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Existence and uniqueness of solutions for stochastic differential equations of fractional-order $q > 1$ with finite delays

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Abstract

This paper is concerned with stochastic differential equations of fractional-order $q \in (m-1, m)$ (where $m \in \mathbb{Z}$ and $m \geq 2$) with finite delay at a space $BC([-\tau, 0]; R^d)$. Some sufficient conditions are obtained for the existence and uniqueness of solutions for these stochastic fractional differential systems by applying the Picard iterations method and the generalized Gronwall inequality.

Keywords: stochastic differential equations; stochastic fractional differential equations; existence of solutions; fractional order

1 Introduction

Stochastic differential equations are valuable tools for description of some systems and processes with stochastic disturbances in many fields of science and engineering. For the general theory of stochastic differential equations, one can refer to the monograph [1]. Moreover, some results of the existence of solutions were obtained for some stochastic differential equations in [2–6], and the exponential stability was considered for a kind of impulsive neutral stochastic partial differential equations in [7]. The existence of mild solutions was addressed for a class of fractional stochastic differential equations with impulses by the fixed point theorem in Hilbert spaces [8]. The approximate controllability is considered for a stochastic fractional differential system in [9].

On the other hand, fractional calculus can effectively characterize the hereditary properties of various materials and processes to be widely studied [10, 11]. The existence of solutions was considered for (impulsive) fractional differential equations in [12–18], and some progress was achieved in controls, stability, chaos synchronization, some other fractional derivatives and some new methods of numerical solutions etc. for fractional differential equations [19–27]; and the general solution was revealed for some impulsive fractional differential equations in [28, 29].

Motivated by the above mentioned works, we will first consider the existence of solution for a d-dimensional stochastic differential equation of fractional-order $q \in (1, 2)$ with finite delay, and then consider broader stochastic differential equations of fractional order $q \in (m-1, m)$ (here $m \in \mathbb{Z}$ and $m \geq 2$) in the present paper.

$$D_t^q X(t) = \delta(t, X_t) + \sigma(t, X_t) \cdot \frac{dB(t)}{dt}, \quad 1 < q < 2, t \in [t_0, T], \tag{1.1}$$

where D_t^q is the Caputo fractional derivative, $X_t = \{X(t+\theta) : -\tau \le \theta \le 0\}$ (where $\tau \in [0,+\infty)$) can be regarded as a $BC([-\tau,0];R^d)$-valued stochastic process, where $\delta : [t_0,T] \times BC([-\tau,0];R^d) \to R^d$ and $\sigma : [t_0,T] \times BC([-\tau,0];R^d) \to R^{d\times m}$. $B(t)$ is a given m-dimensional standard Brownian motion. The initial value is as follows:

$$\begin{aligned} &X_{t_0} = \xi = \{\xi(\theta) : -\tau \le \theta \le 0\} \text{ is an } \mathsf{F}_{t_0}\text{-measurable} \\ &\quad BC([-\tau,0];R^d)\text{-valued random variable such that} \\ &\quad \xi \in \mathsf{M}^2([-\tau,0];R^d) \text{ and } X'_{t_0} = \xi' = d\xi/d\theta \in \mathsf{M}^2([-\tau,0];R^d), \end{aligned} \tag{1.2}$$

where $\mathsf{M}^2([-\tau,0];R^d)$ denotes the family of the process $\{\xi(t)\}_{t\le 0}$ in $L^p([-\tau,0];R^d)$ such that $E\int_{-\tau}^0 |\xi(t)|^2\,dt < \infty$ a.s.

Next, some preliminaries are introduced in Section 2. Finally, some results are obtained for the solution of (1.1) with initial value (1.2), and these results are extended to stochastic differential equations of fractional-order $q \in (m-1,m)$ (here $m \in \mathbb{Z}$ and $m \ge 2$) in Section 3.

2 Preliminaries

We shall give some notations, basic definitions and conclusions which are used throughout this paper.

Let R^d be the d-dimensional Euclidean space with norm $|\cdot|$. A^T denotes the transpose of matrix A, and $|A| = \sqrt{\operatorname{trace}(A^TA)}$ represents the trace norm of matrix A. Let $t_0 \ge 0$ and (Ω,F,P) be a complete probability space with a filtration $\{\mathsf{F}_t\}_{t\in[t_0,+\infty)}$ satisfying the usual conditions (i.e., it is increasing and right continuous). F_{t_0} contains all P-null sets. F_{t_0} is independent of the σ-field generated by $\{B(t)-B(t_0) : t_0 \le t \le T\}$. Let $BC([-\tau,0];R^d)$ denote the family of bounded continuous R^d-value functions ϕ on $[-\tau,0]$ with the norm $\|\phi\| = \sup_{-\tau<\theta\le 0}|\phi(\theta)|$.

Definition 2.1 ([10]) The fractional integral of order q for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)}\int_{t_0}^t \frac{f(s)}{(t-s)^{1-q}}\,ds, \quad t > t_0, q > 0,$$

provided the right-hand side is pointwise defined on $[t_0,\infty)$, where Γ is the gamma function.

Definition 2.2 ([10]) The Caputo derivative of order q for a function f can be written as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)}\int_{t_0}^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}\,ds = I^{n-q}f^{(n)}(t), \quad t > t_0, 0 \le n-1 < q < n.$$

According to Definitions 2.1 and 2.2, system (1.1) with condition (1.2) is transformed into

$$\begin{aligned} X(t) &= \xi(0) + \xi'(0)(t-t_0) + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\delta(s,X_s)\,ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\sigma(s,X_s)\,dB(s), \end{aligned} \tag{2.1}$$

where $1 < q < 2$, $t \in [t_0,T]$.

Therefore, the following definition of the solution of (1.1) with initial value (1.2) is presented according to (2.1).

Definition 2.3 R^d-value stochastic process $X(t)$ defined on $t_0 - \tau < t \le T$ is called a solution of (1.1) with initial value (1.2) if $X(t)$ satisfies the following properties:

(i) $X(t)$ is continuous and $\{X(t)\}_{t_0 \le t \le T}$ is F_t-adapted;

(ii) $\delta(t, X_t) \in \mathsf{L}^1([t_0, T], R^d)$ and $\sigma(t, X_t) \in \mathsf{L}^2([t_0, T]; R^{d\times m})$;

(iii) $X_{t_0} = \xi$ for each $t_0 \le t \le T$,

$$X(t) = \xi(0) + \xi'(0)(t - t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} \delta(s, X_s)\, ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} \sigma(s, X_s)\, dB(s) \quad \text{a.s.}$$

$X(t)$ is named a unique solution if any other solution $\tilde{X}(t)$ is nondistinctive with $X(t)$, that is,

$$P\{X(t) = \tilde{X}(t) \text{ for any } t_0 - \tau < t \le T\} = 1.$$

Lemma 2.4 ([30]) *Suppose $q > 0$, $a(t)$ is a nonnegative function locally integrable on $t_0 \le t < T$ (some $T \le +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $t_0 \le t < T$, $g(t) \le M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $t_0 \le t < T$ with*

$$u(t) \le a(t) + g(t) \int_{t_0}^{t} (t - s)^{q-1} u(s)\, ds$$

on the interval. Then

$$u(t) \le a(t) + \int_{t_0}^{t} \left(\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(q))^n}{\Gamma(nq)} (t - s)^{nq-1} a(s) \right) ds, \quad t_0 \le t < T.$$

Lemma 2.5 ([30]) *Under the hypothesis of Lemma* 2.4, *let $a(t)$ be a nondecreasing function on $[t_0, T)$. Then*

$$u(t) \le a(t) E_q\bigl(g(t)\Gamma(q)(t - t_0)^q\bigr),$$

where E_q is the Mittag-Leffler function *defined by* $E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+1)}$.

3 Main results

Lemma 3.1 *Suppose $q > 1$ and $0 \le t_0 \le T$ and $t \in [t_0, T]$, function g is bounded on $[t_0 - \tau, T]$. If $g(s, X_s) \in \mathsf{M}^2([t_0 - \tau, T], R^{d\times m})$, then $(t - s)^{q-1} g(s, X_s) \in \mathsf{M}^2([t_0 - \tau, t]; R^{d\times m})$.*

Remark 3.1 The conclusion can be obtained by the properties of definite integral, and it is a precondition to use Itô's formula in the proof of Theorem 3.2.

Theorem 3.2 *Let $\overline{K}$ and K be two positive constants. If*

(i) *for all* $\phi, \psi \in BC([-\tau, 0]; R^d)$ *and* $t \in [t_0, T]$,

$$\left|\delta(t,\phi) - \delta(t,\psi)\right|^2 \vee \left|\sigma(t,\phi) - \sigma(t,\psi)\right|^2 \leq \overline{K}\|\phi - \psi\|^2, \tag{3.1}$$

(ii) *for all* $(\phi, t) \in BC([-\tau, 0]; R^d) \times [t_0, T]$,

$$\left|\delta(t,0)\right|^2 \vee \left|\sigma(t,0)\right|^2 \leq K, \tag{3.2}$$

then system (1.1) *with initial value* (1.2) *has a unique solution* $X(t)$ *with* $X(t) \in \mathsf{M}^2([t_0 - \tau, T]; R^d)$.

Firstly, let us prove a conclusion which will be used in the proof of Theorem 3.2.

Lemma 3.3 *Let* (3.1) *and* (3.2) *hold. If* $X(t)$ *is the solution of* (1.1) *with initial value* (1.2), *then*

$$\begin{aligned} &E\Bigl(\sup_{t_0-\tau<s\leq T}\left|X(s)\right|^2\Bigr) \\ &\quad\leq E\|\xi\|^2 + \left\{4E\|\xi\|^2 + 4E\left\|(T-t_0)\xi'\right\|^2 + \frac{4K(T-t_0)^{2q-1}(T-t_0+1)}{q(\Gamma(q))^2}\right\} \\ &\qquad \times E_q\left(\frac{4\overline{K}(T-t_0)^{2q-1}(T-t_0+1)}{\Gamma(q)}\right) \end{aligned} \tag{3.3}$$

and $X(t) \in \mathsf{M}^2([t_0 - \tau, T]; R^d)$.

Proof Define the stopping time $\tau_n = T \wedge \inf\{t \in [t_0, T] : \|X_t\| \geq n\}$ for every integer $n \geq 1$. Obviously, $\tau_n \uparrow T$ a.s. Let $X^n(t) = X(t \wedge \tau_n)$ for $t \in [t_0, T]$. Therefore,

$$\begin{aligned} X^n(t) &= \xi(0) + \xi'(0)(t\wedge\tau_n - t_0) + \frac{1}{\Gamma(q)}\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n - s)^{q-1}\delta\bigl(s, X_s^n\bigr)\,ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n - s)^{q-1}\sigma\bigl(s, X_s^n\bigr)\,dB(s) \quad \text{for } t_0 \leq t \leq T, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \left|X^n(t)\right|^2 &\leq 4\left|\xi(0)\right|^2 + 4\left|\xi'(0)(t\wedge\tau_n - t_0)\right|^2 \\ &\quad + \frac{4}{(\Gamma(q))^2}\left|\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n - s)^{q-1}\delta\bigl(s, X_s^n\bigr)\,ds\right|^2 \\ &\quad + \frac{4}{(\Gamma(q))^2}\left|\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n - s)^{q-1}\sigma\bigl(s, X_s^n\bigr)\,dB(s)\right|^2 \quad \text{for } t_0 \leq t \leq T. \end{aligned} \tag{3.5}$$

Taking the expectation on both sides of (3.5), we obtain

$$\begin{aligned} E\left|X^n(t)\right|^2 &\leq 4E\left|\xi(0)\right|^2 + 4E\left|\xi'(0)(t\wedge\tau_n - t_0)\right|^2 \\ &\quad + \frac{4}{(\Gamma(q))^2}E\left|\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n - s)^{q-1}\delta\bigl(s, X_s^n\bigr)\,ds\right|^2 \\ &\quad + \frac{4}{(\Gamma(q))^2}E\left|\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n - s)^{q-1}\sigma\bigl(s, X_s^n\bigr)\,dB(s)\right|^2. \end{aligned} \tag{3.6}$$

By (3.1) and (3.2), Lemma 3.1, Hölder's inequality and Itô's formula, we get

$$\begin{aligned}
E\sup_{t_0\le s\le t}\left|X^n(s)\right|^2 &\le 4E\|\xi\|^2+4E\left\|(T-t_0)\xi'\right\|^2 \\
&\quad+\frac{4}{(\Gamma(q))^2}E\left[\left|\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n-s)^{q-1}\delta\left(s,X_s^n\right)ds\right|^2\right] \\
&\quad+\frac{4}{(\Gamma(q))^2}E\left[\left|\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n-s)^{q-1}\sigma\left(s,X_s^n\right)dB(s)\right|^2\right] \\
&\le 4E\|\xi\|^2+4E\left\|(T-t_0)\xi'\right\|^2 \\
&\quad+\frac{4(T-t_0)}{(\Gamma(q))^2}E\left[\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n-s)^{(2q-1)-1}\left|\delta\left(s,X_s^n\right)\right|^2ds\right] \\
&\quad+\frac{4}{(\Gamma(q))^2}E\left[\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n-s)^{(2q-1)-1}\left|\sigma\left(s,X_s^n\right)\right|^2ds\right] \\
&\le 4E\|\xi\|^2+4E\left\|(T-t_0)\xi'\right\|^2 \\
&\quad+\frac{4(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n-s)^{(2q-1)-1}\left[\overline{K}E\left\|X_s^n\right\|^2+K\right]ds \\
&\quad+\frac{4}{(\Gamma(q))^2}\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n-s)^{(2q-1)-1}\left[\overline{K}E\left\|X_s^n\right\|^2+K\right]ds \\
&\le 4E\|\xi\|^2+4E\left\|(T-t_0)\xi'\right\|^2+\frac{4K(T-t_0)^{2q-1}(T-t_0+1)}{(2q-1)(\Gamma(q))^2} \\
&\quad+\frac{4\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n-s)^{(2q-1)-1}E\left\|X_s^n\right\|^2ds \\
&\le 4E\|\xi\|^2+4E\left\|(T-t_0)\xi'\right\|^2+\frac{4K(T-t_0)^{2q-1}(T-t_0+1)}{(2q-1)(\Gamma(q))^2} \\
&\quad+\frac{4\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t\wedge\tau_n}(t\wedge\tau_n-s)^{(2q-1)-1}E\sup_{t_0\le r\le s}\left|X^n(r)\right|^2ds.
\end{aligned}$$

Using Lemma 2.5 in the above inequality, we have

$$\begin{aligned}
&E\sup_{t_0\le s\le t}\left|X^n(s)\right|^2 \\
&\quad\le\left\{4E\|\xi\|^2+4E\left\|(T-t_0)\xi'\right\|^2+\frac{4K(T-t_0)^{2q-1}(T-t_0+1)}{(2q-1)(\Gamma(q))^2}\right\} \\
&\qquad\times E_q\left(\frac{4\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\Gamma(q)(t-t_0)^{2q-1}\right) \\
&\quad\le\left\{4E\|\xi\|^2+4E\left\|(T-t_0)\xi'\right\|^2+\frac{4K(T-t_0)^{2q-1}(T-t_0+1)}{(2q-1)(\Gamma(q))^2}\right\} \\
&\qquad\times E_q\left(\frac{4\overline{K}(T-t_0+1)}{\Gamma(q)}(T-t_0)^{2q-1}\right).
\end{aligned}\tag{3.7}$$

Furthermore,

$$\begin{aligned}
&E\sup_{t_0-\tau<s\le t}\left|X^n(s)\right|^2 \\
&\quad\le E\|\xi\|^2+E\sup_{t_0\le s\le t}\left|X^n(s)\right|^2
\end{aligned}$$

$$\leq E\|\xi\|^2 + \left\{4E\|\xi\|^2 + 4E\big\|(T-t_0)\xi'\big\|^2 + \frac{4K(T-t_0)^{2q-1}(T-t_0+1)}{(2q-1)(\Gamma(q))^2}\right\} \times E_q\left(\frac{4\overline{K}(T-t_0+1)}{\Gamma(q)}(T-t_0)^{2q-1}\right). \tag{3.8}$$

Thus,

$$E\sup_{t_0-\tau<s\leq T}\big|X(s\wedge\tau_n)\big|^2 \leq E\|\xi\|^2 + \left\{4E\|\xi\|^2 + 4E\big\|(T-t_0)\xi'\big\|^2 + \frac{4K(T-t_0)^{2q-1}(T-t_0+1)}{(2q-1)(\Gamma(q))^2}\right\} \times E_q\left(\frac{4\overline{K}(T-t_0+1)}{\Gamma(q)}(T-t_0)^{2q-1}\right). \tag{3.9}$$

Letting $n\to+\infty$ in (3.9), we have

$$E\Big(\sup_{t_0-\tau<s\leq T}\big|X(s)\big|^2\Big) \leq E\|\xi\|^2 + \left\{4E\|\xi\|^2 + 4E\big\|(T-t_0)\xi'\big\|^2 + \frac{4K(T-t_0)^{2q-1}(T-t_0+1)}{(2q-1)(\Gamma(q))^2}\right\} \times E_q\left(\frac{4\overline{K}(T-t_0+1)}{\Gamma(q)}(T-t_0)^{2q-1}\right). \tag{3.10}$$

The proof is complete. □

Next, we will prove Theorem 3.2.

Proof *Uniqueness*: Let $X(t)$ and $\tilde{X}(t)$ be two solutions of (2.1). By Lemma 3.3, $X(t)$ and $\tilde{X}(t)$ belong to $\mathsf{M}^2([t_0-\tau,T];R^d)$,

$$X(t)-\tilde{X}(t) = \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\big[\delta(s,X_s)-\delta(s,\tilde{X}_s)\big]\,ds + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\big[\sigma(s,X_s)-\sigma(s,\tilde{X}_s)\big]\,dB(s). \tag{3.11}$$

By (3.1), Lemma 3.1, Hölder's inequality and Itô's formula, we have

$$\begin{aligned}\big|X(t)-\tilde{X}(t)\big|^2 &\leq 2\left|\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\big[\delta(s,X_s)-\delta(s,\tilde{X}_s)\big]\,ds\right|^2 \\ &\quad + 2\left|\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\big[\sigma(s,X_s)-\sigma(s,\tilde{X}_s)\big]\,dB(s)\right|^2 \\ &\leq \frac{2(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{2q-2}\big|\delta(s,X_s)-\delta(s,\tilde{X}_s)\big|^2\,ds \\ &\quad + \frac{2}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{2q-2}\big|\sigma(s,X_s)-\sigma(s,\tilde{X}_s)\big|^2\,ds \\ &\leq \frac{2\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{(2q-1)-1}\|X_s-\tilde{X}_s\|^2\,ds.\end{aligned}$$

Therefore

$$\begin{aligned} &E\sup_{t_0\le s\le t}\bigl|X(s)-\tilde{X}(s)\bigr|^2 \\ &\quad\le \frac{2\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{(2q-1)-1}E\bigl[\bigl\|X(s)-\tilde{X}(s)\bigr\|^2\bigr]\,ds \\ &\quad\le \frac{2\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{(2q-1)-1}E\Bigl\{\sup_{t_0\le s\le t}\bigl[\bigl|X(s)-\tilde{X}(s)\bigr|^2\bigr]\Bigr\}\,ds. \end{aligned} \tag{3.12}$$

By Lemma 2.5, we obtain

$$E\sup_{t_0\le s\le T}\bigl|X(s)-\tilde{X}(s)\bigr|^2=0.$$

This means that $X(t)=\tilde{X}(t)$ for $t_0\le t\le T$. Hence, the solution for system (1.1) with initial value (1.2) is almost surely unique on the interval $[t_0-\tau,T]$.

Existence: Step 1. Suppose that $T-t_0$is sufficiently small such that

$$\Xi=\frac{2\overline{K}(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2}<1. \tag{3.13}$$

Let $X^0(t)=\xi(0)$ and $X^n_{t_0}=\xi$ (here $n=1,2,\ldots$). Define the following Picard sequence:

$$\begin{aligned} X^n(t)&=\xi(0)+(t-t_0)\xi'(0)+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\delta\bigl(s,X_s^{n-1}\bigr)\,ds \\ &\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\sigma\bigl(s,X_s^{n-1}\bigr)\,dB(s)\quad\text{for } t_0\le t\le T. \end{aligned} \tag{3.14}$$

Obviously, $X^0(t)\in\mathsf{M}^2([t_0-\tau,T];R^d)$, then prove $X^n(t)\in\mathsf{M}^2([t_0-\tau,T];R^d)$. By (3.14), we have

$$\begin{aligned} \bigl|X^n(t)\bigr|^2&\le 4\bigl|\xi(0)\bigr|^2+4\bigl|(t-t_0)\xi'(0)\bigr|^2 \\ &\quad+\frac{4}{(\Gamma(q))^2}\biggl[\biggl|\int_{t_0}^{t}(t-s)^{q-1}\delta\bigl(s,X_s^{n-1}\bigr)\,ds\biggr|^2 \\ &\quad+\biggl|\int_{t_0}^{t}(t-s)^{q-1}\sigma\bigl(s,X_s^{n-1}\bigr)\,dB(s)\biggr|^2\biggr]. \end{aligned} \tag{3.15}$$

By (3.1) and (3.2), Lemma 3.1, Hölder's inequality and Itô's formula, we get

$$\begin{aligned} E\bigl|X^n(t)\bigr|^2&\le 4E\|\xi\|^2+4E\bigl\|(T-t_0)\xi'\bigr\|^2 \\ &\quad+\frac{4(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\bigl|\delta\bigl(s,X_s^{n-1}\bigr)\bigr|^2\,ds \\ &\quad+\frac{4}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\bigl|\sigma\bigl(s,X_s^{n-1}\bigr)\bigr|^2\,ds \\ &=4E\|\xi\|^2+4E\bigl\|(T-t_0)\xi'\bigr\|^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{4(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\bigl|\delta\bigl(s,X_s^{n-1}\bigr)-\delta(s,0)+\delta(s,0)\bigr|^2\,ds\\
&+\frac{4}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\bigl|\sigma\bigl(s,X_s^{n-1}\bigr)-\sigma(s,0)+\sigma(s,0)\bigr|^2\,ds\\
\le{}& 4E\|\xi\|^2+4E\bigl\|(T-t_0)\xi'\bigr\|^2\\
&+\frac{4(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\bigl[2\bigl|\delta\bigl(s,X_s^{n-1}\bigr)-\delta(s,0)\bigr|^2+2\bigl|\delta(s,0)\bigr|^2\bigr]\,ds\\
&+\frac{4}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\bigl[2\bigl|\sigma\bigl(s,X_s^{n-1}\bigr)-\sigma(s,0)\bigr|^2+2\bigl|\sigma(s,0)\bigr|^2\bigr]\,ds\\
\le{}& 4E\|\xi\|^2+4E\bigl\|(T-t_0)\xi'\bigr\|^2\\
&+\frac{8(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\bigl[\overline{K}\bigl|X_s^{n-1}\bigr|^2+K\bigr]\,ds\\
\le{}& 4E\|\xi\|^2+4E\bigl\|(T-t_0)\xi'\bigr\|^2+\frac{8K(T-t_0+1)}{(2q-1)(\Gamma(q))^2}(T-t_0)^{2q-1}\\
&+\frac{8\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{(2q-1)-1}E\sup_{t_0\le r\le s}\bigl|X^{n-1}(r)\bigr|^2\,ds.
\end{aligned}
$$

Hence, for any $k\ge 1$, we have

$$
\begin{aligned}
\max_{1\le n\le k}E\bigl|X^n(t)\bigr|^2\le{}& 4E\|\xi\|^2+4E\bigl\|(T-t_0)\xi'\bigr\|^2+\frac{8K(T-t_0+1)}{(2q-1)(\Gamma(q))^2}(T-t_0)^{2q-1}\\
&+\frac{8\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\biggl[\int_{t_0}^{t}(t-s)^{(2q-1)-1}E\Bigl[\max_{1\le n\le k}\bigl|X^{n-1}(r)\bigr|^2\Bigr]\,ds\biggr].
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\max_{1\le n\le k}E\bigl|X^{n-1}(s)\bigr|^2 &= \max\bigl\{E\bigl|\xi(0)\bigr|^2,E\bigl|X^1(s)\bigr|^2,E\bigl|X^2(s)\bigr|^2,\dots,E\bigl|X^{k-1}(s)\bigr|^2\bigr\}\\
&\le \max\bigl\{E\|\xi\|^2,E\bigl|X^1(s)\bigr|^2,E\bigl|X^2(s)\bigr|^2,\dots,E\bigl|X^{k-1}(s)\bigr|^2\bigr\}\\
&= \max\Bigl\{E\|\xi\|^2,\max_{1\le n\le k}E\bigl|X^n(s)\bigr|^2\Bigr\}\\
&\le E\|\xi\|^2+\max_{1\le n\le k}E\bigl|X^n(s)\bigr|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\max_{1\le n\le k}E\bigl|X^n(t)\bigr|^2\le{}& 4E\|\xi\|^2+4E\bigl\|(T-t_0)\xi'\bigr\|^2+\frac{8K(T-t_0+1)}{(2q-1)(\Gamma(q))^2}(T-t_0)^{2q-1}\\
&+\frac{8\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\biggl[\int_{t_0}^{t}(t-s)^{(2q-1)-1}\Bigl[E\|\xi\|^2+\max_{1\le n\le k}E\bigl|X^n(s)\bigr|^2\Bigr]\,ds\biggr]\\
\le{}& 4E\|\xi\|^2+4E\bigl\|(T-t_0)\xi'\bigr\|^2+\frac{8K(T-t_0+1)}{(2q-1)(\Gamma(q))^2}(T-t_0)^{2q-1}\\
&+\frac{8\overline{K}(T-t_0+1)(T-t_0)^{2q-1}E\|\xi\|^2}{(2q-1)(\Gamma(q))^2}\\
&+\frac{8\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\biggl[\int_{t_0}^{t}(t-s)^{(2q-1)-1}\Bigl[\max_{1\le n\le k}E\bigl|X^n(s)\bigr|^2\Bigr]\,ds\biggr].
\end{aligned}
$$

By Lemma 2.5, we have

$$\max_{1\le n\le k} E\left|X^n(t)\right|^2 \le c_1 E_q\left(c_2(T-t_0)^{2q-1}\right),$$

where $c_1 = 4E\|\xi\|^2 + 4E\|(T-t_0)\xi'\|^2 + \frac{8K(T-t_0+1)}{(2q-1)(\Gamma(q))^2}(T-t_0)^{2q-1} + \frac{8\overline{K}(T-t_0+1)(T-t_0)^{2q-1}E\|\xi\|^2}{(2q-1)(\Gamma(q))^2}$ and $c_2 = \frac{8\overline{K}(T-t_0+1)\Gamma(2q-1)}{(\Gamma(q))^2}$. Because of the arbitrary constant k, we obtain

$$E\left|X^n(t)\right|^2 \le c_1 E_q\left(c_2(T-t_0)^{2q-1}\right), \quad t_0\le t\le T, n\ge 1.$$

Next, by (3.13), we have

$$\begin{aligned} X^1(t)-X^0(t) &= X^1(t)-\xi(0) \\ &= (t-t_0)\xi'(0) + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\delta\left(s,X_s^0\right)ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\sigma\left(s,X_s^0\right)dB(s). \end{aligned}$$

With similarity to the proof of uniqueness, we get

$$\begin{aligned} &E\sup_{t_0\le t\le T}\left|X^1(t)-X^0(t)\right|^2 \\ &\quad\le 3E\sup_{t_0\le t\le T}\left\|(T-t_0)\xi'\right\|^2 + 3E\left|\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\delta\left(s,X_s^0\right)ds\right|^2 \\ &\qquad + 3E\left|\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\sigma\left(s,X_s^0\right)dB(s)\right|^2 \\ &\quad\le 3E\sup_{t_0\le t\le T}\left\|(T-t_0)\xi'\right\|^2 + \frac{3(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{2q-2}E\left|\delta\left(s,X_s^0\right)\right|^2 ds \\ &\qquad + \frac{3}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{2q-2}E\left|\sigma\left(s,X_s^0\right)\right|^2 ds \\ &\quad= 3E\sup_{t_0\le t\le T}\left\|(T-t_0)\xi'\right\|^2 \\ &\qquad + \frac{3(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{2q-2}E\left|\delta\left(s,X_s^0\right)-\delta(s,0)+\delta(s,0)\right|^2 ds \\ &\qquad + \frac{3}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{2q-2}E\left|\sigma\left(s,X_s^0\right)-\sigma(s,0)+\sigma(s,0)\right|^2 ds \\ &\quad\le 3E\sup_{t_0\le t\le T}\left\|(T-t_0)\xi'\right\|^2 + \frac{3(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{2q-2}\left[2\overline{K}E\left\|X_s^0\right\|^2+2K\right]ds \\ &\qquad + \frac{3}{(\Gamma(q))^2}\int_{t_0}^t (t-s)^{2q-2}\left[2\overline{K}E\left\|X_s^0\right\|^2+2K\right]ds \\ &\quad\le 3E\left\|(T-t_0)\xi'\right\|^2 + \frac{6\overline{K}(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2}E\|\xi\|^2 \\ &\qquad + \frac{6K(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2} := C. \end{aligned}$$

For arbitrary $n \geq 1$ and $t_0 \leq t \leq T$, we have

$$X^{n+1}(t) - X^n(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \big[\delta\big(s, X_s^n\big) - \delta\big(s, X_s^{n-1}\big)\big]\, ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \big[\sigma\big(s, X_s^n\big) - \sigma\big(s, X_s^{n-1}\big)\big]\, dB(s).$$

With similarity to the proof of uniqueness, we get

$$\begin{aligned} &E \sup_{t_0 \leq t \leq T} \big|X^{n+1}(t) - X^n(t)\big|^2 \\ &\leq 2 \left| \frac{1}{\Gamma(q)} \int_{t_0}^{T} (t-s)^{q-1} \sup_{t_0 \leq s \leq t} \big[\delta\big(s, X_s^n\big) - \delta\big(s, X_s^{n-1}\big)\big]\, ds \right|^2 \\ &\quad + 2 \left| \frac{1}{\Gamma(q)} \int_{t_0}^{T} (t-s)^{q-1} \sup_{t_0 \leq s \leq t} \big[\sigma\big(s, X_s^n\big) - \sigma\big(s, X_s^{n-1}\big)\big]\, dB(s) \right|^2 \\ &\leq \frac{2(T-t_0)}{(\Gamma(q))^2} \int_{t_0}^{T} (t-s)^{2q-2} \sup_{t_0 \leq s \leq t} \big|\delta\big(s, X_s^n\big) - \delta\big(s, X_s^{n-1}\big)\big|^2\, ds \\ &\quad + \frac{2}{(\Gamma(q))^2} \int_{t_0}^{T} (t-s)^{2q-2} \sup_{t_0 \leq s \leq t} \big|\sigma\big(s, X_s^n\big) - \sigma\big(s, X_s^{n-1}\big)\big|^2\, ds \\ &\leq \frac{2\overline{K}(T-t_0+1)}{(\Gamma(q))^2} \int_{t_0}^{T} (t-s)^{2q-2} \sup_{t_0 \leq r \leq t} \big|X^n(r) - X^{n-1}(r)\big|^2\, ds \\ &\leq \frac{2\overline{K}(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2} \sup_{t_0 \leq t \leq T} \big|X^n(t) - X^{n-1}(t)\big|^2 \\ &\leq \Xi^n \sup_{t_0 \leq t \leq T} \big|X^1(t) - X^0(t)\big|^2 \\ &\leq C\Xi^n, \quad t_0 \leq t \leq T. \end{aligned} \tag{3.16}$$

Next, we will verify that $\{X^n(t)\}$ converges to $X(t)$ in the sense of L^2 and probability 1 on $\mathsf{M}^2([t_0 - \tau, T]; R^d)$, and $X(t)$ is the solution of (1.1) with initial value (1.2). By the Chebyshev inequality, we have

$$P\left\{ \sup_{t_0 \leq t \leq T} \big|X^{n+1}(t) - X^n(t)\big|^2 > \frac{1}{2^n} \right\} \leq C(4\Xi)^n. \tag{3.17}$$

By the fact $\sum_{n=0}^{\infty} C(4\Xi)^n < \infty$ and the Borel-Cantelli lemma, there exists a positive integer $n_0 = n_0(\omega)$ for almost all $\omega \in \Omega$ such that

$$\sup_{t_0 \leq t \leq T} \big|X^{n+1}(t) - X^n(t)\big| \leq \frac{1}{2^n}, \quad n \geq n_0.$$

Define the function series

$$X^0(t) + \big[X^1(t) - X^0(t)\big] + \cdots + \big[X^n(t) - X^{n-1}(t)\big] + \cdots \tag{3.18}$$

with the partial sum $X^n(t) = X^0(t) + \sum_{i=1}^{n}[X^i(t) - X^{i-1}(t)]$. Therefore, the absolute value of every item (3.18) is less than the corresponding item of a positive series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots.$$

By Weierstrass's criterion, (3.18) is uniformly convergent on $[t_0 - \tau, T]$. Thus, the approximate sequence $\{X^n(t)\}$ uniformly converges to $X(t)$ (where $X(t)$ is assumed to be the sum function) on $[t_0 - \tau, T]$, and it is F_t-adapted. Thus $X(t)$ is continuous and F_t-adapted. Moreover, (3.16) implies that the sequence $\{X^n(t)\}$ for each t is also a Cauchy sequence in L^2. Hence, $X^n(t) \to^{L^2} X(t)$ as $n \to \infty$, i.e., $E|X^n(t) - X(t)|^2 \to 0$. Letting $n \to \infty$ in $\mathsf{M}^2([t_0 - \tau, T]; R^d)$, we have

$$E\left|X(t)\right|^2 \le c_1 E_q\left(c_2 \Gamma(q)(T - t_0)^{2q-1}\right). \tag{3.19}$$

Using (3.19), we can get

$$\begin{aligned} E\int_{t_0-\tau}^{T} \left|X(s)\right|^2 ds &= E\int_{t_0-\tau}^{t_0} \left|X(s)\right|^2 ds + E\int_{t_0}^{T} \left|X(s)\right|^2 ds \\ &\le E\int_{t_0-\tau}^{t_0} \left|\xi(s)\right|^2 ds + E\int_{t_0}^{T} c_1 E_q\left(c_2 \Gamma(q)(T - t_0)^{2q-1}\right) ds < \infty. \end{aligned}$$

Therefore, $X(t) \in \mathsf{M}^2([t_0 - \tau, T]; R^d)$.

Next, we will verify that $X(t)$ satisfies (1.1).

$$\begin{aligned} &E\left|\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\left[\delta\left(s, X_s^n\right) - \delta(s, X_s)\right] ds \right. \\ &\qquad \left. + \frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\left[\sigma\left(s, X_s^n\right) - \sigma(s, X_s)\right] dB(s)\right|^2 \\ &\quad \le 2E\left|\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\left[\delta\left(s, X_s^n\right) - \delta(s, X_s)\right] ds\right|^2 \\ &\qquad + 2E\left|\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\left[\sigma\left(s, X_s^n\right) - \sigma(s, X_s)\right] dB(s)\right|^2 \\ &\quad \le \frac{2(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^{T}(t-s)^{2q-2} E\left|\delta\left(s, X_s^n\right) - \delta(s, X_s)\right|^2 ds \\ &\qquad + \frac{2}{(\Gamma(q))^2}\int_{t_0}^{T}(t-s)^{2q-2} E\left|\sigma\left(s, X_s^n\right) - \sigma(s, X_s)\right|^2 ds \\ &\quad \le \frac{2\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{T}(t-s)^{2q-2} E\left\|X_s^n - X_s\right\|^2 ds \\ &\quad \le \frac{2\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{T}(t-s)^{2q-2} E\sup_{t_0 \le r \le T}\left|X^n(r) - X(r)\right|^2 ds \\ &\quad \le \frac{2\overline{K}(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{T}(t-s)^{(2q-1)-1} E\left|X^n(s) - X(s)\right|^2 ds. \end{aligned}$$

Thus $\{X^n(t)\}$ is uniformly convergent on $[t_0 - \tau, T]$. This means that there exists an n_0 for any given $\varepsilon > 0$ such that $E|X^n(t) - X(t)|^2 < \varepsilon$ (as $n \geq n_0$ and $\forall t \in [t_0 - \tau, T]$). Therefore

$$\int_{t_0}^{T} (t-s)^{2q-2} E\left|X^n(s) - X(s)\right|^2 ds < \frac{(T-t_0)^{2q-1}}{2q-1}\varepsilon.$$

Hence, for $t \in [t_0, T]$, we have

$$\int_{t_0}^{t} (t-s)^{q-1}\delta\left(s, X_s^n\right) ds \to \int_{t_0}^{t} (t-s)^{q-1}\delta(s, X_s)\, ds \quad \text{in } L^2(\Omega),$$
$$\int_{t_0}^{t} (t-s)^{q-1}\sigma\left(s, X_s^n\right) ds \to \int_{t_0}^{t} (t-s)^{q-1}\sigma(s, X_s)\, ds \quad \text{in } L^2(\Omega).$$

Taking limits on both sides of (3.14), we get

$$\begin{aligned} X(t) = \xi(0) + (t-t_0)\xi'(0) &+ \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1}\delta(s, X_s)\, ds \\ &+ \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1}\sigma(s, X_s)\, dB(s). \end{aligned}$$

Thus $X(t)$ is the solution of (1.1).

Step 2. To remove the limitation of (3.13), suppose that $\gamma > 0$ is sufficiently small to satisfy

$$\frac{2\overline{K}(\gamma + 4)\gamma^{2q-1}}{(2q-1)(\Gamma(q))^2} < 1.$$

Therefore, (1.1) has a solution on $[t_0 - \tau, t_0 + \gamma]$ by Step 1. Next, consider the solution of (1.1) on $[t_0 + \gamma, t_0 + 2\gamma]$. By repeating the process above, it is sure that there is a solution to (1.1) on the entire interval $[t_0 - \tau, T]$. The proof is complete. □

Remark 3.2 Considering (2.1) by the definition of fractional Brownian motion $\beta(t, H)$ on the Maruyama expression in [31] (for details, see [31] and the references therein), we have

$$\begin{aligned} \int_{t_0}^{t} (t-s)^{q-1}\sigma(s, X_s)\, dB(s) &= \int_{t_0}^{t} (t-s)^{q-1}\sigma(s, X_s)\left[\frac{1}{2}\omega(s)(t-s)^{-\frac{1}{2}}\, ds\right] \\ &= \frac{1}{2}\int_{t_0}^{t} \sigma(s, X_s)(t-s)^{(q-\frac{1}{2})-1}\omega(s)\, ds \\ &= \frac{1}{2q-1}\int_{t_0}^{t} \sigma(s, X_s)\omega(s)(ds)^{q-\frac{1}{2}} \\ &= \frac{1}{2q-1}\int_{t_0}^{t} \sigma(s, X_s)\, d\beta\left(s, q-\frac{1}{2}\right), \end{aligned}$$

where $q \in (1, \frac{3}{2})$. This shows that the solution of system (1.1) with initial value (1.2) is disturbed by a fractional Brownian motion $\beta(t, q - \frac{1}{2})$ with Hurst index $H = q - \frac{1}{2} \in (\frac{1}{2}, 1)$.

Next, the estimate of the error will be discussed for the Picard approximation $X^n(t)$ and the exact solution.

Theorem 3.4 *Under the hypothesis of Theorem* 3.2, *suppose that* $X(t)$ *is the unique solution of* (1.1) *with initial value* (1.2) *and* $X^n(t)$ *is defined by* (3.14). *Then, for each* $n \geq 1$,

$$\begin{aligned} &E \sup_{t_0 \leq s \leq t} \left|X^n(s) - X(s)\right|^2 \\ &\quad \leq C\Xi^{n-1} \frac{4\overline{K}(T - t_0 + 1)(T - t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2} E_q\left(\frac{4\overline{K}(T - t_0 + 1)}{\Gamma(q)}(T - t_0)^{2q-1}\right). \end{aligned}$$

Proof With similarity of discussion in Theorem 3.2, we have

$$\begin{aligned} E \sup_{t_0 \leq s \leq t} \left|X^n(s) - X(s)\right|^2 &\leq \frac{2\overline{K}(T - t_0 + 1)}{(\Gamma(q))^2} E \int_{t_0}^t (t-s)^{2q-2}\left[\left\|X_s^{n-1} - X_s\right\|^2\right] ds \\ &\leq \frac{2\overline{K}(T - t_0 + 1)}{(\Gamma(q))^2} \int_{t_0}^t (t-s)^{2q-2} E \sup_{t_0 \leq r \leq s} \left|X^{n-1}(r) - X(r)\right|^2 ds \\ &\leq \frac{4\overline{K}(T - t_0 + 1)}{(\Gamma(q))^2} \int_{t_0}^t (t-s)^{2q-2} E \sup_{t_0 \leq r \leq s} \left|X^n(r) - X^{n-1}(r)\right|^2 ds \\ &\quad + \frac{4\overline{K}(T - t_0 + 1)}{(\Gamma(q))^2} \int_{t_0}^t (t-s)^{2q-2} E \sup_{t_0 \leq r \leq s} \left|X^n(r) - X(r)\right|^2 ds. \end{aligned}$$

Substituting (3.16) into the above inequality, we obtain

$$\begin{aligned} E \sup_{t_0 \leq s \leq t} \left|X^n(s) - X(s)\right|^2 &\leq C\Xi^{n-1} \frac{4\overline{K}(T - t_0 + 1)}{(2q-1)(\Gamma(q))^2}(T - t_0)^{2q-1} \\ &\quad + \frac{4\overline{K}(T - t_0 + 1)}{(\Gamma(q))^2} \int_{t_0}^t (t-s)^{(2q-1)-1} E \sup_{t_0 \leq r \leq s} \left|X^n(r) - X(r)\right|^2 ds. \end{aligned}$$

By Lemma 2.5, we get

$$\begin{aligned} &E \sup_{t_0 \leq s \leq t} \left|X^n(s) - X(s)\right|^2 \\ &\quad \leq C\Xi^{n-1} \frac{4\overline{K}(T - t_0 + 1)(T - t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2} E_q\left(\frac{4\overline{K}(T - t_0 + 1)}{\Gamma(q)}(t - t_0)^{2q-1}\right). \end{aligned}$$

Letting $t = T$ in the above inequality, the conclusion can be drawn. This completes the proof. □

Extending fractional order $q \in (1,2)$ to $q \in (m-1, m)$ (here $m \in \mathbb{Z}$ and $m \geq 2$), consider a d-dimensional stochastic fractional differential equation as follows:

$$D_t^q X(t) = f(t, X_t) + g(t, X_t) \cdot \frac{dB(t)}{dt}, \quad q \in (m-1, m), m \in \mathbb{Z} \text{ and } m \geq 2, t \in [t_0, T], \quad (3.20)$$

where $X_t = \{X(t+\theta) : -\tau \leq \theta \leq 0\}$ can be regarded as a $BC([-\tau, 0]; R^d)$-valued stochastic process, D_t^q is the Caputo fractional derivative, where $f : [t_0, T] \times BC([-\tau, 0]; R^d) \to R^d$ and $g : [t_0, T] \times BC([-\tau, 0]; R^d) \to R^{d \times m}$. The initial value is as follows:

$$\begin{aligned} &X_{t_0} = \xi = \left\{\xi(\theta) : -\tau \leq \theta \leq 0\right\} \text{ is an } \mathsf{F}_{t_0}\text{-measurable} \\ &\quad BC\left([-\tau, 0]; R^d\right)\text{-valued random variable such that} \end{aligned}$$

$$\xi \in \mathsf{M}^2\big([-\tau,0];R^d\big)\text{and}X'_{t_0} = \xi' = d\xi/d\theta \in \mathsf{M}^2\big([-\tau,0];R^d\big),\ldots,$$
$$X_{t_0}^{(m-1)} = \xi^{(m-1)} = d^{(m-1)}\xi/(d\theta)^{(m-1)} \in \mathsf{M}^2\big([-\tau,0];R^d\big), \tag{3.21}$$

where $\mathsf{M}^2([-\tau,0];R^d)$ denotes the family of the process $\{\xi(t)\}_{t\le 0}$ in $L^p([-\tau,0];R^d)$ such that $E\int_{-\tau}^0 |\xi(t)|^2 dt < \infty$ a.s.

According to Definitions 2.1 and 2.2, system (3.20) with initial condition (3.21) can be rewritten as

$$\begin{aligned} X(t) &= \sum_{k=0}^{m-1} \frac{\xi^{(k)}(0)}{k!}(t-t_0)^k \\ &\quad + \frac{1}{\Gamma(q)}\left[\int_{t_0}^t (t-s)^{q-1} f(s,X_s)\,ds + \int_{t_0}^t (t-s)^{q-1} g(s,X_s)\,dB(s)\right] \end{aligned} \tag{3.22}$$

for $t \in [t_0,T]$. Therefore, we give the following definition of the solution of (3.20) with initial value (3.21).

Definition 3.5 R^d-value stochastic process $X(t)$ defined on $t_0 - \tau < t \le T$ is called a solution of (3.20) with initial value (3.21) if $X(t)$ has the following properties:

(i) $X(t)$ is continuous and $\{X(t)\}_{t_0\le t\le T}$ is F_t-adapted;

(ii) $\{f(X_t,t)\} \in \mathsf{L}^1([t_0,T],R^d)$ and $\{g(X_t,t)\} \in \mathsf{L}^2([t_0,T];R^{d\times m})$;

(iii) $X_{t_0} = \xi$ for each $t_0 \le t \le T$,

$$\begin{aligned} X(t) &= \sum_{k=0}^{m-1} \frac{\xi^{(k)}(0)}{k!}(t-t_0)^k + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f(s,X_s)\,ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} g(s,X_s)\,dB(s) \quad \text{a.s.} \end{aligned}$$

$X(t)$ is named a unique solution if any other solution $\tilde{X}(t)$ is nondistinctive with $X(t)$, that is,

$$P\big\{X(t) = \tilde{X}(t) \text{ for any } t_0 - \tau < t \le T\big\} = 1.$$

With similarity to Theorem 3.2 and Lemma 3.3, the following two conclusions can be drawn and their proofs are omitted.

Theorem 3.6 *Let $\overline{K}_2$ and K_2be two positive constants. If*

(H1) *for all $\phi,\psi \in BC([-\tau,0];R^d)$and $t \in [t_0,T]$,*

$$\big|f(t,\phi) - f(t,\psi)\big|^2 \vee \big|g(t,\phi) - g(t,\psi)\big|^2 \le \overline{K}_2\|\phi - \psi\|^2, \tag{3.23}$$

(H2) *for all $(\phi,t) \in BC([-\tau,0];R^d) \times [t_0,T]$,*

$$\big|f(t,0)\big|^2 \vee \big|g(t,0)\big|^2 \le K_2, \tag{3.24}$$

then (3.20) *with initial value* (3.21) *has a unique solution $X(t)$ and $X(t) \in \mathsf{M}^2([t_0-\tau,T];R^d)$.*

Lemma 3.7 *Let* (3.23) *and* (3.24) *hold. If* $X(t)$ *is the solution of* (3.20) *with initial value* (3.21), *then*

$$\begin{aligned} &E\Bigl(\sup_{t_0-\tau<s\le T}\bigl|X^n(s)\bigr|^2\Bigr) \\ &\quad\le E\|\xi\|^2+\left\{(m+2)\sum_{k=0}^{n-1}\frac{\|(T-t_0)^k\xi^{(k)}\|}{k!}+\frac{(m+2)K_2(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2}\right\} \\ &\qquad\times E_q\biggl(\frac{(m+2)\overline{K}_2(T-t_0+1)(T-t_0)^{2q-1}}{\Gamma(q)}\biggr) \end{aligned}$$

and $X(t)\in \mathsf{M}^2([t_0-\tau,T];R^d)$.

Theorem 3.8 *Under the hypothesis of Theorem* 3.6, *suppose that* $X(t)$ *is the unique solution of* (3.20) *with initial value* (3.21). *Then, for each* $n\ge 1$,

$$\begin{aligned} &E\sup_{t_0\le s\le T}\bigl|X^n(s)-X(s)\bigr|^2 \\ &\quad\le C_2\,\Xi_2^{n-1}\frac{4\overline{K}_2(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2}E_q\biggl(\frac{4\overline{K}_2(T-t_0+1)}{\Gamma(q)}(T-t_0)^{2q-1}\biggr), \end{aligned}$$

where $X^n(t)=\sum_{k=0}^{m-1}\frac{\xi^{(k)}(0)}{k!}(t-t_0)^k+\frac{1}{\Gamma(q)}\int_{t_0}^t(t-s)^{q-1}f(s,X_s^{n-1})\,ds+\frac{1}{\Gamma(q)}\int_{t_0}^t(t-s)^{q-1}g(s,X_s^{n-1})\,dB(s)$.

Proof Let $X^0(t)=\xi(0)$ and $X^n_{t_0}=\xi$ (here $n=1,2,\dots$). Define the following Picard sequence:

$$\begin{aligned} X^n(t)={}&\sum_{k=0}^{m-1}\frac{\xi^{(k)}(0)}{k!}(t-t_0)^k+\frac{1}{\Gamma(q)}\int_{t_0}^t(t-s)^{q-1}f\bigl(s,X_s^{n-1}\bigr)\,ds \\ &+\frac{1}{\Gamma(q)}\int_{t_0}^t(t-s)^{q-1}g\bigl(s,X_s^{n-1}\bigr)\,dB(s). \end{aligned}$$

Thus,

$$\begin{aligned} &E\sup_{t_0\le t\le T}\bigl|X^1(t)-X^0(t)\bigr|^2 \\ &\quad\le(m+1)\sum_{k=1}^{m-1}\frac{\|(T-t_0)^k\xi^{(k)}\|}{k!}+(m+1)E\biggl|\frac{1}{\Gamma(q)}\int_{t_0}^t(t-s)^{q-1}f\bigl(s,X_s^0\bigr)\,ds\biggr|^2 \\ &\qquad+(m+1)E\biggl|\frac{1}{\Gamma(q)}\int_{t_0}^t(t-s)^{q-1}g\bigl(s,X_s^0\bigr)\,dB(s)\biggr|^2 \\ &\quad\le(m+1)\sum_{k=1}^{m-1}\frac{\|(T-t_0)^k\xi^{(k)}\|}{k!}+\frac{(m+1)(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^t(t-s)^{2q-2}E\bigl|f\bigl(s,X_s^0\bigr)\bigr|^2\,ds \\ &\qquad+\frac{(m+1)}{(\Gamma(q))^2}\int_{t_0}^t(t-s)^{2q-2}E\bigl|g\bigl(s,X_s^0\bigr)\bigr|^2\,ds \\ &\quad\le(m+1)\sum_{k=1}^{m-1}\frac{\|(T-t_0)^k\xi^{(k)}\|}{k!}+\frac{(m+1)(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^t(t-s)^{2q-2}\bigl[2\overline{K}_2E\bigl\|X_s^0\bigr\|^2+2K_2\bigr]\,ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{(m+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}\bigl[2\overline{K}_2E\bigl\|X_s^0\bigr\|^2+2K_2\bigr]\,ds\\
&\le (m+1)\sum_{k=1}^{m-1}\frac{\|(T-t_0)^k\xi^{(k)}\|}{k!}+\frac{2(m+1)(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2}\bigl[\overline{K}_2E\|\xi\|^2+K_2\bigr]\\
&:= C_2.
\end{aligned}
$$

Next, for $n \ge 1$ and $t_0 \le t \le T$, we have

$$
\begin{aligned}
X^{n+1}(t)-X^n(t)={}&\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\bigl[f\bigl(s,X_s^n\bigr)-f\bigl(s,X_s^{n-1}\bigr)\bigr]\,ds\\
&+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\bigl[g\bigl(s,X_s^n\bigr)-g\bigl(s,X_s^{n-1}\bigr)\bigr]\,dB(s).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&E\sup_{t_0\le t\le T}\bigl|X^{n+1}(t)-X^n(t)\bigr|^2\\
&\le 2\left|\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\sup_{t_0\le s\le t}\bigl[f\bigl(s,X_s^n\bigr)-f\bigl(s,X_s^{n-1}\bigr)\bigr]\,ds\right|^2\\
&\quad+2\left|\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\sup_{t_0\le s\le t}\bigl[g\bigl(s,X_s^n\bigr)-g\bigl(s,X_s^{n-1}\bigr)\bigr]\,dB(s)\right|^2\\
&\le \frac{2(T-t_0)}{(\Gamma(q))^2}\int_{t_0}^{T}(t-s)^{2q-2}\sup_{t_0\le s\le t}\bigl|f\bigl(s,X_s^n\bigr)-f\bigl(s,X_s^{n-1}\bigr)\bigr|^2\,ds\\
&\quad+\frac{2}{(\Gamma(q))^2}\int_{t_0}^{T}(t-s)^{2q-2}\sup_{t_0\le s\le t}\bigl|g\bigl(s,X_s^n\bigr)-g\bigl(s,X_s^{n-1}\bigr)\bigr|^2\,ds\\
&\le \frac{2\overline{K}_2(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{T}(t-s)^{2q-2}\sup_{t_0\le r\le t}\bigl|X^n(r)-X^{n-1}(r)\bigr|^2\,ds\\
&\le \frac{2\overline{K}_2(T-t_0+1)}{(2q-1)(\Gamma(q))^2}(T-t_0)^{2q-1}\sup_{t_0\le t\le T}\bigl|X^n(t)-X^{n-1}(t)\bigr|^2\\
&\le \Xi_2^n\sup_{t_0\le t\le T}\bigl|X^1(t)-X^0(t)\bigr|^2\\
&\le C_2\,\Xi_2^n,\quad t_0\le t\le T.
\end{aligned}
$$

Substituting the above two results into the following inequality, we have

$$
\begin{aligned}
&E\sup_{t_0\le s\le t}\bigl|X^n(s)-X(s)\bigr|^2\\
&\le 2\left|\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\sup_{t_0\le s\le t}\bigl[f\bigl(s,X_s^{n-1}\bigr)-f(s,X_s)\bigr]\,ds\right|^2\\
&\quad+2\left|\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\sup_{t_0\le s\le t}\bigl[g\bigl(s,X_s^{n-1}\bigr)-g(s,X_s)\bigr]\,dB(s)\right|^2\\
&\le \frac{2\overline{K}_2(T-t_0+1)}{(\Gamma(q))^2}E\int_{t_0}^{t}(t-s)^{2q-2}\bigl[\bigl\|X_s^{n-1}-X_s\bigr\|^2\bigr]\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{2\overline{K}_2(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\sup_{t_0\leq r\leq s}\left|X^{n-1}(r)-X(r)\right|^2 ds \\
&\leq \frac{4\overline{K}_2(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\sup_{t_0\leq r\leq s}\left|X^{n}(r)-X^{n-1}(r)\right|^2 ds \\
&\quad + \frac{4\overline{K}_2(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{2q-2}E\sup_{t_0\leq r\leq s}\left|X^{n}(r)-X(r)\right|^2 ds \\
&\leq C_2\,\Xi_2^{n-1}\frac{4\overline{K}_2(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2} \\
&\quad + \frac{4\overline{K}_2(T-t_0+1)}{(\Gamma(q))^2}\int_{t_0}^{t}(t-s)^{(2q-1)-1}E\sup_{t_0\leq r\leq s}\left|X^{n}(r)-X(r)\right|^2 ds.
\end{aligned}
$$

By Lemma 2.5, we have

$$
\begin{aligned}
&E\sup_{t_0\leq s\leq T}\left|X^{n}(s)-X(s)\right|^2 \\
&\quad\leq C_2\,\Xi_2^{n-1}\frac{4\overline{K}_2(T-t_0+1)(T-t_0)^{2q-1}}{(2q-1)(\Gamma(q))^2}E_q\left(\frac{4\overline{K}_2(T-t_0+1)}{\Gamma(q)}(T-t_0)^{2q-1}\right).
\end{aligned}
$$

The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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