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On the $2k$-step Jordan-Fibonacci sequence

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Abstract

In this paper, we define the $2k$-step Jordan-Fibonacci sequence, and then we study the $2k$-step Jordan-Fibonacci sequence modulo m. Also, we obtain the cyclic groups from the multiplicative orders of the generating matrix of the $2k$-step Jordan-Fibonacci sequence when read modulo m, and we give the relationships among the orders of the cyclic groups obtained and the periods of the $2k$-step Jordan-Fibonacci sequence modulo m. Furthermore, we extend the $2k$-step Jordan-Fibonacci sequence to groups, and then we examine this sequence in the finite groups. Finally, we obtain the period of the $2k$-step Jordan-Fibonacci sequence in the generalized quaternion group Q_{2^n} as applications of the results produced.

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Keywords: Jordan-Fibonacci sequence; group; matrix; period

1 Introduction

Suppose that the $(n+k)$th term of a sequence is defined consecutively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [1], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows.

Let the matrix A be defined by

$$A = [a_{ij}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

The k-step Fibonacci sequence $\{F_n^k\}$ is defined by initial values $F_1^k = \cdots = F_{k-1}^k = 0$, $F_k^k = 1$ and the recurrence relation

$$F_{n+k}^k = \sum_{i=0}^{k-1} F_{n+i}^k$$

for $n \geq 1$.

It is clear that the characteristic polynomial of the k-step Fibonacci sequence is as follows:

$$P_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

A square matrix of the form

$$\begin{bmatrix} \lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda & 1 & \ldots & 0 & 0 \\ 0 & 0 & \lambda & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda \end{bmatrix}$$

is called a Jordan block. The Jordan block of order k is denoted by $J_k(\lambda)$. A Jordan matrix is a block diagonal matrix whose blocks are all Jordan blocks.

Work on Fibonacci sequences in groups began with the previous study of Wall [2] where the ordinary Fibonacci sequences in cyclic groups were investigated. Later, Wilcox [3] extended the problem to Abelian groups. Recently, many authors have studied some special linear recurrence sequences in algebraic structures; see, for example, [4–14]. In [7, 11, 13–15]. The authors obtained the cyclic groups via some special matrices.

In this paper, we obtain the $(k) \times (2k)$ Jordan-type matrix J with the aid of the characteristic polynomial of the k-step Fibonacci sequence, and then we define the $2k$-step Jordan-Fibonacci sequence by using the Jordan-Fibonacci matrix F_J^k of order $2k$ which is produced from the matrix J. Then we study the $2k$-step Jordan-Fibonacci sequence modulo m, and we obtain the cyclic groups from the multiplicative orders of the Jordan-Fibonacci matrix F_J^k of order $2k$ such that the elements of the matrix F_J^k when read modulo m. Also, we derive the relationships among the orders of the cyclic groups obtained and the periods of the $2k$-step Jordan-Fibonacci sequence modulo m. Furthermore, we redefine the $2k$-step Jordan-Fibonacci sequence by means of the elements of the groups which have two or more generators, and then we examine this sequence in the finite groups. Finally, we obtain the period of the $2k$-step Jordan-Fibonacci sequence in the generalized quaternion group Q_{2^n} as applications of the results produced.

2 Main results and proofs

We consider the $(k) \times (2k)$ Jordan-type matrix J which is defined by using the characteristic polynomial of the k-step Fibonacci sequence:

$$\begin{bmatrix} 1 & -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & -1 & -1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & & & & \\ 0 & \dots & 0 & 1 & -1 & -1 & -1 & \dots & -1 \end{bmatrix}.$$

Now we define the Jordan-Fibonacci matrix $F_J^k = [f_{ij}]_{2k\times 2k}$ by using the last row of the matrix J as follows:

$$\begin{array}{c} \begin{array}{ccccccccc} & & & & k\text{th} & & & & \\ & & & & \downarrow & & & & \end{array} \\ \begin{bmatrix} 1 & 0 & \dots & 0 & 1 & -1 & \dots & -1 \\ 1 & 0 & 0 & & \dots & & 0 & 0 \\ 0 & 1 & 0 & & \dots & & 0 & 0 \\ & & & \ddots & & & & \\ 0 & & \dots & & 0 & 1 & 0 & 0 \\ 0 & & \dots & & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

For example,

$$F_J^2 = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Next, we define the $2k$-step Jordan-Fibonacci sequence with the aid of the matrix F_J^k as follows:

$$a_{n+2k}^k = a_{n+2k-1}^k + a_{n+k}^k - a_{n+k-1}^k - \cdots - a_n^k \tag{1}$$

for $n \geq 1$, with initial conditions $a_1^k = \cdots = a_{2k-1}^k = 0$ and $a_{2k}^k = 1$.

By mathematical induction on α, we may write

$$\left(F_J^2\right)^\alpha = \begin{bmatrix} a_{\alpha+4}^2 & a_{\alpha+5}^2 - a_{\alpha+4}^2 & -a_{\alpha+3}^2 - a_{\alpha+2}^2 & -a_{\alpha+3}^2 \\ a_{\alpha+3}^2 & a_{\alpha+4}^2 - a_{\alpha+3}^2 & -a_{\alpha+2}^2 - a_{\alpha+1}^2 & -a_{\alpha+2}^2 \\ a_{\alpha+2}^2 & a_{\alpha+3}^2 - a_{\alpha+2}^2 & -a_{\alpha+1}^2 - a_{\alpha}^2 & -a_{\alpha+1}^2 \\ a_{\alpha+1}^2 & a_{\alpha+2}^2 - a_{\alpha+1}^2 & a_{\alpha}^2 - a_{\alpha-1}^2 & -a_{\alpha}^2 \end{bmatrix} \tag{2}$$

and

$$\left(F_J^k\right)^\alpha = \begin{bmatrix} a^k_{\alpha+2k} & a^k_{\alpha+2k+1} - a^k_{\alpha+2k} & a^k_{\alpha+2k+2} - a^k_{\alpha+2k+1} & \cdots & a^k_{\alpha+3k-1} - a^k_{\alpha+3k-2} & & -a^k_{\alpha+2k-1} \\ a^k_{\alpha+2k-1} & a^k_{\alpha+2k} - a^k_{\alpha+2k-1} & a^k_{\alpha+2k+1} - a^k_{\alpha+2k} & \cdots & a^k_{\alpha+3k-2} - a^k_{\alpha+3k-3} & & -a^k_{\alpha+2k-2} \\ a^k_{\alpha+2k-2} & a^k_{\alpha+2k-1} - a^k_{\alpha+2k-2} & a^k_{\alpha+2k} - a^k_{\alpha+2k-1} & \cdots & a^k_{\alpha+3k-3} - a^k_{\alpha+3k-4} & M & -a^k_{\alpha+2k-3} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a^k_{\alpha+1} & a^k_{\alpha+2} - a^k_{\alpha+1} & a^k_{\alpha+3} - a^k_{\alpha+2} & \cdots & a^k_{\alpha+k} - a^k_{\alpha+k-1} & & -a^k_{\alpha} \end{bmatrix} \tag{3}$$

for $k \geq 3$, where M is a $(2k) \times (k)$ matrix as follows:

$$M = \begin{bmatrix} -a^k_{\alpha+2k-1} - a^k_{\alpha+2k-2} - \cdots - a^k_{\alpha+k} & -a^k_{\alpha+2k-1} - a^k_{\alpha+2k-2} - \cdots - a^k_{\alpha+k+1} & \cdots & -a^k_{\alpha+2k-1} - a^k_{\alpha+2k-2} \\ -a^k_{\alpha+2k-2} - a^k_{\alpha+2k-3} - \cdots - a^k_{\alpha+k-1} & -a^k_{\alpha+2k-2} - a^k_{\alpha+2k-3} - \cdots - a^k_{\alpha+k} & \cdots & -a^k_{\alpha+2k-2} - a^k_{\alpha+2k-3} \\ -a^k_{\alpha+2k-3} - a^k_{\alpha+2k-4} - \cdots - a^k_{\alpha+k-2} & -a^k_{\alpha+2k-3} - a^k_{\alpha+2k-4} - \cdots - a^k_{\alpha+k-1} & \cdots & -a^k_{\alpha+2k-3} - a^k_{\alpha+2k-4} \\ \vdots & \vdots & & \vdots \\ -a^k_{\alpha} - a^k_{\alpha-1} - \cdots - a^k_{\alpha-k+1} & -a^k_{\alpha} - a^k_{\alpha-1} - \cdots - a^k_{\alpha-k+2} & \cdots & -a^k_{\alpha} - a^k_{\alpha-1} \end{bmatrix}.$$

Now we consider the $2k$-step Jordan-Fibonacci sequence $\{a_n^k\}$ modulo m. If we reduce the sequence $\{a_n^k\}$ by a modulus m, then we get the repeating sequence, denoted by

$$\left\{a_n^k(m)\right\} = \left\{a_1^k(m), a_2^k(m), \ldots, a_i^k(m), \ldots\right\},$$

where we denote $a_i^k \pmod m$ by $a_i^k(m)$. It has the same recurrence relation as in (1).

A sequence is said to be *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. In particular, if the first n elements in the sequence form a repeating subsequence, then this sequence is *simply periodic* and its period is n.

Theorem 2.1 $\{a_n^k(m)\}$ *forms a simply periodic sequence.*

Proof Let $X = \{(x_1, x_2, \ldots, x_{2k}) | x_i\text{'s be integers such that } 0 \leq x_i \leq m-1\}$. Since there are m^{2k} distinct $2k$-tuples of elements of Z_m, at least one of the $2k$-tuples appears twice in the sequence $\{a_n^k(m)\}$. Therefore, the subsequence following this $2k$-tuple repeats; hence, the sequence is periodic. Assume that $u > v$ and

$$a^k_{u+1}(m) = a^k_{v+1}(m), a^k_{u+2}(m) = a^k_{v+2}(m), \ldots, a^k_{u+2k}(m) = a^k_{v+2k}(m),$$

then $u \equiv v \pmod{2k}$. By (1), we may write

$$a_n^k = -a^k_{n+2k} + a^k_{n+2k-1} + a^k_{n+k} - a^k_{n+k-1} - \cdots - a^k_{n+1}.$$

Then we obtain

$$a^k_u(m) = a^k_v(m), a^k_{u-1}(m) = a^k_{v-1}(m), \ldots, a^k_{u-v+1}(m) = a^k_1(m).$$

Thus it is verified that the sequence is simply periodic. □

The period of the sequence $\{a_n^k(m)\}$ is denoted by $P^k(m)$.

Given an integer matrix $A = [a_{ij}]$, $A \pmod m$ means that all entries of A are modulo m, that is,

$$A \pmod m = \bigl(a_{ij} \pmod m\bigr).$$

Let us consider the set

$$\langle A\rangle_m = \bigl\{A^i \pmod m \mid i \geq 0\bigr\}.$$

If $\gcd(m, \det A) = 1$, then $\langle A\rangle_m$ is a cyclic group; if $\gcd(m, \det A) \neq 1$, then $\langle A\rangle_m$ is a semigroup. Let the notation $|\langle A\rangle_m|$ denote the order of the set $\langle A\rangle_m$. Since $\det F_J^k = 1$, $|\langle F_J^k\rangle_m|$ is a cyclic group for every positive integer m. From (2) and (3), it is easy to see that $P^k(m) = |\langle F_J^k\rangle_m|$.

Now we give some properties of the period $P^k(m)$ by the following theorems.

Theorem 2.2 *Let t be a prime and suppose that u is the largest positive integer with $P^k(t) = P^k(t^u)$. Then*

$$P^k\bigl(t^v\bigr) = t^{v-u} \cdot P^k(t)$$

for every $v \geq u$. In particular if $P^k(t^2) \neq P^k(t)$, then $P^k(t^v) = t^{v-1} \cdot P^k(t)$.

Proof Since $P^k(m) = |\langle F_J^k\rangle_m|$, we have a positive integer n such that

$$\bigl(F_J^k\bigr)^{P^k(t^{n+1})} \equiv I \bigl(\bmod t^{n+1}\bigr),$$

where I is the $(2k) \times (2k)$ identity matrix. Then it is clear that

$$\bigl(F_J^k\bigr)^{P^k(t^{n+1})} \equiv I \bigl(\bmod t^{n}\bigr),$$

which implies that $P^k(t^{n+1})$ is divisible by $P^k(t^n)$. Furthermore, if we denote

$$\bigl(F_J^k\bigr)^{P^k(t^{n})} = I + \bigl(f_{ij}^{(t)} \cdot t^n\bigr),$$

then by the binomial expansion, we may write

$$\begin{aligned}\bigl(F_J^k\bigr)^{P^k(t^{n})\cdot t} &= \bigl(I + \bigl(f_{ij}^{(t)} \cdot t^n\bigr)\bigr)^t \\ &= \sum_{i=0}^{t} \binom{t}{i} \bigl(f_{ij}^{(t)} \cdot t^n\bigr)^i \equiv I \bigl(\bmod t^{n+1}\bigr).\end{aligned}$$

This yields that $P^k(t^{n+1})$ divides $P^k(t^n) \cdot t$. Then it is clear that

$$P^k\bigl(t^{n+1}\bigr) = P^k\bigl(t^n\bigr)$$

or

$$P^k\bigl(t^{n+1}\bigr) = P^k\bigl(t^{n+1}\bigr) \cdot t.$$

It is easy to see that the latter holds if and only if there is $f_{ij}^{(t)}$ which is not divisible by t. Since u is the largest positive integer such that

$$P^k(t) = P^k(t^u),$$
$$P^k(t^u) \neq P^k(t^{u+1}),$$

then there is $f_{ij}^{(u+1)}$ which is not divisible by t. Therefore, we obtain

$$P^k(t^{u+1}) \neq P^k(t^{u+2}),$$

and so

$$P^k(t^{n+2}) = P^k(t^{n+1}) \cdot t = P^k(t^n) \cdot t^2.$$

To complete the proof we may use an inductive method. □

Theorem 2.3 *If m has the prime factorization $m = \prod_{i=1}^{u} t_i^{r_i}$, where $(u \geq 1)$, then $P^k(m)$ equals the least common multiple of $P^k(t_i^{r_i})'s$.*

Proof Since $P^k(t_i^{r_i})$ is a period of the sequence $\{a_n^k(t_i^{r_i})\}$, it is easy to see that the sequence $\{a_n^k(t_i^{r_i})\}$ repeats only after blocks of length $n \cdot P^k(t_i^{r_i})$, $(n \in \mathbb{N})$. Since also $P^k(m)$ is a period of the sequence $\{a_n^k(m)\}$, the sequence $\{a_n^k(t_i^{r_i})\}$ repeats after terms $P^k(m)$ for all values i. Then we get that $P^k(m)$ is the form $n \cdot P^k(p_i^{r_i})$ for all values of i, and since any such number gives a period of $P^k(m)$, we conclude that

$$P^k(m) = \mathrm{lcm}\big[P^k(t_1^{r_1}), \ldots, P^k(t_u^{r_u})\big].$$ □

Now we extend the $2k$-step Jordan-Fibonacci sequence to k-generator groups such that $k \geq 2$. Let G be a finite k-generator group and suppose that

$$X = \Big\{(x_1, x_2, \ldots, x_k) \in \underbrace{G \times G \times \cdots \times G}_{k} \Big| \big\langle \{x_1, x_2, \ldots, x_k\} \big\rangle = G\Big\}.$$

We call $(x_1, x_2, \ldots, x_k)$ a generating k-tuple for G.

Definition 2.1 For a k-tuple $(x_1, x_2, \ldots, x_k) \in X$, we define the Jordan-Fibonacci orbit

$$\mathrm{JO}(G; x_1, x_2, \ldots, x_k) = \{b_i^k\}$$

by

$$b_1^k = x_1, \ldots, b_k^k = x_k, b_{k+1}^k = (x_1)^{-1}(x_2)^{-1} \cdots (x_k)^{-1},$$
$$b_{k+2}^k = \cdots = b_{2k}^k = (x_1)^{-1}(x_2)^{-1} \cdots (x_k)^{-1}(x_{k+1})$$

and

$$b_{n+2k}^k = \big(b_n^k\big)^{-1} \cdots \big(b_{n+k-1}^k\big)^{-1}\big(b_{n+k}^k\big)\big(b_{n+2k-1}^k\big)$$

for $n \geq 1$.

Theorem 2.4 *If the group G is finite, then the Jordan-Fibonacci orbit* $\mathrm{JO}(G;x_1,x_2,\ldots,x_k)$ *is simply periodic.*

Proof Let β be order of the group G, then it is clear that there are β^{2k} distinct $2k$-tuples of elements of G. Thus it is verified that at least one of the $2k$-tuples appears twice in the Jordan-Fibonacci orbit $\mathrm{JO}(G;x_1,x_2,\ldots,x_k)$. Because of the repeating, the sequence is periodic. Since the Jordan-Fibonacci orbit $\mathrm{JO}(G;x_1,x_2,\ldots,x_k)$ is periodic, there exist natural numbers m and l with $l \equiv m \pmod{2k}$ such that

$$b^k_{m+1} = b^k_{l+1}, b^k_{m+2} = b^k_{l+2}, \ldots, b^k_{m+2k} = b^k_{l+2k}.$$

From the definition of the Jordan-Fibonacci orbit $\mathrm{JO}(G;x_1,x_2,\ldots,x_k)$, we may write

$$b^k_n = \left(b^k_{n+1}\right)^{-1}\cdots\left(b^k_{n+k-1}\right)^{-1}\left(b^k_{n+k}\right)\left(b^k_{n+2k-1}\right)\left(b^k_{n+2k}\right)^{-1}.$$

Thus it is verified that

$$b^k_m = b^k_l, b^k_{m-1} = b^k_{l-1}, \ldots, b^k_1 = b^k_{l-m+1}.$$

So the proof is complete. □

The period of the Jordan-Fibonacci orbit $\mathrm{JO}(G;x_1,x_2,\ldots,x_k)$ is denoted by $\mathrm{PJO}(G;x_1,x_2,\ldots,x_k)$.

It is well known that the generalized quaternion group Q_{2^n} is defined by presentation

$$\left\langle x,y \mid x^{2^{n-1}} = e, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1}\right\rangle,$$

for $n \geq 3$. We will now address the Jordan-Fibonacci orbit of the generalized quaternion group Q_{2^n} for generating pair (x,y).

Theorem 2.5 *The period of the Jordan-Fibonacci orbit* $\mathrm{JO}(Q_{2^n};x,y)$ *is* $P^2(2)\cdot 2^{n-2} = 5\cdot 2^{n-2}$.

Proof We prove this by direct calculation. We first note that $|x| = 2^{n-1}$, $|y| = 4$ and $xy = yx^{-1}$. Since

$$\left\{a^2_n(2)\right\} = \{0,0,0,1,1,0,0,0,1,\ldots\},$$

it is clear that $P^2(2) = 5$. From Definition 2.1, it is easy to see that the Jordan-Fibonacci orbit $\mathrm{JO}(Q_{2^n};x,y)$ is as follows:

$$b^2_{n+4} = \left(b^2_n\right)^{-1}\left(b^2_{n+1}\right)^{-1}\left(b^2_{n+2}\right)\left(b^2_{n+3}\right)$$

for $n \geq 1$, with initial conditions

$$b^2_1 = x, b^2_2 = y, b^2_3 = x^{2^{n-2}-1}, b^2_4 = x^{2^{n-2}}.$$

Then we have the sequence

$$x, y, x^{2^{n-2}-1}y, x^{2^{n-2}}, e, x^{2^{n-2}+1}, yx^2,$$
$$yx, x^{-2}, x^{-2^{n-2}-4}, x^{-7}, yx^{-8}, x^{2^{n-2}+7}, x^{2^{n-2}+12}, \ldots.$$

So, the Jordan-Fibonacci orbit $\mathrm{JO}(Q_{2^n}; x, y)$ can be said to form layers of length 10. Using the above, the sequence becomes

$$b_1^2 = x, b_2^2 = y, b_3^2 = x^{2^{n-2}-1}, b_4^2 = x^{2^{n-2}}, \ldots,$$
$$b_{11}^2 = x^{-7}, b_{12}^2 = x^{-8}y, b_{13}^2 = x^{2^{n-2}+7}y, b_{14}^2 = x^{2^{n-2}+12}, \ldots,$$
$$b_{10i+1}^2 = x^{-8i+1}, b_{10i+2}^2 = x^{-8i}y, b_{10i+3}^2 = x^{8i+2^{n-2}-1}, b_{10i+4}^2 = x^{8i+2^{n-2}}, \ldots.$$

Hence, we need the smallest $i \in \mathbb{N}$ such that $8i = 2^{n-1}\alpha_1$ and $12i = 2^{n-1}\alpha_2$. If we choose $i = 2^{n-3}$, we obtain

$$b_{5\cdot2^{n-2}+1}^2 = x, b_{5\cdot2^{n-2}+2}^2 = y, b_{5\cdot2^{n-2}+3}^2 = x^{2^{n-2}-1}, b_{5\cdot2^{n-2}+4}^2 = x^{2^{n-2}}.$$

Since the elements succeeding $b_{5\cdot2^{n-2}+1}^2$, $b_{5\cdot2^{n-2}+2}^2$, $b_{5\cdot2^{n-2}+3}^2$ and $b_{5\cdot2^{n-2}+4}^2$ depend on x, y, $x^{2^{n-2}-1}$ and $x^{2^{n-2}}$ for their values, the cycle begins again with the $(5 \cdot 2^{n-2})$nd element. Then we get that

$$\mathrm{PJO}(Q_{2^n}; x, y) = P^2(2) \cdot 2^{n-2} = 5 \cdot 2^{n-2}.$$

Thus it is verified that $PA(Q_8 : x, y) = 434$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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