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The stable equilibrium of a system of piecewise linear difference equations

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Abstract

In this article we consider the global behavior of the system of first order piecewise linear difference equations: $x_{n+1} = |x_n| - y_n + b$ and $y_{n+1} = x_n - |y_n| - d$ where the parameters b and d are any positive real numbers. We show that for any initial condition in R^2 the solution to the system is eventually the equilibrium, $(2b + d, b)$. Moreover, the solutions of the system will reach the equilibrium within six iterations.

MSC: 39A10; 65Q10

Keywords: difference equations; piecewise linear; equilibrium; stability

1 Introduction

In applications, difference equations usually describe the evolution of a certain phenomenon over the course of time. In mathematics, a difference equation produces a sequence of numbers where each term of the sequence is defined as a function of the preceding terms. For the convenience of the reader we supply the following definitions. See [1, 2]. A *system of difference equations of the first order* is a system of the form

$$\begin{cases} x_{n+1} = f(x_n, y_n), \\ y_{n+1} = g(x_n, y_n), \end{cases} \quad n = 0, 1, \ldots, \tag{1}$$

where f and g are continuous functions which map $\mathbf{R}^2$ into $\mathbf{R}$.

A *solution* of the system of difference equations (1) is a sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ which satisfies the system for all $n \geq 0$. If we prescribe an *initial condition*

$$(x_0, y_0) \in \mathbf{R}^2$$

then

$$\begin{cases} x_1 = f(x_0, y_0), \\ y_1 = g(x_0, y_0), \end{cases}$$

$$\begin{cases} x_2 = f(x_1, y_1), \\ y_2 = g(x_1, y_1), \end{cases}$$

$$\vdots$$

and so the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of the system of difference equations (1) exists for all $n \geq 0$ and is uniquely determined by the initial condition (x_0, y_0).

A solution of the system of difference equations (1) which is constant for all $n \geq 0$ is called an *equilibrium solution*. If

$$(x_n, y_n) = (\bar{x}, \bar{y}) \quad \text{for all } n \geq 0$$

is an equilibrium solution of the system of difference equations (1), then $(\bar{x}, \bar{y})$ is called an *equilibrium point*, or simply an *equilibrium* of the system of difference equations (1).

Known methods to determine the local asymptotic stability and global stability are not easily applied to piecewise systems. This is why two of the most famous and enigmatic systems of difference equations are piecewise: the Lozi Map

$$\begin{cases} x_{n+1} = -a|x_n| + y_n + 1, \\ y_{n+1} = bx_n, \end{cases} \quad n = 0, 1, \ldots,$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$ and the parameters $a, b \in \mathbf{R}$, and the Gingerbreadman map

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1, \\ y_{n+1} = x_n, \end{cases} \quad n = 0, 1, \ldots,$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$. See [3–5] for more information regarding the Lozi map and Gingerbreadman map. In the last 30 years there has been progress in determining the local behavior of such systems but only limited progress in determining the global behavior. See [1, 6].

Ladas and Grove developed the following family of 81 piecewise linear systems:

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b, \\ y_{n+1} = x_n + c|y_n| + d, \end{cases} \quad n = 0, 1, \ldots, \tag{2}$$

where the initial condition $(x_0, y_0) \in R^2$ and the parameters a, b, c, and $d \in \{-1, 0, 1\}$, in the hope of creating prototypes that will help us understand the global behavior of more complicated systems such as the Lozi map and the Gingerbreadman map. See ([7–9]).

In 2013, Lapierre found in [8] that the solutions of the following system of piecewise linear difference equations:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1, \\ y_{n+1} = x_n - |y_n| - 1, \end{cases} \quad n = 0, 1, \ldots, \tag{3}$$

are eventually the unique equilibrium for every initial condition $(x_0, y_0) \in \mathbf{R}^2$. In this paper we extend the results by examining a generalization of System (3), that is,

$$\begin{cases} x_{n+1} = |x_n| - y_n + b, \\ y_{n+1} = x_n - |y_n| - d, \end{cases} \quad n = 0, 1, \ldots, \tag{4}$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$ and the parameters b and d are any positive real numbers.

2 Main results

Set

$$\begin{aligned}
&\text{Condition (1)} = \{(x,y) : |x| - x + |y| - y + 2b - d \geq \big|x - |y| - d\big| - \big||x| - y + b\big|\},\\
&\text{Condition (2)} = \{(x,y) : x + |x| \geq y + |y| - b + 2d\},\\
&\text{Condition (3)} = \{(x,y) : x \geq |y| + d\},\\
&\text{Condition (4)} = \{(x,y) : x \geq 0, y \geq 0 \text{ and } x \geq |y| + d\},\\
&\text{Condition (5)} = \{(x,y) : x \geq 0, y \geq 0 \text{ and } x = y + b + d\}.
\end{aligned}$$

The proof of the theorem below uses the result from the four lemmas that follow. They show that if $(x_0, y_0) \in R^2$ then (x_1, y_1) is an element of Condition (1).

Theorem 1 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be the solution of System* (4).

$$\begin{cases} x_{n+1} = |x_n| - y_n + b, \\ y_{n+1} = x_n - |y_n| - d, \end{cases} \quad n = 0, 1, \ldots,$$

with $(x_0, y_0) \in \mathbf{R}^2$ *and* $b, d \in (0, \infty)$. *Then* $\{(x_n, y_n)\}_{n=6}^{\infty}$ *is the equilibrium* $(2b + d, d)$.

Proof Suppose $(x_0, y_0) \in R^2$. First we will show that (x_2, y_2) is an element of Condition (2), that is,

$$x_2 + |x_2| \geq y_2 + |y_2| - b + 2d.$$

By Lemmas 1 through 4 we know that (x_1, y_1) is an element of Condition (1), so we have

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d \geq \big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big|.$$

Then

$$|x_1| - y_1 + b + \big||x_1| - y_1 + b\big| \geq x_1 - |y_1| - d + \big|x_1 - |y_1| - d\big| - b + 2d.$$

Therefore (x_2, y_2) is an element of Condition (2), as required.

Next, we will show that (x_3, y_3) is an element of Condition (3), that is

$$x_3 \geq |y_3| + d.$$

Since (x_2, y_2) is an element of Condition (2), we have

$$x_2 - |y_2| - d \geq -|x_2| + y_2 - b + d$$

and we have

$$x_2 - |y_2| - d \le |x_2| - y_2 + b - d.$$

Then

$$\big|x_2 - |y_2| - d\big| \le |x_2| - y_2 + b - d.$$

Therefore (x_3, y_3) is an element of Condition (3), as required.

Next, we will show that (x_4, y_4) is an element of Condition (4), that is

$$x_4 \ge 0, \qquad y_4 \ge 0 \quad \text{and} \quad x_4 \ge |y_4| + d.$$

Since (x_3, y_3) is an element of Condition (3) and $x_4 = |x_3| - y_3 + b$ and $y_4 = x_3 - |y_3| - d$, we see that $x_4 \ge 0$ and $y_4 \ge 0$. Also since (x_3, y_3) is an element of Condition (3), we have

$$|x_3| + x_3 \ge y_3 + |y_3| - b + 2d,$$

and so

$$x_3 - |y_3| - d \ge -|x_3| + y_3 - b + d.$$

Note that

$$x_3 - |y_3| - d \le |x_3| - y_3 + b - d.$$

Then

$$\big|x_3 - |y_3| - d\big| \le |x_3| - y_3 + b - d.$$

Therefore (x_4, y_4) is an element of Condition (4), as required.

Next, we will show that (x_5, y_5) is an element of Condition (5), that is

$$x_5 \ge 0, \qquad y_5 \ge 0 \quad \text{and} \quad x_5 = y_5 + b + d.$$

Since (x_4, y_4) is an element of Condition (4) and $x_5 = |x_4| - y_4 + b$ and $y_5 = x_4 - |y_4| - d$, we see that $x_5 \ge 0$ and $y_5 \ge 0$. Consider

$$x_5 - y_5 = |x_4| - y_4 + b - x_4 + |y_4| + d = b + d \quad \text{and so} \quad x_5 = y_5 + b + d.$$

Therefore (x_5, y_5) is an element of Condition (5), as required.

Finally, it is easy to show by direct computations that $(x_6, y_6) = (2b + d, b)$. This completes the proof of the theorem. □

The following four lemmas will show that if $(x_0, y_0) \in R^2$ then (x_1, y_1) is an element of Condition (1). Set

$$\mathcal{Q}_1 = \{(x, y)|x \geq 0 \text{ and } y \geq 0\},$$
$$\mathcal{Q}_2 = \{(x, y)|x \leq 0 \text{ and } y \geq 0\},$$
$$\mathcal{Q}_3 = \{(x, y)|x \leq 0 \text{ and } y \leq 0\},$$
$$\mathcal{Q}_4 = \{(x, y)|x \geq 0 \text{ and } y \leq 0\},$$

and recall that

$$\text{Condition (1)} = \{(x, y) : |x| - x + |y| - y + 2b - d \geq \big|x - |y| - d\big| - \big||x| - y + b\big|\}.$$

Lemma 1 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (4) *with* (x_0, y_0) *in* $\mathcal{Q}_1$. *Then* (x_1, y_1) *is an element of Condition* (1).

Proof Suppose $(x_0, y_0) \in \mathcal{Q}_1$ then $x_0 \geq 0$ and $y_0 \geq 0$. Thus

$$\begin{cases} x_1 = |x_0| - y_0 + b = x_0 - y_0 + b, \\ y_1 = x_0 - |y_0| - d = x_0 - y_0 - d. \end{cases}$$

Case 1 Suppose further $x_0 \geq y_0 + d$. We have $x_1 = x_0 - y_0 + b > 0$ and $y_1 = x_0 - y_0 - d \geq 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d = 2b - d$$

and

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = -b - d.$$

Hence (x_1, y_1) is an element of Condition (1) and Case 1 is complete.

Case 2 Suppose $x_0 < y_0 + d$ but $x_0 + b \geq y_0$. We have $x_1 = x_0 - y_o + b \geq 0$ and $y_1 = x_0 - y_0 - d < 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d = -2x_0 + 2y_0 + 2b + d$$

and

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = |2x_0 - 2y_0 + b - 2d| - 2b - d.$$

Case 2A Suppose further $2x_0 - 2y_0 + b - 2d \geq 0$. Then

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = 2x_0 - 2y_0 - 2d - b - d.$$

Since $y_1 = x_0 - y_0 - d < 0$, we have $2x_0 - 2y_0 - 2d - b < 0$. Also note that $|y_1| - y_1 + 2b > 0$, so

$$\begin{aligned}|x_1| - x_1 + |y_1| - y_1 + 2b - d &= |y_1| - y_1 + 2b - d \\ &> 2x_0 - 2y_0 - 2d - b - d \\ &= \bigl|x_1 - |y_1| - d\bigr| - \bigl||x_1| - y_1 + b\bigr|.\end{aligned}$$

Case 2A is complete.

Case 2B Suppose $2x_0 - 2y_0 + b - 2d < 0$. Then

$$\begin{aligned}\bigl|x_1 - |y_1| - d\bigr| - \bigl||x_1| - y_1 + b\bigr| &= -2x_0 + 2y_0 - 3b + d \\ &< -2x_0 + 2y_0 + 2b + d \\ &= |x_1| - x_1 + |y_1| - y_1 + 2b - d.\end{aligned}$$

Hence (x_1, y_1) is an element of Condition (1) and Case 2 is complete.

Case 3 Finally suppose $x_0 < y_0 + d$ and $x_0 + b < y_0$. We have $x_1 = x_0 - y_o + b < 0$ and $y_1 = x_0 - y_0 - d < 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d = -4x_0 + 4y_0 + d$$

and

$$\bigl|x_1 - |y_1| - d\bigr| - \bigl||x_1| - y_1 + b\bigr| = d - b.$$

Since $x_0 + b < y_0$, we have $y_0 > x_0$. Thus $4y_0 - 4x_0 > 0$. We note that $-b < 0$. Then

$$\begin{aligned}|x_1| - x_1 + |y_1| - y_1 + 2b - d &= -4x_0 + 4y_0 + d \\ &> d - b \\ &= \bigl|x_1 - |y_1| - d\bigr| - \bigl||x_1| - y_1 + b\bigr|.\end{aligned}$$

Hence (x_1, y_1) is an element of Condition (1) and Case 3 is complete. □

Lemma 2 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System* (4) *with (x_0, y_0) in $\mathcal{Q}_2$. Then (x_1, y_1) is an element of Condition* (1).

Proof Suppose $(x_0, y_0) \in \mathcal{Q}_2$ then $x_0 \leq 0$ and $y_0 \geq 0$. Thus

$$\begin{cases} x_1 = |x_0| - y_0 + b = -x_0 - y_0 + b, \\ y_1 = x_0 - |y_0| - d = x_0 - y_0 - d < 0. \end{cases}$$

Case 1 Suppose further $-x_0 + b < y_0$. We have $x_1 = -x_0 - y_0 + b < 0$ and $y_1 = x_0 - y_0 - d < 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d = -2x_1 - 2y_1 + 2b - d = 4y_0 + d$$

and

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = d - b.$$

Since $y_0 \geq 0$ and $-b < 0$, we see that (x_1, y_1) is an element of Condition (1) and so Case 1 is complete.

Case 2 Suppose $-x_0 + b \geq y_0$. We have $x_1 = -x_0 - y_0 + b \geq 0$ and $y_1 = x_0 - y_0 - d < 0$. Note that

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d = -2x_0 + 2y_0 + 2b + d > 0$$

and

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = |-2y_0 + b - 2d| + 2x_0 - 2b - d.$$

Case 2A Suppose further $b \geq 2y_0 + d$. Then

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = 2x_0 - 2y_0 - b - 3d.$$

Since $y_1 = x_0 - y_0 - d < 0$, we have $2x_0 - 2y_0 - b - 3d < 0$. Hence (x_1, y_1) is an element of Condition (1). Case 2A is complete.

Case 2B Finally suppose $b < 2y_0 + d$. Then

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = 2x_0 + 2y_0 - 3b + d.$$

Since $x_1 = -x_0 - y_0 + b \geq 0$, we have $2x_0 + 2y_0 - 3b < 0$. Hence (x_1, y_1) is an element of Condition (1) and the proof to Lemma 2 is complete. □

Lemma 3 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (4) *with* (x_0, y_0) *in* $\mathcal{Q}_3$. *Then* (x_1, y_1) *is an element of Condition* (1).

Proof Suppose $(x_0, y_0) \in \mathcal{Q}_3$ then $x_0 \leq 0$ and $y_0 \leq 0$. Thus

$$\begin{cases} x_1 = |x_0| - y_0 + b = -x_0 - y_0 + b > 0, \\ y_1 = x_0 - |y_0| - d = x_0 + y_0 - d < 0. \end{cases}$$

Then

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d = -2x_0 - 2y_0 + 2b + d > 0$$

and

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = |b - 2d| - (-2x_0 - 2y_0 + 2b + d).$$

Case 1 Suppose $b - 2d \geq 0$. Then

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = 2x_0 + 2y_0 - b - 3d < 0.$$

Hence (x_1, y_1) is an element of Condition (1) and Case 1 is complete.

Case 2 Suppose further $b - 2d < 0$. Then

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = -b + 2d + 2x_0 + 2y_0 - 2b - d = 2x_0 + 2y_0 - 3b + d.$$

Since $-2x_0 - 2y_0 + 2b > 0$, we have $2x_0 + 2y_0 - 3b < 0$. Hence (x_1, y_1) is an element of Condition (1) and the proof to Lemma 3 is complete. □

Lemma 4 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (4) *with* (x_0, y_0) *in* $\mathcal{Q}_4$. *Then* (x_1, y_1) *is an element of Condition* (1).

Proof Suppose $(x_0, y_0) \in \mathcal{Q}_4$ then $x_0 \geq 0$ and $y_0 \leq 0$. Thus

$$\begin{cases} x_1 = |x_0| - y_0 + b = x_0 - y_0 + b > 0, \\ y_1 = x_0 - |y_0| - d = x_0 + y_0 - d. \end{cases}$$

Case 1 Suppose further $y_1 = x_0 + y_0 - d \geq 0$. Then

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d = 2b - d$$

and

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = -b - d.$$

Hence (x_1, y_1) is an element of Condition (1) and Case 1 is complete.

Case 2 Suppose $y_1 = x_0 + y_0 - d < 0$. Then

$$|x_1| - x_1 + |y_1| - y_1 + 2b - d = -2y_1 + 2b - d = -2x_0 - 2y_0 + 2b + d$$

and

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = |2x_0 + b - 2d| + 2y_0 - 2b - d.$$

Case 2A Suppose further $2x_0 + b - 2d \geq 0$. Then

$$\big|x_1 - |y_1| - d\big| - \big||x_1| - y_1 + b\big| = 2x_0 + b - 2d + 2y_0 - 2b - d = 2x_0 + 2y_0 - 3d - b.$$

Since $2x_0 + b - 2d \geq 0$, $b > -2x_0$. Thus $-2x_0 - 2y_0 + 2b + d > 0$. Since $y_1 = x_0 + y_0 - d < 0$, we have $2x_0 + 2y_0 - 3d - b < 0$. Hence (x_1, y_1) is an element of Condition (1) and Case 2A is complete.

Case 2B Now suppose $2x_0 + b - 2d < 0$. Then

$$\left|x_1 - |y_1| - d\right| - \left||x_1| - y_1 + b\right| = -2x_0 - b + 2d + 2y_0 - 2b - d = -2x_0 + 2y_0 - 3b + d.$$

Since $y_0 \leq 0$ and $b > 0$, we see that (x_1, y_1) is an element of Condition (1) and the proof of Lemma 4 is complete. □

3 Discussion and conclusion

In this paper we showed that for any initial value $(x_0, y_0) \in R^2$ we have the following sequence:

$$\begin{aligned}
&(x_1, y_1) \in \left\{(x, y) : |x| - x + |y| - y + 2b - d \geq \left|x - |y| - d\right| - \left||x| - y + b\right|\right\},\\
&(x_2, y_2) \in \left\{(x, y) : x + |x| \geq y + |y| - b + 2d\right\},\\
&(x_3, y_3) \in \left\{(x, y) : x \geq |y| + d\right\},\\
&(x_4, y_4) \in \left\{(x, y) : x \geq 0, y \geq 0 \text{ and } x \geq |y| + d\right\},\\
&(x_5, y_5) \in \left\{(x, y) : x \geq 0, y \geq 0 \text{ and } x = y + b + d\right\},\\
&(x_6, y_6) = (\bar{x}, \bar{y}) = (2b + d, b).
\end{aligned}$$

In addition, if we begin with an initial condition that is an element of Condition (N) for $N \in \{1, 2, 3, 4, 5\}$, then it requires $6 - N$ iterations to reach the equilibrium point.

The generalized system of piecewise linear difference equations examined in this paper was created as a prototype to understand the global behavior of more complicated systems. We believe that this paper contributes broadly to the overall understanding of systems whose global behavior still remains unknown.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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