|\, d_{q,\omega}s\bigg) + \frac{(pt+\theta)(1-p)+\theta}{T-\omega_0} \\
&\quad \times \int_{\omega_0}^{T} \big[T-(qs+\omega)\big]\big|\hat{f}_x(t)-\hat{f}_y(t)\big|\, d_{q,\omega}s\Bigg\}\Bigg| \\
&\leq \Bigg\{ \frac{1}{p(1-p)(t-\omega_0)}\Bigg| \int_{\omega_0}^{pt+\theta} \big[(pt+\theta)-(qs+\omega)\big]\, d_{q,\omega}s \\
&\quad - \int_{\omega_0}^{p^2t+(p+1)\theta} \big[p^2t+(p+1)\theta-(qs+\omega)\big]\, d_{q,\omega}s\Bigg| \\
&\quad + \frac{1}{T-\omega_0}\int_{\omega_0}^{T}\big[T-(qs+\omega)\big]\, d_{q,\omega}s\Bigg\} \\
&\quad \times \big(\|h_1\| + \|h_2\| + \|h_3\|\varphi_0(T-\omega_0)\big)\|x-y\|_X \\
&\leq \Bigg\{ \frac{(pt+\theta-\omega_0)^2-(p^2t+(p+1)\theta-\omega_0)^2}{p(1-p)(t-\omega_0)(1+q)} + \frac{T-\omega_0}{1+q}\Bigg\} \\
&\quad \times \big(\|h_1\| + \|h_2\| + \|h_3\|\varphi_0(T-\omega_0)\big)\|x-y\|_X \\
&\leq \frac{(T-\omega_0)(1+p+p^2)}{1+q}\big(\|h_1\| + \|h_2\| + \|h_3\|\varphi_0(T-\omega_0)\big)\|x-y\|_X \\
&= \Phi_0\Phi_2\|x-y\|_X.
\end{aligned}$$

Therefore

$$\|\mathcal{F}x - \mathcal{F}y\|_X \leq \Phi_0(\Phi_1+\Phi_2)\|x-y\|_X.$$

This implies, by (H_3), that $\mathcal{F}$ is a contraction. Therefore, by Banach's fixed point theorem, $\mathcal{F}$ has a unique fixed point, which is the unique solution of problem (1.2) on $[\omega_0, T]_{q,\omega}$. The proof is completed. □

Corollary 3.1 *If* $h_i = L$, $i = 1,2,3$ *with* $L < \frac{1}{[2+\varphi_0(T-\omega_0)](\Phi_1+\Phi_2)}$ *in Theorem* 3.1, *then problem* (1.2) *has a unique solution on* $[\omega_0, T]_{q,\omega}$.

Example 3.1 Consider the following boundary value problem for second-order Hahn integro-difference equation:

$$\begin{cases} D^2_{1/4,3/2}x(t) = f(t, x(t), D_{p,\theta}x(pt+\theta), \Psi_{p,\theta}x(pt+\theta)), \\ x(2) = x(4), \qquad x(1{,}025/512) = (4/3)\int_2^4 e^{\cos(2\pi s)}x(s)\, d_{1/4,3/2}s, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} &f\bigl(t, x(t), D_{p,\theta}x(pt+\theta), \Psi_{p,\theta}x(pt+\theta)\bigr) \\ &\quad= \frac{1}{(30+t^3)(1+|x(t)|)}\bigl[e^{-\sin^2(2\pi t)}\bigl(x^2+2|x|\bigr) + e^{-\cos^2(2\pi t)}|D_{1/256,255/128}x| \\ &\qquad + e^{-t^2}|\Psi_{1/256,255/128}x|\bigr] \end{aligned} \tag{3.5}$$

and

$$\Psi_{1/256,255/128}x(t) = \int_2^t \frac{e^{-((1/256)s+255/128-t)}}{15} x\bigl((1/256)s+255/128\bigr)\, d_{1/256,255/128}s.$$

Here $q = 1/4$, $\omega = 3/2$, $\omega_0 = 2$, $p = 1/256$, $m = 4$, $\theta = 255/128$, $T = 4$, $\eta = 1{,}025/512$, $\mu = 4/3$, $g(t) = e^{\cos(2\pi t)}$, $\varphi(t, ps+\theta) = \frac{e^{-((1/256)s+255/128-t)}}{15}$. By direct computation, we find that

$$\begin{aligned} &\bigl|f(t, x, D_{p,\theta}x, \Psi_{p,\theta}x) - f(t, y, D_{p,\theta}y, \Psi_{p,\theta}y)\bigr| \\ &\quad\le \frac{2e^{-\sin^2(2\pi t)}}{30+t^3}\|x-y\| + \frac{e^{-\cos^2(2\pi t)}}{30+t^3}\|D_{p,\theta}x - D_{p,\theta}y\| + \frac{e^{-t^2}}{30+t^3}\|\Psi_{p,\theta}x - \Psi_{p,\theta}y\|, \end{aligned}$$

$\varphi_0 = 1/15$, and (H_1) is satisfied with $h_1(t) = \frac{2e^{-\sin^2(2\pi t)}}{30+t^3}$, $h_2(t) = \frac{e^{-\cos^2(2\pi t)}}{30+t^3}$, $h_3(t) = \frac{e^{-t^2}}{30+t^3}$. So, $\Phi_0 = 0.06238$. From $0 < g(t) < e$, (H_2) is satisfied with $N = e$. By the given data we find $\Phi_1 = 13.66619$, $\Phi_2 = 1.60627$. Therefore, we can compute that

$$\mathfrak{S} = \Phi_0(\Phi_1 + \Phi_2) = 0.9527.$$

Hence, by Theorem 3.1, problem (3.4) with (3.5) has a unique solution on $[2,4]_{1/4,3/2}$.

3.2 Existence result via Leray-Schauder's nonlinear alternative

Lemma 3.1 (Nonlinear alternative for single valued maps [26]) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 3.2 *Let $f : [\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. In addition, we assume that*:

(H_4) *there exist a function* $m \in C([\omega_0, T]_{q,\omega}, \mathbb{R})$, *and continuous nondecreasing functions* $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2, 3$ *such that*

$$\left|f(t,x,y,z)\right| \leq m(t)\left(\psi_1\left(|x|\right) + \psi_2\left(|y|\right) + \psi_3\left(|z|\right)\right),$$

$\forall (t,x,y,z) \in [\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$;

(H_5) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{(\psi_1(M) + \psi_2(M) + \psi_3(\varphi_0(T-\omega_0)M))\|m\|(\Phi_1 + \Phi_2)} > 1,$$

where φ_0 *is defined by* (3.2) *and* Φ_1, Φ_2 *are defined by* (3.3).

Then the boundary value problem (1.2) *has at least one solution on* $[\omega_0, T]_{q,\omega}$.

Proof Consider the operator $\mathcal{F} : X \to X$ defined by (3.1).

Firstly, we show that the operator $\mathcal{F}$ maps bounded sets into bounded sets in the space X. Let $B_r = \{x \in X : \|x\|_X \leq r\}$, $r > 0$. For any $x \in B_r$, putting

$$\hat{f}_x(t) := f\left(t, x(t), D_{p,\theta}x(pt+\theta), \Psi_{p,\theta}x(pt+\theta)\right),$$

and using the inequalities

$$\begin{aligned}
\left|\hat{f}_x(t)\right| &\leq m(t)\left(\psi_1\left(\left|x(t)\right|\right) + \psi_2\left(\left|D_{p,\theta}x(pt+\theta)\right|\right) + \psi_3\left(\left|\Psi_{p,\theta}x(pt+\theta)\right|\right)\right) \\
&\leq m(t)\left(\psi_1\left(\|x\|\right) + \psi_2\left(\|D_{p,\theta}x\|\right) + \psi_3\left(\varphi_0(T-\omega_0)\|x\|\right)\right) \\
&\leq \|m\|\left(\psi_1\left(\|x\|_X\right) + \psi_2\left(\|x\|_X\right) + \psi_3\left(\varphi_0(T-\omega_0)\|x\|_X\right)\right) \\
&\leq \|m\|\left(\psi_1(r) + \psi_2(r) + \psi_3\left(\varphi_0(T-\omega_0)r\right)\right),
\end{aligned}$$

we have

$$\begin{aligned}
&\left|(\mathcal{F}x)(t)\right| \\
&\quad\leq \int_{\omega_0}^{t} \left[t-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s + \frac{(t-\omega_0)}{T-\omega_0}\int_{\omega_0}^{T}\left[T-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s \\
&\qquad+ \frac{1}{|\Omega|}\Bigg[\bigg|\mu\int_{\omega_0}^{T}\int_{\omega_0}^{r} g(r)\left[r-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s\, d_{q,\omega}r \\
&\qquad- \int_{\omega_0}^{\eta}\left[\eta-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s\bigg| \\
&\qquad+ \frac{1}{T-\omega_0}\int_{\omega_0}^{T}\left[T-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s\left|\eta-\omega_0-\mu\int_{\omega_0}^{T} g(r)(r-\omega_0)\, d_{q,\omega}r\right|\Bigg] \\
&\quad\leq \int_{\omega_0}^{T}\left[T-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s + \int_{\omega_0}^{T}\left[T-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s \\
&\qquad+ \frac{1}{|\Omega|}\Bigg[|\mu|N\int_{\omega_0}^{T}\int_{\omega_0}^{r}\left[r-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s\, d_{q,\omega}r + \int_{\omega_0}^{\eta}\left[\eta-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s \\
&\qquad+ \frac{1}{T-\omega_0}\int_{\omega_0}^{T}\left[T-(qs+\omega)\right]\left|\hat{f}_x(t)\right| d_{q,\omega}s\left((\eta-\omega_0) + |\mu|N\int_{\omega_0}^{T}(r-\omega_0)\, d_{q,\omega}r\right)\Bigg]
\end{aligned}$$

$$\leq \left\{ \frac{2(T-\omega_0)^2}{1+q} + \frac{1}{|\Omega|}\left[\frac{|\mu|N(T-\omega_0)^3}{(1+q)(1+q+q^2)} + \frac{(\eta-\omega_0)^2}{1+q} + \frac{T-\omega_0}{1+q} \right.\right.$$
$$\left.\left. \times \left((\eta-\omega_0) + \frac{|\mu|N(T-\omega_0)^2}{1+q} \right)\right]\right\} \|m\|\big(\psi_1(r)+\psi_2(r)+\psi_3\big(\varphi_0(T-\omega_0)r\big)\big)$$
$$\leq \left\{ \frac{2(T-\omega_0)^2}{1+q} + \frac{1}{(1+q)|\Omega|}\left[\frac{|\mu|N(T-\omega_0)^3(2+2q+q^2)}{(1+q)(1+q+q^2)} \right.\right.$$
$$\left.\left. + (\eta-\omega_0)(T+\eta-2\omega_0) \right]\right\} \|m\|\big(\psi_1(r)+\psi_2(r)+\psi_3\big(\varphi_0(T-\omega_0)r\big)\big).$$

Taking the p,θ-derivative for (3.1) where $p = q^m$, $m \in \mathbb{N}$ and $\theta = \omega(\frac{1-p}{1-q})$, we obtain

$$\big|(D_{p,\theta}\mathcal{F}x)(pt+\theta)\big|$$
$$\leq \left| \frac{1}{[-(pt+\theta)(1-p)+\theta]}\left\{ \left(\int_{\omega_0}^{p(pt+\theta)+\theta} \big[p(pt+\theta)+\theta-(qs+\omega)\big]\big|\hat{f}_x(t)\big|\,d_{q,\omega}s \right.\right.\right.$$
$$\left. - \int_{\omega_0}^{pt+\theta} \big[(pt+\theta)-(qs+\omega)\big]\big|\hat{f}_x(t)\big|\,d_{q,\omega}s \right)$$
$$\left.\left. + \frac{(pt+\theta)(1-p)+\theta}{T-\omega_0} \int_{\omega_0}^{T} \big[T-(qs+\omega)\big]\big|\hat{f}_x(t)\big|\,d_{q,\omega}s \right\}\right|$$
$$\leq \left\{ \frac{(pt+\theta-\omega_0)^2-(p^2t+(p+1)\theta-\omega_0)^2}{p(1-p)(t-\omega_0)(1+q)} + \frac{T-\omega_0}{1+q} \right\}$$
$$\times \|m\|\big(\psi_1(r)+\psi_2(r)+\psi_3\big(\varphi_0(T-\omega_0)r\big)\big)$$
$$\leq \left\{ \frac{(1+p+p^2)(T-\omega_0)}{1+q} \right\} \|m\|\big(\psi_1(r)+\psi_2(r)+\psi_3\big(\varphi_0(T-\omega_0)r\big)\big).$$

Consequently

$$\|Fx\|_X \leq \|m\|\big(\psi_1(r)+\psi_2(r)+\psi_3\big(\varphi_0(T-\omega_0)r\big)\big)(\Phi_1+\Phi_2).$$

Next, we shall show that $\mathcal{F}: B_r \to B_r$ is equicontinuous. For any $t_1, t_2 \in [\omega_0, T]_{q,\omega}$, $t_1 < t_2$. Then we have

$$\big|(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)\big|$$
$$\leq \int_{\omega_0}^{t_2} \big[t_2-(qs+\omega)\big]\big|\hat{f}_x(s)\big|\,d_{q,\omega}s - \int_{\omega_0}^{t_1} \big[t_1-(qs+\omega)\big]\big|\hat{f}_x(s)\big|\,d_{q,\omega}s$$
$$+ \frac{|t_2-t_1|}{T-\omega_0} \int_{\omega_0}^{T} \big[T-(qs+\omega)\big]\big|\hat{f}_x(s)\big|\,d_{q,\omega}s$$
$$\leq \int_{\omega_0}^{t_1} [t_2-t_1]\big|\hat{f}_x(s)\big|\,d_{q,\omega}s + \int_{t_1}^{t_2} \big[t_2-(qs+\omega)\big]\big|\hat{f}_x(s)\big|\,d_{q,\omega}s$$
$$+ \frac{|t_2-t_1|}{T-\omega_0} \int_{\omega_0}^{T} \big[T-(qs+\omega)\big]\big|\hat{f}_x(s)\big|\,d_{q,\omega}s$$
$$\leq \left(\int_{\omega_0}^{t_1} [t_2-t_1]\,d_{q,\omega}s + \int_{t_1}^{t_2} \big[t_2-(qs+\omega)\big]\,d_{q,\omega}s + \frac{|t_2-t_1|(T-\omega_0)}{1+q} \right)$$
$$\times \|m\|\big(\psi_1(r)+\psi_2(r)+\psi_3\big(\varphi_0(T-\omega_0)r\big)\big)$$

and

$$\begin{aligned}
&\bigl|(D_{p,\theta}\mathcal{F}x)(pt_2+\theta)-(D_{p,\theta}\mathcal{F}x)(pt_1+\theta)\bigr|\\
&\quad\le\Bigg|\frac{1}{[-(pt_2+\theta)(1-p)+\theta]}\Bigg\{\Bigg(\int_{\omega_0}^{p(pt_2+\theta)+\theta}\bigl[p(pt_2+\theta)+\theta-(qs+\omega)\bigr]\,d_{q,\omega}s\\
&\qquad-\int_{\omega_0}^{pt_2+\theta}\bigl[(pt_2+\theta)-(qs+\omega)\bigr]\,d_{q,\omega}s\Bigg)\\
&\qquad+\frac{(pt_2+\theta)(1-p)+\theta}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\,d_{q,\omega}s\Bigg\}\\
&\qquad-\frac{1}{[-(pt_1+\theta)(1-p)+\theta]}\Bigg\{\Bigg(\int_{\omega_0}^{p(pt_1+\theta)+\theta}\bigl[p(pt_1+\theta)+\theta-(qs+\omega)\bigr]\,d_{q,\omega}s\\
&\qquad-\int_{\omega_0}^{pt_1+\theta}\bigl[(pt_1+\theta)-(qs+\omega)\bigr]\,d_{q,\omega}s\Bigg)\\
&\qquad+\frac{(pt_1+\theta)(1-p)+\theta}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\,d_{q,\omega}s\Bigg\}\Bigg|\\
&\qquad\times\|m\|\bigl(\psi_1(r)+\psi_2(r)+\psi_3\bigl(\varphi_0(T-\omega_0)r\bigr)\bigr)\\
&\quad\le\Bigg\{\frac{(pt_2+\theta-\omega_0)^2-(p^2t_2+(p+1)\theta-\omega_0)^2}{p(1-p)(t_2-\omega_0)(1+q)}\\
&\qquad-\frac{(pt_1+\theta-\omega_0)^2-(p^2t_1+(p+1)\theta-\omega_0)^2}{p(1-p)(t_1-\omega_0)(1+q)}\Bigg\}\\
&\qquad\times\|m\|\bigl(\psi_1(r)+\psi_2(r)+\psi_3\bigl(\varphi_0(T-\omega_0)r\bigr)\bigr)\\
&\quad\le\Bigg\{\frac{p(1+p)(t_2-t_1)}{1+q}\Bigg\}\|m\|\bigl(\psi_1(r)+\psi_2(r)+\psi_3\bigl(\varphi_0(T-\omega_0)r\bigr)\bigr).
\end{aligned}$$

Hence

$$\max_{x\in\bar{B}_r}\bigl|(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)\bigr|+\max_{x\in\bar{B}_r}\bigl|D_{p,\theta}(\mathcal{F}x)(pt_2+\theta)-D_{p,\theta}(\mathcal{F}x)(pt_1+\theta)\bigr|\to 0,$$

as $t_2\to t_1$ and the limit is independent of $x\in\bar{B}_r$. Therefore the operator $\mathcal{F}:B_r\to B_r$ is equicontinuous and uniformly bounded. The Arzelá-Ascoli theorem implies that $\mathcal{F}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 3.1) once we have proved the boundedness of the set of all solutions to equations $x=\lambda\mathcal{F}x$ for $\lambda\in(0,1)$.

Let x be a solution. Then, for $t\in[\omega_0,T]_{q,\omega}$, and using the computations in proving that $\mathcal{F}$ is bounded, for $\lambda\in(0,1)$, let $x=\lambda\mathcal{F}x$. Then we have

$$\|x\|_X\le\bigl(\psi_1\bigl(\|x\|_X\bigr)+\psi_2\bigl(\|x\|_X\bigr)+\psi_3\bigl(\varphi_0(T-\omega_0)\|x\|_X\bigr)\bigr)\|m\|(\Phi_1+\Phi_2),$$

or

$$\frac{\|x\|_X}{(\psi_1(\|x\|_X)+\psi_2(\|x\|_X)+\psi_3(\varphi_0(T-\omega_0)\|x\|_X))\|m\|(\Phi_1+\Phi_2)}\le 1.$$

In view of (H_5), there exists M such that $\|x\|_X \neq M$. Let us set

$$U = \{x \in C([\omega_0, T]_{q,\omega}, \mathbb{R}) : \|x\|_X < M\}.$$

Note that the operator $\mathcal{F} : \overline{U} \to C([\omega_0, T]_{q,\omega}, \mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda \mathcal{F} x$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.1), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{U}$ which is a solution of problem (1.2). This completes the proof. □

Corollary 3.2 *Suppose that a continuous function f satisfies $|f(t,x,y,z)| \leq K(|x| + |y| + |z|)$, $K \geq 0$. If $K[2 + \varphi_0(T - \omega_0)](\Phi_1 + \Phi_2) < 1$, then problem* (1.2) *has at least one solution on $[\omega_0, T]_{q,\omega}$.*

Example 3.2 Consider the following boundary value problem for second-order Hahn integro-difference equation:

$$\begin{cases} D^2_{1/3,4/3} x(t) = f(t, x(t), D_{p,\theta} x(pt + \theta), \Psi_{p,\theta} x(pt + \theta)), \\ x(2) = x(4), \qquad x(1{,}460/729) = (5/3) \int_2^4 e^{\cos(2\pi s)} x(s)\, d_{1/3,4/3} s, \end{cases} \tag{3.6}$$

where

$$\begin{aligned} &f(t, x(t), D_{p,\theta} x(pt + \theta), \Psi_{p,\theta} x(pt + \theta)) \\ &\quad = \frac{1}{(t+1)^2} \left[\frac{1}{10} e^{-|x|} \sin|x| + \frac{|D_{1/81,160/81} x|^2}{12(1 + |D_{1/81,160/81} x|)} + \frac{|\Psi_{1/81,160/81} x|^5}{15(1 + |\Psi_{1/81,160/81} x|^4)} + 3 \right] \end{aligned} \tag{3.7}$$

and

$$\Psi_{1/81,160/81} x(t) = \int_2^t \frac{e^{-((1/81)s + 160/81 - t)}}{150} x((1/81)s + 160/81)\, d_{1/81,160/81} s.$$

Here $q = 1/3$, $\omega = 4/3$, $\omega_0 = 2$, $p = 1/81$, $m = 4$, $\theta = 160/81$, $T = 4$, $\eta = 1{,}460/729$, $\mu = 5/3$, $g(t) = e^{\cos(2\pi t)}$, $\varphi(t, ps + \theta) = \frac{e^{-((1/81)s + 160/81 - t)}}{150}$. Since

$$\begin{aligned} &|f(t, x, D_{p,\theta} x, \Psi_{p,\theta} x)| \\ &\quad \leq \frac{1}{(t+1)^2} \left[\frac{1}{10}|x| + \frac{1}{12}|D_{1/81,160/81} x| + \frac{1}{15}|\Psi_{1/81,160/81} x| + 3 \right], \end{aligned}$$

$\varphi_0 = 1/150$, (H_4) is satisfied with $m(t) = \frac{1}{(t+1)^2}$, $\psi_1(|x|) = \frac{1}{10}|x|$, $\psi_2(|D_{1/81,160/81} x|) = \frac{1}{12}|D_{1/81,160/81} x| + 1$, $\psi_3(|\Psi_{1/81,160/81} x|) = \frac{1}{15}|\Psi_{1/81,160/81} x| + 2$. In addition, we see that $0 < g(t) < e$, then (H_2) is satisfied with $N = e$. From the above information, we find that $\Phi_1 = 13.86461$, $\Phi_2 = 1.51875$. Therefore, there exists a constant $M > 7.48455$ satisfying (H_5). Hence, by Theorem 3.2, problem (3.6) with (3.7) has at least one solution on $[2, 4]_{1/3,4/3}$.

3.3 Existence result via Krasnoselskii's fixed point theorem

The final existence result is based on Krasnoselskii's fixed point theorem.

Lemma 3.2 (Krasnoselskii's fixed point theorem [27]) *Let S be a closed, convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax + By \in S$ *whenever* $x, y \in S$; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in S$ such that $z = Az + Bz$.*

Theorem 3.3 *Assume that* (H_1) *and* (H_2) *hold. In addition we assume that*:

(H_6) $|f(t,x,y,z)| \leq \mu(t)$, *for each* $t \in [\omega_0, T]_{q,\omega}$, $x, y, z \in \mathbb{R}$ *and* $\mu \in C([\omega_0, T], \mathbb{R}^+)$.

Then problem (1.2) *has at least one solution on* $[\omega_0, T]_{q,\omega}$, *provided*

$$\frac{T-\omega_0}{1+q}(1+T-\omega_0)\Phi_0 < 1, \tag{3.8}$$

where Φ_0 *is defined by* (3.3).

Proof Consider the operator $\mathcal{F} : X \to X$ defined by (3.1) as

$$(\mathcal{F}x)(t) = (\mathcal{F}_1 x)(t) + (\mathcal{F}_2 x)(t), \quad t \in [\omega_0, T]_{q,\omega}, \tag{3.9}$$

where

$$(\mathcal{F}_1 x)(t) = -\frac{(t-\omega_0)}{T-\omega_0}\int_{\omega_0}^{T}\big[T-(qs+\omega)\big]\hat{f}_x(s)\,d_{q,\omega}s$$

and

$$\begin{aligned}
&(\mathcal{F}_2 x)(t)\\
&\quad = \int_{\omega_0}^{t}\big[t-(qs+\omega)\big]\hat{f}_x(s)\,d_{q,\omega}s\\
&\qquad + \frac{1}{\Omega}\bigg[\mu\int_{\omega_0}^{T}\int_{\omega_0}^{r} g(r)\big[r-(qs+\omega)\big]\hat{f}_x(s)\,d_{q,\omega}s\,d_{q,\omega}r - \int_{\omega_0}^{\eta}\big[\eta-(qs+\omega)\big]\hat{f}_x(s)\,d_{q,\omega}s\\
&\qquad + \frac{1}{T-\omega_0}\int_{\omega_0}^{T}\big[T-(qs+\omega)\big]\hat{f}_x(s)\,d_{q,\omega}s\bigg(\eta-\omega_0-\mu\int_{\omega_0}^{T} g(r)(r-\omega_0)\,d_{q,\omega}r\bigg)\bigg].
\end{aligned}$$

Setting $\max\{\mu(t) : t \in [\omega_0, T]_{q,\omega}\} = \|\mu\|$ and choosing $\rho \geq \|\mu\|(\Phi_1 + \Phi_2)$ we consider $B_\rho = \{x \in C([\omega_0, T]_{q,\omega}, \mathbb{R}) : \|x\|_X \leq \rho\}$. For any $x, y \in B_\rho$ we have

$$\begin{aligned}
&\big|(\mathcal{F}_1 x)(t) + (\mathcal{F}_2 y)(t)\big|\\
&\quad \leq \int_{\omega_0}^{t}\big[t-(qs+\omega)\big]\big|\hat{f}_y(t)\big|\,d_{q,\omega}s + \frac{(t-\omega_0)}{T-\omega_0}\int_{\omega_0}^{T}\big[T-(qs+\omega)\big]\big|\hat{f}_x(t)\big|\,d_{q,\omega}s\\
&\qquad + \frac{1}{|\Omega|}\bigg[\bigg|\mu\int_{\omega_0}^{T}\int_{\omega_0}^{r} g(r)\big[r-(qs+\omega)\big]\big|\hat{f}_y(t)\big|\,d_{q,\omega}s\,d_{q,\omega}r\\
&\qquad - \int_{\omega_0}^{\eta}\big[\eta-(qs+\omega)\big]\big|\hat{f}_y(t)\big|\,d_{q,\omega}s\bigg|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_y(t)\bigr|\,d_{q,\omega}s\left|\eta-\omega_0-\mu\int_{\omega_0}^{T}g(r)(r-\omega_0)\,d_{q,\omega}r\right|\Bigg]\\
&\leq\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_y(t)\bigr|\,d_{q,\omega}s+\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)\bigr|\,d_{q,\omega}s\\
&+\frac{1}{|\Omega|}\Bigg[|\mu|N\int_{\omega_0}^{T}\int_{\omega_0}^{r}\bigl[r-(qs+\omega)\bigr]\bigl|\hat{f}_y(t)\bigr|\,d_{q,\omega}s\,d_{q,\omega}r+\int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]\bigl|\hat{f}_y(t)\bigr|\,d_{q,\omega}s\\
&+\frac{1}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_y(t)\bigr|\,d_{q,\omega}s\left((\eta-\omega_0)+|\mu|N\int_{\omega_0}^{T}(r-\omega_0)\,d_{q,\omega}r\right)\Bigg]\\
&\leq\|\mu\|\Bigg\{\frac{2(T-\omega_0)^2}{1+q}+\frac{1}{|\Omega|}\Bigg[\frac{|\mu|N(T-\omega_0)^3}{(1+q)(1+q+q^2)}+\frac{(\eta-\omega_0)^2}{1+q}\\
&+\frac{T-\omega_0}{1+q}\left((\eta-\omega_0)+\frac{|\mu|N(T-\omega_0)^2}{1+q}\right)\Bigg]\Bigg\}.
\end{aligned}
$$

Taking the p,θ-derivative for (3.9) where $p=q^m$, $m\in\mathbb{N}$ and $\theta=\omega(\frac{1-p}{1-q})$, we obtain

$$
\begin{aligned}
&\bigl|(D_{p,\theta}\mathcal{F}_1x)(pt+\theta)+(D_{p,\theta}\mathcal{F}_2y)(pt+\theta)\bigr|\\
&\leq\left|\frac{1}{[-(pt+\theta)(1-p)+\theta]}\Bigg\{\Bigg(\int_{\omega_0}^{p(pt+\theta)+\theta}\bigl[p(pt+\theta)+\theta-(qs+\omega)\bigr]\bigl|\hat{f}_y(t)\bigr|\,d_{q,\omega}s\right.\\
&-\int_{\omega_0}^{pt+\theta}\bigl[(pt+\theta)-(qs+\omega)\bigr]\bigl|\hat{f}_y(t)\bigr|\,d_{q,\omega}s\Bigg)\\
&\left.+\frac{(pt+\theta)(1-p)+\theta}{T-\omega_0}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\bigl|\hat{f}_x(t)\bigr|\,d_{q,\omega}s\Bigg\}\right|\\
&\leq\|\mu\|\left\{\frac{(1+p+p^2)(T-\omega_0)}{1+q}\right\}.
\end{aligned}
$$

Consequently

$$
\|\mathcal{F}_1x+\mathcal{F}_2y\|_X\leq\|\mu\|(\Phi_1+\Phi_2)\leq\rho,
$$

which shows that $\mathcal{F}_1x+\mathcal{F}_2y\in B_\rho$.

It is easy to prove that

$$
\|\mathcal{F}_1x-\mathcal{F}_1y\|\leq\frac{(T-\omega_0)^2}{1+q}\Phi_0\|x-y\|_X,\qquad\|D_{p,\theta}\mathcal{F}_1x-D_{p,\theta}\mathcal{F}_1y\|\leq\frac{T-\omega_0}{1+q}\Phi_0\|x-y\|_X,
$$

and consequently

$$
\|\mathcal{F}_1x-\mathcal{F}_1y\|_X\leq\frac{T-\omega_0}{1+q}(1+T-\omega_0)\Phi_0\|x-y\|_X,
$$

which implies, by (3.8), that $\mathcal{F}_1$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{F}_2$ is continuous. Also, $\mathcal{F}_2$ is uniformly bounded on B_ρ and equicontinuous, as proved in Theorem 3.2. So $\mathcal{F}_2$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{F}_2$ is compact on B_ρ. Thus all the assumptions of Lemma 3.2 are satisfied. So the conclusion of Lemma 3.2 implies that problem (1.2) has at least one solution on $[\omega_0,T]_{q,\omega}$. □

Corollary 3.3 *Suppose that a continuous function f satisfies (H_1) with $h_i = L$, $i = 1,2,3$, and $|f(t,x,y,z)| \leq K$, $K > 0$. If $\frac{T-\omega_0}{1+q}(1+T-\omega_0)(2+\varphi_0(T-\omega_0))L < 1$ then problem (1.2) has at least one solution on $[\omega_0, T]_{q,\omega}$.*

Example 3.3 Consider the boundary value problem for second-order Hahn integro-difference equation in Example 3.1 with

$$\begin{aligned} &f\bigl(t,x(t),D_{p,\theta}x(pt+\theta),\Psi_{p,\theta}x(pt+\theta)\bigr) \\ &\quad= \frac{1}{(25+t^3)(1+|x(t)|)}\bigl[e^{-\sin^2(2\pi t)}\bigl(x^2+2|x|\bigr)+e^{-\cos^2(2\pi t)}|D_{1/256,255/128}x| \\ &\qquad + e^{-t^2}|\Psi_{1/256,255/128}x|\bigr]. \end{aligned} \tag{3.10}$$

Now, we see that

$$\begin{aligned} &\bigl|f(t,x,D_{p,\theta}x,\Psi_{p,\theta}x)-f(t,y,D_{p,\theta}y,\Psi_{p,\theta}y)\bigr| \\ &\quad\leq \frac{2e^{-\sin^2(2\pi t)}}{25+t^3}\|x-y\| + \frac{e^{-\cos^2(2\pi t)}}{25+t^3}\|D_{p,\theta}x-D_{p,\theta}y\| + \frac{e^{-t^2}}{25+t^3}\|\Psi_{p,\theta}x-\Psi_{p,\theta}y\|. \end{aligned}$$

Then (H_1) is satisfied with $h_1(t) = \frac{2e^{-\sin^2(2\pi t)}}{25+t^3}$, $h_2(t) = \frac{e^{-\cos^2(2\pi t)}}{25+t^3}$, $h_3(t) = \frac{e^{-t^2}}{25+t^3}$. Therefore, we can find that $\Phi_0 = 0.07183$, and

$$\mathfrak{S} = \Phi_0(\Phi_1+\Phi_2) = 1.09702 > 1.$$

Thus, Theorem 3.1 cannot be applied in this case. However, (H_6) is satisfied with $\mu(t) = \frac{1}{25+t^3}$, by $|f(t,x,D_{p,\theta}x,\Psi_{p,\theta}x)| \leq \frac{1}{25+t^3}$. Indeed, we find that

$$\frac{T-\omega_0}{(1+q)}(1+T-\omega_0)\Phi_0 = 0.34478 < 1.$$

Then, by using Theorem 3.3, problem (3.4) with (3.10) has at least one solution on $[9/2,8]_{2/3,3/2}$.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Author details
[1]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand. [2]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece. [3]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematices, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [4]Centre of Excellence in Mathematics, CHE, Sri Ayutthaya Rd., Bangkok, 10400, Thailand. [5]Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, 10300, Thailand.

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