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A non-local problem with discontinuous matching condition for loaded mixed type equation involving the Caputo fractional derivative

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Abstract

This research work is devoted to investigations of the existence and uniqueness of the solution of a non-local boundary value problem with discontinuous matching condition for the loaded equation. Considering parabolic-hyperbolic type equations involves the Caputo fractional derivative and loaded part joins in Riemann-Liouville integrals. The uniqueness of a solution is proved by the method of integral energy and the existence is proved by the method of integral equations.

MSC: 35M10

Keywords: loaded equation; parabolic-hyperbolic type; Caputo fractional derivative; existence and uniqueness of solution; non-local condition; discontinuous matching condition; integral energy; integral equations

1 Introduction and formulation of a problem

It is well known that fractional derivatives have been successfully applied to problems in system biology [1], physics [2–5] and hydrology [6, 7]. Physical models fractional differential operators have recently renewed attention from scientist which is mainly due to applications as models for physical phenomena exhibiting anomalous diffusion.

Note that investigations of fractional analogs of main ODE and PDEs appear as a result of the mathematic models for real-life processes [8], and they have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering [9, 10].

In the monographs of Kilbas *et al.* [11], Miller and Ross [12], Podlubny [13], and Samko *et al.* [14] we can see significant development of fractional differential equations.

Very recently some basic theory for the initial boundary value problem (BVP)s of fractional differential equations involving a Riemann-Liouville differential operator of order $0 < \alpha \leq 1$ has been discussed by Lakshmikantham and Vatsala [15, 16]. In a series of papers (see [17, 18]) the authors considered some classes of initial value problems for functional differential equations involving Riemann-Liouville and Caputo fractional derivatives of order $0 < \alpha \leq 1$.

It is well known that most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The numerical solutions based on finite difference methods and several spectral algorithms for fractional differential equations were reported in Refs. [19–25].

It should be noted that problems for a class of fractional differential system and for the non-line differential equations with integral conditions were investigated in [26–30] and BVPs for the mixed type equations involving the Caputo and the Riemann-Liouville fractional differential operators were investigated by many authors; see for instance [31–33].

BVPs discounting matching conditions for the loaded equations with fractional derivative have not been investigated yet.

This paper deals the existence and uniqueness of a solution of the non-local problem with discontinuous matching condition for a loaded mixed type equation:

$$0 = \begin{cases} u_{xx} - {}_C D_{oy}^{\alpha} u + p(x,y) \int_x^1 (t-x)^{\beta-1} u(t,0)\,dt, & \text{at } y > 0, \\ u_{xx} - u_{yy} + q(x,y) \int_{x+y}^1 (t-x-y)^{\gamma-1} u(t,0)\,dt, & \text{at } y < 0 \end{cases} \tag{1}$$

involving the Caputo fractional derivative operator [34]:

$${}_C D_{oy}^{\alpha} f = \frac{1}{\Gamma(1-\alpha)} \int_0^y (y-t)^{-\alpha} f'(t)\,dt, \tag{2}$$

where $0 < \alpha, \beta, \gamma < 1$.

Definition The Riemann-Liouville integral-differential operator of fractional order α ($\alpha \in R$), starting from the point a, is represented as follows [34]:

$$D_{ax}^{\alpha} f(x) = \frac{\operatorname{sign}(x-a)}{\Gamma(-\alpha)} \int_a^x \frac{f(t)}{|x-t|^{\alpha+1}}\,dt, \quad \alpha < 0; \tag{3}$$

$$D_{ax}^{\alpha} f(x) = f(x), \quad \alpha = 0;$$

$$D_{ax}^{\alpha} f(x) = \operatorname{sign}^k(x-a) \frac{d^k}{dt^k} D_{ax}^{\alpha-k} f(x), \quad k-1 < \alpha \le k, k \in N.$$

Definition The Caputo differential operator of fractional order α ($\alpha > 0$) is represented as follows [34]:

$${}_C D_{ax}^{\alpha} f(x) = \operatorname{sign}^k(x-a) D_{ax}^{\alpha-k} f^{(k)}(x), \quad k-1 < \alpha \le k, k \in N.$$

Let us take Ω, a domain, bounded with segments : $A_1A_2 = \{(x,y) : x = 1, 0 < y < h\}$, $B_1B_2 = \{(x,y) : x = 0, 0 < y < h\}$, $B_2A_2 = \{(x,y) : y = h, 0 < x < 1\}$ at the $y > 0$, and characteristics: $A_1C : x - y = 1$; $B_1C : x + y = 0$ of equation (1) at $y < 0$, where $A_1(1;0)$, $A_2(1;h)$, $B_1(0;0)$, $B_2(0;h)$, $C(\frac{1}{2}; -\frac{1}{2})$.

Introduce the notations: $\theta(x) = \frac{x+1}{2} + i \cdot \frac{x-1}{2}$, $i^2 = -1$. We have

$$\Omega^+ = \Omega \cap (y > 0), \quad \Omega^- = \Omega \cap (y < 0), \quad I_1 = \left\{x : \frac{1}{2} < x < 1\right\}, \quad I_2 = \{y : 0 < y < h\}.$$

In the domain of Ω the following problem is investigated.

Problem I Find a solution $u(x,y)$ of equation (1) from the following class of functions:

$$W = \{u(x,y) : u(x,y) \in C(\bar{\Omega}) \cap C^2(\Omega^-), u_{xx} \in C(\Omega^+), {}_C D_{oy}^{\alpha} u \in C(\Omega^+)\}$$

satisfying the boundary conditions

$$u(x,y)|_{A_1A_2} = \varphi(y), \quad 0 \le y \le h, \tag{4}$$

$$u(x,y)|_{B_1B_2} = \psi(y), \quad 0 \le y \le h, \tag{5}$$

$$\frac{d}{dx} u(\theta(x)) = a(x)u_y(x,0) + b(x)u_x(x,0) + c(x)u(x,0) + d(x), \quad x \in I_1, \tag{6}$$

and the gluing condition:

$$\lim_{y\to+0} y^{1-\alpha} u_y(x,y) = \lambda(x)u_y(x,-0), \quad (x,0) \in A_1B_1, \tag{7}$$

where $\varphi(y)$, $\psi(y)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$, and $\lambda(x)$ ($\lambda(x) \neq 0$) are given functions.

2 Results and discussion

The uniqueness of solution of Problem I.

In the sequel, we assume that $q(x,y) = -q_1(x+y)q_2(x-y)$. In fact, equation (1) at $y \le 0$ and on the characteristics coordinate $\xi = x + y$ and $\eta = x - y$ in summary looks like:

$$u_{\xi\eta} = \frac{q_1(\xi)q_2(\eta)}{4} \int_{\xi}^{1} (t-\xi)^{\gamma-1} u(t,0)\, dt. \tag{8}$$

Let us denote $u(x,0) = \tau(x)$, $0 \le x \le 1$; $u_y(x,-0) = \nu^-(x)$, $0 < x < 1$;

$$\lim_{y\to+0} y^{1-\alpha} u_y(x,y) = \nu^+(x), \quad 0 < x < 1.$$

It is well known that a solution of the Cauchy problem for equation (1) in the domain Ω^- can be represented as follows:

$$\begin{aligned} u(x,y) = {} & \frac{\tau(x+y) + \tau(x-y)}{2} - \frac{1}{2} \int_{x+y}^{x-y} \nu^-(t)\, dt \\ & + \frac{1}{4} \int_{x+y}^{x-y} q_1(\xi)\, d\xi \int_{\xi}^{x-y} q_2(\eta)\, d\eta \int_{\xi}^{1} (t-\xi)^{\gamma-1} \tau(t)\, dt. \end{aligned} \tag{9}$$

After using condition (6) and taking (3) into account from (9) we will get

$$(2a(x) - 1)\nu^-(x) = \Gamma(\gamma) q_1(x)\tilde{q}_2(x) D_{x1}^{-\gamma} \tau(x) + (1 - 2b(x))\tau'(x) - 2c(x)\tau(x) - 2d(x), \tag{10}$$

where $\tilde{q}_2(x) = \int_x^1 q_2(\eta)\, d\eta$.

Considering the notations and gluing condition (7) we have

$$\nu^+(x) = \lambda(x)\nu^-(x). \tag{11}$$

Further from equation (1) at $y \to +0$ taking (2), (11) into account, and

$$\lim_{y\to 0} D_{0y}^{\alpha-1} f(y) = \Gamma(\alpha) \lim_{y\to 0} y^{1-\alpha} f(y)$$

we get [28]

$$\tau''(x) - \lambda(x)\Gamma(\alpha)\nu^-(x) + \Gamma(\beta)p(x,0)D_{x1}^{-\beta}\tau(x) = 0. \tag{12}$$

Theorem 1 *If the following conditions are satisfied*:

$$\frac{\lambda(0)q_1(0)\tilde{q}_2(0)}{2a(0)-1} \geq 0, \qquad p(0,0) \leq 0, \qquad p'(x,0) \leq 0; \tag{13}$$

$$\left(\frac{q_1(x)\tilde{q}_2(x)}{2a(x)-1}\lambda(x)\right)' \geq 0, \qquad \frac{\lambda(x)c(x)}{2a(x)-1} \leq 0, \qquad \left(\frac{1-2b(x)}{2a(x)-1}\lambda(x)\right)' \leq 0, \tag{14}$$

then the solution $u(x,y)$ *of Problem* I *is unique*.

Proof It is well known that, if a homogeneous problem has only a trivial solution, then we can state that the original problem has a unique solution. To this aim we assume that Problem I has two solutions, then denoting the difference of these as $u(x,y)$ we will get an appropriate homogeneous problem.

We multiply equation (12) by $\tau(x)$ and integrate from 0 to 1:

$$\int_0^1 \tau''(x)\tau(x)\,dx - \Gamma(\alpha)\int_0^1 \lambda(x)\tau(x)\nu^-(x)\,dx + \Gamma(\beta)\int_0^1 \tau(x)p(x,0)D_{x1}^{-\beta}\tau(x)\,dx = 0. \tag{15}$$

We will investigate the integral

$$I = \Gamma(\alpha)\int_0^1 \lambda(x)\tau(x)\nu^-(x)\,dx - \Gamma(\beta)\int_0^1 \tau(x)p(x,0)D_{x1}^{-\beta}\tau(x)\,dx.$$

Taking (10) into account $d(x) = 0$ we get

$$\begin{aligned}
I ={}& \frac{\Gamma(\alpha)\Gamma(\gamma)}{2}\int_0^1 \frac{q_1(x)\tilde{q}_2(x)}{2a(x)-1}\lambda(x)\tau(x)D_{x1}^{-\gamma}\tau(x)\,dx \\
&+ \Gamma(\alpha)\int_0^1 \frac{(1-2b(x))\lambda(x)}{2a(x)-1}\tau(x)\tau'(x)\,dx \\
&- \Gamma(\alpha)\int_0^1 \frac{\lambda(x)c(x)}{2a(x)-1}\tau^2(x)\,dx - \Gamma(\beta)\int_0^1 \tau(x)p(x,0)D_{x1}^{-\beta}\tau(x)\,dx \\
={}& \frac{\Gamma(\alpha)}{2}\int_0^1 \frac{q_1(x)\tilde{q}_2(x)}{2a(x)-1}\lambda(x)\tau(x)\,dx\int_x^1 (t-x)^{\gamma-1}\tau(t)\,dt \\
&- \frac{\Gamma(\alpha)}{2}\int_0^1 \frac{1-2b(x)}{2a(x)-1}\lambda(x)\,d\bigl(\tau^2(x)\bigr) \\
&- \Gamma(\alpha)\int_0^1 \frac{\lambda(x)c(x)}{2a(x)-1}\tau^2(x)\,dx - \int_0^1 \tau(x)p(x,0)\,dx\int_x^1 (t-x)^{\beta-1}\tau(t)\,dt.
\end{aligned} \tag{16}$$

Considering $\tau(1) = 0$, $\tau(0) = 0$ (deduced from the conditions (4), (5) in the homogeneous case) and on the base of the formula [35] we have

$$|x-t|^{-\gamma} = \frac{1}{\Gamma(\gamma)\cos\frac{\pi\gamma}{2}} \int_0^\infty z^{\gamma-1}\cos\bigl[z(x-t)\bigr]\,dz, \quad 0 < \gamma < 1.$$

After some simplifications from (16) we will get

$$\begin{aligned}
I ={}& \frac{\Gamma(\alpha)q_1(0)\tilde{q}_2(0)\lambda(0)}{4(2a(0)-1)\Gamma(1-\gamma)\sin\frac{\pi\gamma}{2}} \int_0^\infty z^{-\gamma}\biggl[\biggl(\int_0^1 \tau(t)\cos zt\,dt\biggr)^2 \\
&+ \biggl(\int_0^1 \tau(t)\sin zt\,dt\biggr)^2\biggr]\,dz \\
&+ \frac{\Gamma(\alpha)}{4\Gamma(1-\gamma)\sin\frac{\pi\gamma}{2}} \int_0^\infty z^{-\gamma}\,dz \int_0^1 \frac{\partial}{\partial x}\biggl[\lambda(x)\frac{q_1(x)\tilde{q}_2(x)}{2a(x)-1}\biggr] \\
&\times \biggl[\biggl(\int_x^1 \tau(t)\cos zt\,dt\biggr)^2 + \biggl(\int_x^1 \tau(t)\sin zt\,dt\biggr)^2\biggr]\,dx \\
&- \frac{\Gamma(\alpha)}{2}\int_0^1 \tau^2(x)\biggl(\lambda(x)\frac{1-2b(x)}{2a(x)-1}\biggr)'\,dx - 2\Gamma(\alpha)\int_0^1 \frac{\lambda(x)c(x)}{2a(x)-1}\tau^2(x)\,dx \\
&- \frac{p(0,0)}{2\Gamma(1-\beta)\sin\frac{\pi\beta}{2}} \int_0^\infty z^{-\beta}\biggl[\biggl(\int_0^1 \tau(t)\cos zt\,dt\biggr)^2 + \biggl(\int_0^1 \tau(t)\sin zt\,dt\biggr)^2\biggr]\,dz \\
&- \frac{1}{2\sin\frac{\pi\beta}{2}\Gamma(1-\beta)} \int_0^\infty z^{-\beta}\,dz \int_0^1 \frac{\partial}{\partial x}\bigl[p(x,0)\bigr] \\
&\times \biggl[\biggl(\int_x^1 \tau(t)\cos zt\,dt\biggr)^2 + \biggl(\int_x^1 \tau(t)\sin zt\,dt\biggr)^2\biggr]\,dx.
\end{aligned} \tag{17}$$

Thus, due to conditions (13), (14) from (17) we infer that $\tau(x) \equiv 0$. Hence, based on the solution of the first boundary problem for equation (1) [32, 33] by using conditions (4) and (5) we will get $u(x,y) \equiv 0$ in $\overline{\Omega}^+$. Further, from the functional relations (10), taking into account $\tau(x) \equiv 0$, we deduce that $\nu^-(x) \equiv 0$. Consequently, based on the solution (9) we obtain $u(x,y) \equiv 0$ in a closed domain $\overline{\Omega}^-$. □

The existence of a solution of Problem I.

Theorem 2 *If conditions* (13), (14) *are satisfied and*

$$\varphi(y), \psi(y) \in C(\overline{I_2}) \cap C^1(I_2), \quad p(x,0) \in C(\overline{A_1B_1}) \cap C^2(A_1B_1), \tag{18}$$

$$q(x,y) \in C\bigl(\overline{\Omega^-}\bigr) \cap C^2\bigl(\Omega^-\bigr), \quad a(x), b(x), c(x), d(x) \in C^1(\overline{I_1}) \cap C^2(I_1), \tag{19}$$

then the solution of the investigated problem exists.

Proof Taking (10) into account, from equation (12) we will obtain

$$\tau''(x) - A(x)\tau'(x) = f(x) - B(x)\tau(x), \tag{20}$$

where

$$f(x)=\frac{\Gamma(\alpha)\Gamma(\gamma)\lambda(x)q_1(x)\tilde{q}_2(x)}{2(2a(x)-1)}D_{x1}^{-\gamma}\tau(x)-\Gamma(\beta)p(x,0)D_{x1}^{-\beta}\tau(x)-\frac{2\Gamma(\alpha)\lambda(x)d(x)}{2a(x)-1}, \tag{21}$$

$$A(x)=\frac{\Gamma(\alpha)\lambda(x)(1-2b(x))}{2a(x)-1}, \qquad B(x)=\frac{2\Gamma(\alpha)\lambda(x)c(x)}{2a(x)-1}. \tag{22}$$

The solution of equation (20) together with the conditions

$$\tau(0)=\psi(0), \qquad \tau(1)=\varphi(0) \tag{23}$$

has the form

$$\begin{aligned}\tau(x)={}&A_1(x)\left(\int_x^1\big(B(t)\tau(t)-f(t)\big)A_1'(t)\,dt+\frac{\varphi(0)-\psi(0)}{A_1(1)}\right)\\&-\frac{A_1(x)}{A_1(1)}\int_0^1\big(B(t)\tau(t)-f(t)\big)\frac{A_1(t)}{A_1'(t)}\,dt\\&+\int_0^x\big(B(t)\tau(t)-f(t)\big)\frac{A_1(t)}{A_1'(t)}\,dt+\psi(0),\end{aligned} \tag{24}$$

where

$$A_1(x)=\int_0^x\exp\left(\int_0^t A(z)\,dz\right)dt. \tag{25}$$

Further, considering (21) and using (3) from (24) we will get

$$\begin{aligned}\tau(x)={}&A_1(x)\left[\int_x^1 A_1'(t)B(t)\tau(t)\,dt-\frac{\Gamma(\alpha)}{2}\int_x^1\frac{\lambda(t)q_1(t)\tilde{q}_2(t)}{2a(t)-1}A_1'(t)\,dt\int_t^1(s-t)^{\gamma-1}\tau(s)\,ds\right]\\&+A_1(x)\int_x^1 A_1'(t)p(t,0)\,dt\int_t^1(s-t)^{\beta-1}\tau(s)\,ds-\frac{A_1(x)}{A_1(1)}\int_0^1\frac{A_1(t)}{A_1'(t)}B(t)\tau(t)\,dt\\&+\frac{\Gamma(\alpha)}{2}\frac{A_1(x)}{A_1(1)}\int_0^1\frac{A_1(t)\lambda(t)q_1(t)\tilde{q}_2(t)}{(2a(t)-1)A_1'(t)}\,dt\int_t^1(s-t)^{\gamma-1}\tau(s)\,ds\\&-\frac{A_1(x)}{A_1(1)}\int_0^1\frac{A_1(t)}{A_1'(t)}p(t,0)\,dt\int_t^1(s-t)^{\beta-1}\tau(s)\,ds+\int_0^x\frac{A_1(t)}{A_1'(t)}B(t)\tau(t)\,dt\\&-\frac{\Gamma(\alpha)}{2}\int_0^x\frac{A_1(t)\lambda(t)q_1(t)\tilde{q}_2(t)}{(2a(t)-1)A_1'(t)}\,dt\int_t^1(s-t)^{\gamma-1}\tau(s)\,ds\\&+\int_0^x\frac{A_1(t)}{A_1'(t)}p(t,0)\,dt\int_t^1(s-t)^{\beta-1}\tau(s)\,ds+f_1(x),\end{aligned} \tag{26}$$

where

$$\begin{aligned}f_1(x)={}&\left(1-\frac{A_1(x)}{A_1(1)}\right)\int_0^x\frac{2\Gamma(\alpha)d(t)A_1(t)\lambda(t)}{A_1'(t)(2a(t)-1)}\,dt+2\Gamma(\alpha)A_1(x)\int_x^1\frac{d(t)A_1'(t)\lambda(t)}{2a(t)-1}\,dt\\&-\frac{A_1(x)}{A_1(1)}\int_x^1\frac{2\Gamma(\alpha)d(t)A_1(t)\lambda(t)}{A_1'(t)(2a(t)-1)}\,dt-\frac{A_1(x)}{A_1(1)}\big(\psi(0)-\varphi(0)\big)+\psi(0).\end{aligned} \tag{27}$$

After some simplifications (26) we will rewrite our expression in the form

$$\begin{aligned}\tau(x) &= A_1(x)\left[\int_x^1 A_1'(t)B(t)\tau(t)\,dt - \frac{\Gamma(\alpha)}{2}\int_x^1 \tau(s)\,ds\int_x^s (s-t)^{\gamma-1}\frac{\lambda(t)q_1(t)\tilde{q}_2(t)}{2a(t)-1}A_1'(t)\,dt\right] \\ &\quad + A_1(x)\int_x^1 \tau(s)\,ds\int_x^s (s-t)^{\beta-1}A_1'(t)p(t,0)\,dt - \frac{A_1(x)}{A_1(1)}\int_0^1 \frac{A_1(t)}{A_1'(t)}B(t)\tau(t)\,dt \\ &\quad + \frac{\Gamma(\alpha)}{2}\frac{A_1(x)}{A_1(1)}\int_0^1 \tau(s)\,ds\int_0^s (s-t)^{\gamma-1}\frac{A_1(t)\lambda(t)q_1(t)\tilde{q}_2(t)}{(2a(t)-1)A_1'(t)}\,dt \\ &\quad - \frac{A_1(x)}{A_1(1)}\int_0^1 \tau(s)\,ds\int_0^s (s-t)^{\beta-1}\frac{A_1(t)}{A_1'(t)}p(t,0)\,dt + \int_0^x \frac{A_1(t)}{A_1'(t)}B(t)\tau(t)\,dt \\ &\quad - \frac{\Gamma(\alpha)}{2}\left(\int_0^x \tau(s)\,ds\int_0^s \frac{A_1(t)\lambda(t)q_1(t)\tilde{q}_2(t)}{A_1'(t)(2a(t)-1)(s-t)^{1-\gamma}}\,dt\right. \\ &\quad \left. + \int_x^1 \tau(s)\,ds\int_0^x \frac{A_1(t)\lambda(t)q_1(t)\tilde{q}_2(t)}{A_1'(t)(2a(t)-1)(s-t)^{1-\gamma}}\,dt\right) \\ &\quad + \int_0^x \tau(s)\,ds\int_0^s \frac{A_1(t)(s-t)^{\beta-1}}{A_1'(t)}p(t,0)\,dt \\ &\quad + \int_x^1 \tau(s)\,ds\int_0^x \frac{A_1(t)(s-t)^{\beta-1}}{A_1'(t)}p(t,0)\,dt + f_1(x)\end{aligned}$$

i.e., in summary, we have the integral equation

$$\tau(x) = \int_0^1 K(x,t)\tau(t)\,dt + f_1(x). \tag{28}$$

Here

$$K(x,t) = \begin{cases} K_1(x,s), & 0 \le t \le x, \\ K_2(x,s), & x \le t \le 1, \end{cases} \tag{29}$$

$$\begin{aligned}K_1(x,s) &= \left(\frac{A_1(x)}{A_1(1)} - 1\right)\left[\frac{\Gamma(\alpha)}{2}\int_0^s (s-t)^{\gamma-1}\frac{A_1(t)\lambda(t)q_1(t)\tilde{q}_2(t)}{(2a(t)-1)A_1'(t)}\,dt - \frac{A_1(s)}{A_1'(s)}B(s)\right] \\ &\quad - \left(\frac{A_1(x)}{A_1(1)} - 1\right)\int_0^s (s-t)^{\beta-1}\frac{A_1(t)}{A_1'(t)}p(t,0)\,dt,\end{aligned} \tag{30}$$

$$\begin{aligned}K_2(x,s) &= A_1(x)\left(A_1'(s)B(s) - \frac{\Gamma(\alpha)}{2}\int_x^s (s-t)^{\gamma-1}\frac{\lambda(t)q_1(t)\tilde{q}_2(t)}{2a(t)-1}A_1'(t)\,dt\right) \\ &\quad + A_1(x)\int_x^s (s-t)^{\beta-1}A_1'(t)p(t,0)\,dt - \frac{A_1(x)}{A_1(1)}\frac{A_1(s)}{A_1'(s)}B(s) \\ &\quad + \frac{\Gamma(\alpha)}{2}\frac{A_1(x)}{A_1(1)}\int_0^s (s-t)^{\gamma-1}\frac{A_1(t)\lambda(t)q_1(t)\tilde{q}_2(t)}{(2a(t)-1)A_1'(t)}\,dt \\ &\quad - \frac{\Gamma(\alpha)}{2}\int_0^x \frac{A_1(t)\lambda(t)q_1(t)\tilde{q}_2(t)}{A_1'(t)(2a(t)-1)}(s-t)^{\gamma-1}\,dt \\ &\quad + \left(1 - \frac{A_1(x)}{A_1(1)}\right)\int_0^x \frac{A_1(t)(s-t)^{\beta-1}}{A_1'(t)}p(t,0)\,dt.\end{aligned} \tag{31}$$

Due to the class (18), (19) of the given functions and after some evaluations, from (30), (31) and (27), (29) we will conclude that $|K(x,t)| \le \text{const}$, $|f_1(x)| \le \text{const}$.

Since the kernel $K(x,t)$ is continuous and the function on the right-hand side $F(x)$ is continuously differentiable, we can write the solution of integral equation (28) via the resolvent-kernel:

$$\tau(x) = f_1(x) - \int_0^1 \Re(x,t) f_1(t)\,dt, \tag{32}$$

where $\Re(x,t)$ is the resolvent-kernel of $K(x,t)$.

The unknown functions $\nu^-(x)$ and $\nu^+(x)$ we will find accordingly from (10) and (11):

$$\begin{aligned}\nu^-(x) = {}& \frac{q_1(x)\tilde{q}_2(x)}{2(1-2a(x))}\int_x^1 (t-x)^{\gamma-1}\,dt \int_0^1 \Re(t,s) f_1(s)\,ds \\ &+ \frac{q_1(x)\tilde{q}_2(x)}{2(2a(x)-1)}\int_x^1 (t-x)^{\gamma-1} f_1(t)\,dt \\ &+ \frac{1-2b(x)}{2a(x)-1} f_1'(x) - \frac{1-2b(x)}{2a(x)-1}\int_0^1 \frac{\partial \Re(x,t)}{\partial x} f_1(t)\,dt - \frac{2c(x)}{2a(x)-1} f_1(x) \\ &+ \frac{2c(x)}{2a(x)-1}\int_0^1 \Re(x,t) f_1(t)\,dt - \frac{2d(x)}{2a(x)-1}\end{aligned}$$

and $\nu^+(x) = \lambda(x)\nu^-(x)$.

Considering the solution of Problem I in the domain Ω^+ we write our expression as follows [33, 36]:

$$\begin{aligned}u(x,y) = {}& \int_0^y G_\xi(x,y,0,\eta)\psi(\eta)\,d\eta - \int_0^y G_\xi(x,y,1,\eta)\varphi(\eta)\,d\eta + \int_0^1 G_0(x-\xi,y)\tau(\xi)\,d\xi \\ &- \int_0^y \int_0^1 G(x,y,0,\eta) p(\xi)\,d\xi\,d\eta \int_\xi^1 (t-\xi)^{\beta-1}\tau(t)\,dt.\end{aligned} \tag{33}$$

Here $G_0(x-\xi,y) = \frac{1}{\Gamma(1-\alpha)}\int_0^y \eta^{-\alpha} G(x,y,\xi,\eta)\,d\eta$,

$$G(x,y,\xi,\eta) = \frac{(y-\eta)^{\alpha/2-1}}{2}\sum_{n=-\infty}^{\infty}\left[e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\alpha/2}}\right) - e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\alpha/2}}\right)\right].$$

Here the Green's function of the first boundary problem equation (1) in the domain Ω^+ with the Riemanne-Liouville fractional differential operator instead of the Caputo ones [34],

$$e^{1,\delta}_{1,\delta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\delta-\delta n)},$$

is a Wright type function [36]. □

3 Conclusion

If conditions (13), (14), (18), and (19) are satisfied, then the solution of Problem I is unique and exists, and this solution in the domains Ω^- and Ω^+ will be found by equations (9) and (33), respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors have participated into the results obtained. The collaboration of each one cannot be separated in different parts of the paper. Both of them have made substantial contributions to the theoretical results. The two authors have been involved in drafting the manuscript and revising it critically for important intellectual content. Both authors have given final approval of the version to be published.

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